On the Modeling of Robots in Contact with a Dynamic Environment

Alessandro De Luca   Costanzo Manes
Dipartimento di Informatica e Sistemistica
Università degli Studi di Roma "La Sapienza"
Via Eudossiana 18, 00184 Roma,Italy

Abstract—**A new modeling approach for describing motion of robots in contact with a possibly dynamic environment is presented. The proposed technique allows to model all those cases in which purely kinematic constraints imposed on the robot end-effector live together with dynamic interactions. Suitable parametrizations are introduced for the environment configurations and constrained end-effector poses and for the exchanged forces. The generalized directions of 'static' environment reaction forces (orthogonal to the directions of admissible end-effector motion) and those of 'active' forces (responsible for energy transfer between robot and environment) are formally characterized. The overall dynamics of the robot-environment system is then derived in a unique framework. The obtained model structure is shown to be suitable for the design of hybrid control laws. Simple but significative examples are reported to illustrate the modeling procedure.**

I. INTRODUCTION

Many industrial tasks involve intentional contact among objects, as in assembly operations, or between tools and workpieces, as in deburring, grinding, polishing or cutting. It has been recognized [1,2] that such tasks cannot be executed by industrial robots under traditional position control, without relying on expensive special purpose effectors. Therefore, several compliant control strategies have been investigated in advanced robotics [3] and the handling of simple situations is becoming mature for industrial transfer. This paper gives a contribution in the field of modeling, providing a technique for the kinematic and dynamic description of more general classes of interactions than usually considered.

Contact between robot and environment may or may not imply energy exchange. When the environment imposes purely kinematic constraints on the end-effector motion, only a static balance of forces and torques occurs at the contact and thus without energy transfer or dissipation —friction effects being neglected. This modeling assumption underlies the constrained approach of Yoshikawa [4] and McClamroch and Wang [5], where an algebraic vector equation restricts the set of feasible end-effector poses. On the other hand, the presence of contact dynamics with energy exchange between robot and environment is commonly treated using a full-dimensional linear impedance model (Hogan [6], Kazerooni *et al.* [7]). Small deformations are implicitly assumed for the environment.

As opposed to 'completely static' or 'limited dynamic' interaction, there is a whole class of tasks that is accurately modeled only by considering a more general dynamic behavior for the environment. With this respect, a robot may exert *active forces* at the tip, i.e. forces not compensated by a constraint reaction and producing work on the environment geometry, still being completely constrained along other directions. The original modeling approach proposed here deals with all those cases in which the end-effector is kinematically constrained and/or dynamically coupled with the external world.

The kinematic description of the robot-environment system is revisited, expressing the overall configuration in terms of a proper set of independent parameters. Generalized (6-dimensional) admissible directions of end-effector motion are geometrically characterized and, accordingly, generalized reaction forces are determined using energy transfer arguments. When the dynamics of the environment is included, additional active contact forces have to be introduced. Both types of forces will be conveniently expressed in terms of another set of parameters. The dynamic equations of the robot-environment system are derived in a unique framework. The various modeling steps will be illustrated using, as a case study, different variations of the paradigmatic robotic task of turning a crank in the vertical plane, with and without considering crank dynamics.

A. Standing Assumptions

The class of robot-environment interactions considered in this paper satisfies the following assumptions:

(A1) a *one-to-one* relationship exists between the end-effector pose of the robot in contact and the environment configuration;

(A2) the environment is an autonomous mechanical system, i.e. not externally driven;

(A3) possible kinematic constraints imposed by the environment on the robot end-effector are holonomic and frictionless.

The first assumption can be restated saying that the environment geometry is *non-redundant* in the usual robotic

This work is supported by the *Consiglio Nazionale delle Ricerche*, grant no. 90.00381.PF67 (*Progetto Finalizzato Robotica*).

7803-0078/91/0600-0568$01.00 ©1991 IEEE

sense. Holonomy is required in order to simplify model derivation, since velocity constraints are then simply obtained from positional ones. Moreover, unilateral constraints can be treated as bilateral once the contact is assured. Finally, absence of friction is a common assumption at this stage of modeling.

II. KINEMATIC MODELING

Consider a robot with n degrees of freedom, consisting of an open kinematic chain of rigid bodies. Robot arm configurations are identified by the joint variables vector $\mathbf{q} \in \mathbb{R}^n$. In general, $n \geq 6$ so that arbitrary positioning and orientation of the end-effector is allowed, together with the possible presence of kinematic redundancy for the arm. A world reference frame ${}^0S$ is fixed e.g. at the robot base, while a frame ${}^nS$ is attached to the arm tip. In the following, ${}^i\mathbf{x}$ denotes the expression of a vector $\mathbf{x}$ in ${}^iS$ coordinates. Superscripts will be dropped when assertions involving vectors are coordinate-independent.

Let $\mathbf{r}$ be the position vector of the origin of ${}^nS$ with respect to ${}^0S$. A minimal representation is used for the orientation of ${}^nS$ with respect to ${}^0S$, e.g. Euler angles $\mathbf{o} = (\phi, \theta, \psi)$. Position and orientation can be organized in a single 6-dimensional pose vector $\mathbf{p}^T = (\mathbf{r}^T, \mathbf{o}^T)$. As a consequence, end-effector direct and differential kinematics are defined *from the robot side* by

$$ {}^0\mathbf{p} = \mathbf{k}(\mathbf{q}), \qquad {}^0\dot{\mathbf{p}} = \frac{\partial \mathbf{k}(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} = \mathbf{J_k}(\mathbf{q})\dot{\mathbf{q}}. \tag{1} $$

The generalized end-effector velocity ${}^0\mathbf{v}^T = ({}^0\dot{\mathbf{r}}^T, {}^0\omega^T)$, composed of linear velocity ${}^0\dot{\mathbf{r}}$ and angular velocity ${}^0\omega$, is related to ${}^0\dot{\mathbf{p}}$ by means of a matrix $\mathbf{G}$ depending on the set of orientation angles, so that

$$ {}^0\mathbf{v} = \mathbf{G}({}^0\mathbf{p}){}^0\dot{\mathbf{p}} \quad \text{with} \quad \mathbf{G}({}^0\mathbf{p}) = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{O} \\ \mathbf{O} & \widehat{\mathbf{G}}(\phi, \theta, \psi) \end{bmatrix}. \tag{2} $$

As a result one has

$$ {}^0\mathbf{v} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \quad \text{with} \quad \mathbf{J}(\mathbf{q}) = \mathbf{G}(\mathbf{k}(\mathbf{q}))\mathbf{J_k}(\mathbf{q}), \tag{3} $$

where $\mathbf{J}(\mathbf{q})$ is the standard robot Jacobian. For simplicity, end-effector orientation is always supposed to be in nonsingular configurations for the chosen set of angles ($det\, \mathbf{G}(\mathbf{k}(\mathbf{q})) \neq 0$).

Different types of contacts are allowed between the robot end-effector and the environment, ranging from simple point contact to power grasps. The basic issue is that the end-effector pose can be expressed *from the environment side* in terms of a parameter vector $\mathbf{s} \in \mathbb{R}^e$, with $e \leq 6$. Since this choice is not unique, in order to determine a convenient parametrization we will proceed as follows. A first set of variables $\mathbf{s}_D \in \mathbb{R}^d$ is needed to describe environment dynamics, when present, and will appear in the associated equations of motion. An additional set of purely kinematic variables $\mathbf{s}_K \in \mathbb{R}^k$, $k = e - d$, may be required to specify uniquely the end-effector pose ${}^0\mathbf{p}$. This aspect is not considered in the common literature (see e.g. [8,9]). The two vectors are then merged as $\mathbf{s} = (\mathbf{s}_K, \mathbf{s}_D)$ so that

$$ {}^0\mathbf{p} = \Gamma(\mathbf{s}), \qquad {}^0\dot{\mathbf{p}} = \frac{\partial \Gamma}{\partial \mathbf{s}} \dot{\mathbf{s}}. \tag{4} $$

Using (2) and (4), one has

$$ {}^0\mathbf{v} = \mathbf{T}(\mathbf{s})\dot{\mathbf{s}}, \qquad \text{with} \quad \mathbf{T}(\mathbf{s}) = \mathbf{G}\big(\Gamma(\mathbf{s})\big)\frac{\partial \Gamma(\mathbf{s})}{\partial \mathbf{s}}. \tag{5} $$

According to (A1), matrix $\mathbf{T}$ is assumed to be full rank. The parametrization of end-effector pose transfers directly to velocity, allowing to separate contributions due to *kinematic* and to *dynamic* degrees of freedom of the environment

$$ {}^0\mathbf{v} = \mathbf{T}_K(\mathbf{s})\,\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\,\dot{\mathbf{s}}_D. \tag{6} $$

Remark 1. Equations (1) and (4) couple together robot and environment kinematics. If $e = 6$, this coupling does not impose any kinematic constraint on the robot end-effector motion, resulting just in a mapping between robot pose and environment parameters. Viceversa, if $e < 6$ and $k = e$ there are actually $6 - e$ kinematic constraints imposed on the end-effector. ■

Remark 2. Equating the two expressions of velocity ${}^0\mathbf{v}$ in terms of robot coordinates (3) and of environment coordinates (5), it follows that the number of degrees of freedom for the constrained end-effector will be $m = dim\,(span\,[\mathbf{J}(\mathbf{q})] \cap span\,[\mathbf{T}(\mathbf{s})])$. Obviously, first-order motion is inhibited if this intersection is empty. ■

In association with (5), *generalized reaction forces* $\mathbf{F}_R$ are defined as those which do not deliver power on admissible velocities at the contact, or

$$ \mathbf{v}^T\mathbf{F}_R = [\dot{\mathbf{r}}^T \;\; \omega^T] \begin{bmatrix} \mathbf{f}_R \\ \mathbf{m}_R \end{bmatrix} = 0, \tag{7} $$

where $\mathbf{f}_R$ are reaction forces and $\mathbf{m}_R$ are reaction torques at the robot tip. By convention, these are assumed to be acting from the robot to the environment. Dual to the parametrization of velocity, a full column rank matrix $\mathbf{Y}_R$ can be determined so that reaction forces are expressed as

$$ {}^0\mathbf{F}_R = \mathbf{Y}_R(\mathbf{s})\lambda_R. \tag{8} $$

Since *all* reaction forces should belong to $span\,[\mathbf{Y}_R]$, the number of columns defining this matrix is maximal, and thus $\lambda_R \in \mathbb{R}^{6-e}$. Vector λ_R will parametrize static reaction forces in the same way as $\dot{\mathbf{s}}$ parametrizes velocities. From the orthogonality condition (7), it follows that

$$ \mathbf{T}^T(\mathbf{s})\mathbf{Y}_R(\mathbf{s}) = \mathbf{0}_{e\times(6-e)}. \tag{9} $$

Since $\mathbf{T}$ is full rank, the 6×6 matrix $[\mathbf{T}(\mathbf{s})\ \mathbf{Y}_R(\mathbf{s})]$ will be nonsingular.

Remark 3. The columns of $\mathbf{T}$ and $\mathbf{Y}_R$ specify 6-dimensional directions for velocities and forces, intended in a generalized sense. Therefore, a column of $\mathbf{T}$ may represent an angular velocity *together with* a related linear one. Similarly, a column of $\mathbf{Y}_R$ contains a direction for linear force and a momentum axis. ∎

Remark 4. Equation (9) expresses orthogonality between force and velocity subspaces. Being based on energetic considerations, this is an invariant property and its definition is coordinate-independent. Conversely, orthogonality *among* generalized velocity directions (or among generalized forces) is not canonically defined due to the non-homogeneity of the vector components. An improper use of this concept has led researchers to basic questioning of conventional hybrid control approaches [10]. ∎

III. DYNAMIC MODELING

A robot interacts with a dynamic environment not only through the balance of reaction forces associated with purely kinematic constraints. Instead, an energy exchange between robot and environment is allowed when *active contact forces* come into play, defined along dynamic directions that are no more orthogonal to those of admissible motions. These forces will appear as inputs both to the dynamic model of the environment and of the robot. A Lagrangian approach will be followed for deriving the equations of motion, using the set of generalized coordinates $\mathbf{s}_D$ for the environment in the same way as $\mathbf{q}$ for the manipulator. Let the kinetic energy, potential energy, and Lagrangian of the robot be

$$K = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}, \quad P = P(\mathbf{q}), \quad L = K - P, \tag{10a}$$

and the corresponding quantities for the environment

$$\begin{aligned} K_E &= \frac{1}{2}\dot{\mathbf{s}}_D^{\mathrm{T}}\mathbf{B}_E(\mathbf{s}_D)\dot{\mathbf{s}}_D, \\ P_E = P_E(\mathbf{s}_D), &\qquad L_E = K_E - P_E, \end{aligned} \tag{10b}$$

so that the total Lagrangian of the interacting system is $L_T = L + L_E$. A symmetric form is taken for the positive definite inertia matrices $\mathbf{B}$ and $\mathbf{B}_E$. The robot potential energy P usually collects only gravitational terms, while P_E may contain also elastic energy, whenever some deformation is possible. Non-conservative forces performing work on $\mathbf{q}$ are the torques $\mathbf{u}$ supplied by the motors and viscous friction modeled by a dissipative term $-\mathbf{D}\dot{\mathbf{q}}$ (with $\mathbf{D} > 0$). Similarly, the only non-conservative force performing work on $\mathbf{s}_D$ is $-\mathbf{D}_E\dot{\mathbf{s}}_D$ (with $\mathbf{D}_E > 0$). The dynamic variables $\mathbf{q}$ and $\mathbf{s}_D$ are related by

$${}^0\mathbf{p} = \mathbf{k}(\mathbf{q}) = \Gamma(\mathbf{s}) \quad \Rightarrow \quad \Gamma(\mathbf{s}) - \mathbf{k}(\mathbf{q}) = \mathbf{0}, \tag{11}$$

in which *also* the kinematic variables $\mathbf{s}_K$ appear. In the presence of (11), the composite Lagrangian becomes

$$L_C = L_T + \eta^{\mathrm{T}}[\Gamma(\mathbf{s}) - \mathbf{k}(\mathbf{q})], \tag{12}$$

where $\eta \in \mathbb{R}^6$ is the vector of Lagrange multipliers. The equations of motion are

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\mathbf{q}}}\Big) - \frac{\partial L}{\partial \mathbf{q}} + \eta^{\mathrm{T}}\frac{\partial \mathbf{k}}{\partial \mathbf{q}} &= (\mathbf{u} - \mathbf{D}\dot{\mathbf{q}})^{\mathrm{T}}, \\ \frac{d}{dt}\Big(\frac{\partial L_E}{\partial \dot{\mathbf{s}}_D}\Big) - \frac{\partial L_E}{\partial \mathbf{s}_D} - \eta^{\mathrm{T}}\frac{\partial \Gamma}{\partial \mathbf{s}_D} &= -(\mathbf{D}_E\dot{\mathbf{s}}_D)^{\mathrm{T}}, \end{aligned} \tag{13}$$

together with $\partial L_C/\partial \eta = \mathbf{0}$, which gives back (11). Developing computations,

$$\begin{aligned} \mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) &= \mathbf{u} - \mathbf{D}\dot{\mathbf{q}} - \Big(\frac{\partial \mathbf{k}}{\partial \mathbf{q}}\Big)^{\mathrm{T}}\eta, \\ \mathbf{B}_E(\mathbf{s}_D)\ddot{\mathbf{s}}_D + \mathbf{c}_E(\mathbf{s}_D, \dot{\mathbf{s}}_D) &= -\mathbf{D}_E\dot{\mathbf{s}}_D + \Big(\frac{\partial \Gamma}{\partial \mathbf{s}_D}\Big)^{\mathrm{T}}\eta, \end{aligned} \tag{14}$$

where $\mathbf{c}$ and $\mathbf{c}_E$ collects Coriolis, centrifugal, gravitational and elastic contributions.

Using (1), (4), and the virtual work principle, multipliers η can be interpreted as generalized forces associated to ${}^0\mathbf{p}$. Moreover, $\eta = \mathbf{G}^{\mathrm{T}}({}^0\mathbf{p}){}^0\mathbf{F}$ from (3) and (5). Collecting terms, the dynamic model can be rewritten as

$$\begin{aligned} \mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) &= \mathbf{u} - \mathbf{J}^{\mathrm{T}}(\mathbf{q}){}^0\mathbf{F}, && \text{robot} \\ \mathbf{B}_E(\mathbf{s}_D)\ddot{\mathbf{s}}_D + \mathbf{n}_E(\mathbf{s}_D, \dot{\mathbf{s}}_D) &= \mathbf{T}_D^{\mathrm{T}}(\mathbf{s}){}^0\mathbf{F}, && \text{environment} \end{aligned} \tag{15}$$

together with the algebraic relation (11) or, in its differential form,

$$\mathbf{T}_K(\mathbf{s})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\dot{\mathbf{s}}_D = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}. \tag{16}$$

From (15), it follows that only those contact forces ${}^0\mathbf{F}$ not orthogonal to the columns of $\mathbf{T}_D$ will affect environment dynamics. Since $\mathbf{T}_D$ is part of $\mathbf{T}$, from (9) these *active* forces will certainly not belong to *span* $[\mathbf{Y}_R]$. Therefore, they can be generated as combinations of the columns of another matrix $\mathbf{Y}_A$ such that *span* $[\mathbf{Y}_R] \cap$ *span* $[\mathbf{Y}_A] = \emptyset$. Any contact force, being composed of reaction and active terms, will be parametrized as

$${}^0\mathbf{F} = {}^0\mathbf{F}_R + {}^0\mathbf{F}_A = \mathbf{Y}_R(\mathbf{s})\lambda_R + \mathbf{Y}_A(\mathbf{s})\lambda_A = \mathbf{Y}(\mathbf{s})\lambda, \tag{17}$$

where $\mathbf{Y}_R$ and λ_R are the same as in (8). The power transfer from robot to environment takes now the form

$${}^0\mathbf{v}^{\mathrm{T}}\,{}^0\mathbf{F} = \big(\mathbf{T}_K(\mathbf{s})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\dot{\mathbf{s}}_D\big)^{\mathrm{T}}\big(\mathbf{Y}_R(\mathbf{s})\lambda_R + \mathbf{Y}_A(\mathbf{s})\lambda_A\big). \tag{18}$$

By definition of reaction forces (see also (9)),

$$[\mathbf{T}_K \quad \mathbf{T}_D]^{\mathrm{T}}\,\mathbf{Y}_R = \mathbf{0}. \tag{19}$$

Being kinematic displacements orthogonal to *all* forces,

$$\mathbf{T}_K^{\mathrm{T}}\left[\mathbf{Y}_R \quad \mathbf{Y}_A\right] = \mathbf{0}, \tag{20}$$

because there cannot be work performed on a kinematic variable. Since $\mathbf{T}$ (and thus $\mathbf{T}_K$) has full rank by assumption, $\mathbf{Y}$ and $\mathbf{Y}_R$ have $6-k$ and $6-e$ columns, so that $\mathbf{Y}_A$ has d independent columns and $\lambda_A \in I\!R^d$. Accordingly, the power transfer at the contact simplifies to

$${}^0\mathbf{v}^{\mathrm{T}}\,{}^0\mathbf{F} = \dot{\mathbf{s}}_D^{\mathrm{T}}\mathbf{T}_D^{\mathrm{T}}(\mathbf{s})\mathbf{Y}_A(\mathbf{s})\lambda_A. \tag{21}$$

Remark 5. It can be easily seen that

$$span\left[\mathbf{T}_D \quad \mathbf{T}_K \quad \mathbf{Y}_R\right] = span\left[\mathbf{Y}_A \quad \mathbf{T}_K \quad \mathbf{Y}_R\right] = I\!R^6, \tag{22}$$

but in general $span\left[\mathbf{T}_D\right] \neq span\left[\mathbf{Y}_A\right]$. This means that directions of active forces may differ from those of dynamic motion. For shortness, both will be referred as *dynamic directions*. Under (A1), matrix $\mathbf{T}_D^{\mathrm{T}}\mathbf{Y}_A$ is always nonsingular, since for any $\lambda_A \neq \mathbf{0}$ a vector $\dot{\mathbf{s}}_D$ exists such that (21) is not zero. Otherwise, $\mathbf{Y}_A\lambda_A$ would be a reaction force, contrary to its definition. Moreover, for a given parametrization $\mathbf{s}$, viz. for a given $\mathbf{T}_D$, one can always select matrix $\mathbf{Y}_A$ so that $\mathbf{T}_D^{\mathrm{T}}\mathbf{Y}_A = \mathbf{I}_{d\times d}$. ∎

Remark 6. The obtained dynamic model (15), with constraint (16) and ${}^0\mathbf{F}$ expressed by (17), is in a useful format for hybrid task specification and control. In fact, the recognition of a suitable 'task space' is generally required (see [11-13]). The parameter space of $\mathbf{s}$ and λ serves this purpose. For, observe that a task can be viewed as a sequence of environment modifications specified by means of desired values for $\mathbf{s}$ (a trajectory $\mathbf{s}_{des}(t)$). Using (4), this results in a cartesian trajectory for the robot end-effector. Also, forces needed to correctly execute the task can be specified through a trajectory $\lambda_{des}(t)$ and (17). A hybrid controller can then be designed so to keep $\mathbf{s}$ and λ at their desired values. The control issues associated to this approach are investigated in [14]. ∎

Starting from (15), the dynamic model can be manipulated so to automatically satisfy (16) and eliminate the explicit appearance of the force vector ${}^0\mathbf{F}$. This model format is useful for simulation and further analysis. To this aim, differentiate both sides of (16)

$$\begin{aligned}\mathbf{T}_K(\mathbf{s})\ddot{\mathbf{s}}_K + \dot{\mathbf{T}}_K(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}}_K + \mathbf{T}_D(\mathbf{s})\ddot{\mathbf{s}}_D + \dot{\mathbf{T}}_D(\mathbf{s},\dot{\mathbf{s}})\dot{\mathbf{s}}_D \\ = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}},\end{aligned} \tag{23}$$

and solve for acceleration in the dynamic equations (15)

$$\begin{aligned}\ddot{\mathbf{q}} &= \mathbf{B}^{-1}\mathbf{u} - \mathbf{B}^{-1}\mathbf{J}^{\mathrm{T}}\,{}^0\mathbf{F} - \mathbf{B}^{-1}\mathbf{n}, \\ \ddot{\mathbf{s}}_D &= \mathbf{B}_E^{-1}\mathbf{T}_D^{\mathrm{T}}\,{}^0\mathbf{F} - \mathbf{B}_E^{-1}\mathbf{n}_E.\end{aligned} \tag{24}$$

where term dependence is dropped. Substituting (24) and the force parametrization (17) into (23), yields

$$\mathbf{Q}(\mathbf{q},\mathbf{s})\begin{bmatrix}\lambda_A \\ \lambda_R \\ \ddot{\mathbf{s}}_K\end{bmatrix} = \mathbf{m}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \mathbf{J}(\mathbf{q})\mathbf{B}^{-1}(\mathbf{q})\mathbf{u}, \tag{25}$$

with

$$\mathbf{Q} = \left[(\mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{T}_D^{\mathrm{T}} + \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^{\mathrm{T}})\mathbf{Y}_A \quad \mathbf{J}\mathbf{B}^{-1}\mathbf{J}^{\mathrm{T}}\mathbf{Y}_R \quad \mathbf{T}_K\right] \tag{26}$$

and

$$\mathbf{m} = -\dot{\mathbf{T}}_K\dot{\mathbf{s}}_K - \dot{\mathbf{T}}_D\dot{\mathbf{s}}_D + \dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{T}_D\mathbf{B}_E^{-1}\mathbf{n}_E - \mathbf{J}\mathbf{B}^{-1}\mathbf{n}, \tag{27}$$

where $\mathbf{T}_D^{\mathrm{T}}\mathbf{Y}_R = \mathbf{0}$ was used. It can be shown that invertibility of $\mathbf{Q}$ is assured if

$$ker\left[\mathbf{T}^{\mathrm{T}}(\mathbf{s})\right] \cap ker\left[\mathbf{J}^{\mathrm{T}}(\mathbf{q})\right] = \emptyset, \tag{28}$$

i.e. if $\mathbf{J}^{\mathrm{T}}\mathbf{Y}_R$ is full rank. When a singular condition occurrs, reaction forces are not determined and jamming occurs between the robot and the environment. Note that a duality is recovered with the condition $span\left[\mathbf{J}(\mathbf{q})\right] \cap span\left[\mathbf{T}(\mathbf{s})\right] \neq \emptyset$, for the existence of admissible directions of end-effector motion.

Partitioning the inverse of $\mathbf{Q}$ in blocks and defining $\overline{\mathbf{m}}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}})$ and $\overline{\mathbf{N}}(\mathbf{q},\mathbf{s})$ in the following way

$$\mathbf{Q}^{-1} = \begin{bmatrix}\mathbf{P}_A \\ \mathbf{P}_R \\ \mathbf{P}_K\end{bmatrix}, \quad \begin{aligned}\overline{\mathbf{m}} &= (\mathbf{Y}_R\mathbf{P}_R + \mathbf{Y}_A\mathbf{P}_A)\mathbf{m}, \\ \overline{\mathbf{N}} &= (\mathbf{Y}_R\mathbf{P}_R + \mathbf{Y}_A\mathbf{P}_A)\mathbf{J}\mathbf{B}^{-1},\end{aligned} \tag{29}$$

the contact force can be rewritten as a function of the robot-environment state $(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}})$ and of the input $\mathbf{u}$:

$${}^0\mathbf{F} = \overline{\mathbf{m}}(\mathbf{q},\dot{\mathbf{q}},\mathbf{s},\dot{\mathbf{s}}) + \overline{\mathbf{N}}(\mathbf{q},\mathbf{s})\mathbf{u}. \tag{30}$$

Replacing (30) into (15) yields finally

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{n} + \mathbf{J}^{\mathrm{T}}\overline{\mathbf{m}} = \left[\mathbf{I} - \mathbf{J}^{\mathrm{T}}\overline{\mathbf{N}}\right]\mathbf{u}, \tag{31a}$$

$$\mathbf{B}_E\ddot{\mathbf{s}}_D + \mathbf{n}_E - \mathbf{T}_D^{\mathrm{T}}\overline{\mathbf{m}} = \mathbf{T}_D^{\mathrm{T}}\overline{\mathbf{N}}\,\mathbf{u}, \tag{31b}$$

which should be completed with a third set of differential equations obtained from (25)

$$\ddot{\mathbf{s}}_K = \mathbf{P}_K\mathbf{m} + \mathbf{P}_K\mathbf{J}\mathbf{B}^{-1}\mathbf{u}. \tag{31c}$$

Remark 7. Terms may be simplified in (31b), taking into account the orthogonality condition (19) and using as specific parametrization of active forces the one inducing $\mathbf{T}_D^{\mathrm{T}}\mathbf{Y}_A = \mathbf{I}_{d\times d}$. Thus, (31b) can be rewritten as

$$\mathbf{B}_E\ddot{\mathbf{s}}_D + \mathbf{n}_E - \mathbf{P}_A\mathbf{m} = \mathbf{P}_A\mathbf{J}\mathbf{B}^{-1}\mathbf{u}.$$ ∎

The number of differential equations describing the system has moved from $n+d$ in (15) to $n+e$ (where $e=d+k$) in (31). The additional dynamics is relative to the kinematic variables $\mathbf{s}_K$ and has allowed to eliminate the $6-k$ parameters λ appearing in (15) via (17). Using constraint (11), the total number of independent equations could be decreased. For, let h be the rank of matrix $[\partial\mathbf{k}(\mathbf{q})/\partial\mathbf{q}\ \partial\Gamma(\mathbf{s})/\partial\mathbf{s}]$. From the Implicit Function Theorem it is possible to locally express h of the $n+e$ variables $(\mathbf{q},\mathbf{s})$ in terms of the remaining ones, thus reducing (31) to $n+e-h$ equations. Assumption (A1) implies that the mapping Γ has full rank e, and so $h \geq e$. As a result, the elimination process can be performed so that *all* the components of $\mathbf{s}$ are expressed in terms of the robot variables $\mathbf{q}$. With a slight abuse of notation, the dynamics of the robot in contact with the environment is then described by

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{J}^{\mathrm{T}}(\mathbf{q})\overline{\mathbf{m}}(\mathbf{q},\dot{\mathbf{q}}) = [\mathbf{I} - \mathbf{J}^{\mathrm{T}}(\mathbf{q})\overline{\mathbf{N}}(\mathbf{q})]\mathbf{u}, \tag{32}$$

obtained by formally replacing $\mathbf{s} = \mathbf{s}(\mathbf{q})$ and $\dot{\mathbf{s}} = \dot{\mathbf{s}}(\mathbf{q},\dot{\mathbf{q}})$ in (31a). Moreover, if h is strictly greater than e, the above system can be further *reduced*, having $h-e$ of the $\mathbf{q}$ components as functions of the other ones. In any case, (32) shows how the motion behavior of the robot will be modified by the contact with the environment, both in the dynamic and non-dynamic case.

IV. CASE STUDY

The modeling approach is illustrated for the task of a robot turning a crank in the vertical plane. Different operative conditions will be presented, in order to gradually introduce cases of increasing complexity.

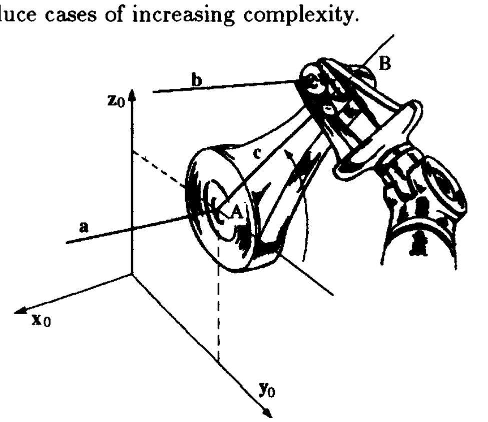

Fig. 1 – Robot turning a crank with a free knob

With reference to Fig. 1, suppose first that the crank has negligible mass and inertia, and consider the case of a *free rotating knob* at the crank pin. An inertial frame ${}^0S$ is fixed with the robot base. Let $\mathbf{a}$ be the crankshaft axis, $\mathbf{b}$ the rotation axis of the knob, both parallel to $\mathbf{x}_0$, and $\mathbf{c}$ the axis of the crank web, normal to both $\mathbf{a}$ and $\mathbf{b}$ and intersecting $\mathbf{b}$ at a point B. The crank web has length r.

The robot end-effector frame ${}^nS$ is located in B and the hand grasp is such that axis $\mathbf{x}_n$ is always kept parallel to $\mathbf{b}$. This leaves one degree of freedom to the end-effector orientation.

As kinematic variables $\mathbf{s}_K$, two absolute angles are chosen so to determine uniquely the contact configuration between robot end-effector and environment (see Fig. 2): $s_{K,1}$, the angle between $\mathbf{c}$ and $\mathbf{y}_0$, and $s_{K,2}$, the angle between $\mathbf{z}_n$ and $\mathbf{z}_0$.

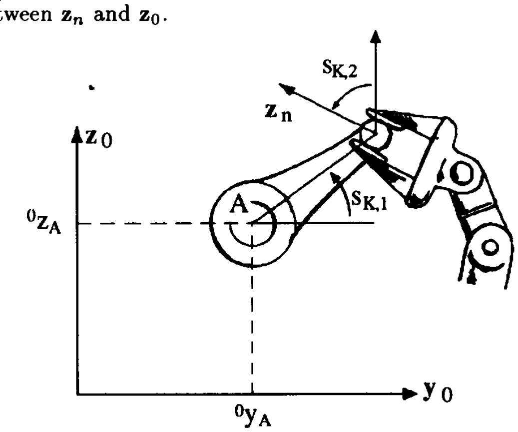

Fig. 2 – Kinematic variables for the case of a free knob

Representing end-effector orientations with zxz-Euler angles, the absolute pose ${}^0\mathbf{p}$ is written as a function of $\mathbf{s} \equiv \mathbf{s}_K$

$${}^0\mathbf{p} = \begin{bmatrix} {}^0x \\ {}^0y \\ {}^0z \\ \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ {}^0y_A + r\cos s_{K,1} \\ {}^0z_A + r\sin s_{K,1} \\ 0 \\ s_{K,2} \\ 0 \end{bmatrix} = \Gamma(\mathbf{s}). \tag{33}$$

Differentiating (33) and multiplying it by $\mathbf{G}$ as in (5), one obtains the parametrization of the admissible end-effector velocity ${}^0\mathbf{v}$, as well as the columns of matrix $\mathbf{T}(\mathbf{s}) \equiv \mathbf{T}_K(\mathbf{s}_K)$:

$${}^0\mathbf{v} = \begin{bmatrix} {}^0\dot{x} \\ {}^0\dot{y} \\ {}^0\dot{z} \\ {}^0\omega_x \\ {}^0\omega_y \\ {}^0\omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ -r\sin s_{K,1} \\ r\cos s_{K,1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{s}_{K,1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \dot{s}_{K,2} = \mathbf{T}(\mathbf{s})\dot{\mathbf{s}}. \tag{34}$$

Using the orthogonality condition (9), reaction forces can be parametrized as

$${}^0\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos s_{K,1} & 0 & 0 \\ 0 & \sin s_{K,1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \lambda_R = \mathbf{Y}_R(\mathbf{s}_K)\lambda_R, \tag{35}$$

where λ_R is a 4-vector. Restricting attention to $\mathbb{R}^2$, Fig. 3 shows a condition under which only a zero cartesian linear velocity is allowed for a two-link planar arm performing the above task.

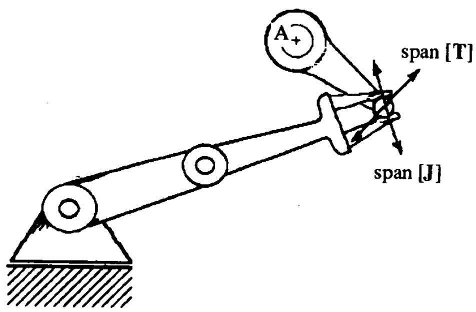

Fig. 3 – Kinematic singularity: $span[\mathbf{J}] \cap span[\mathbf{T}] = \emptyset$ in $\mathbb{R}^2$

A situation in which contact forces in $\mathbb{R}^2$ are not uniquely determined is illustrated in Fig. 4. Note that a slight relative displacement between the robot base point in the plane and the crank axis would induce extremely high forces.

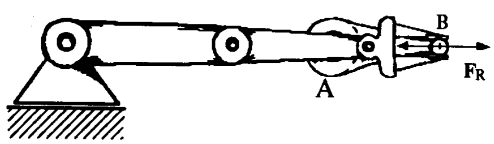

Fig. 4 – Force singularity: $ker[\mathbf{T}^T] \cap ker[\mathbf{J}^T] \neq \emptyset$ in $\mathbb{R}^2$

Although other definitions of $\mathbf{Y}_R$ in (35) are possible, the chosen columns have a 'natural' interpretation in terms of forces along axes $\mathbf{b}$ and $\mathbf{c}$, and moments around $\mathbf{y}_0$ and $\mathbf{z}_0$. The whole description could be made using a coordinate-dependent *task-frame* ${}^tS$ located at point B, and with the axes $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t)$ directed as $(\mathbf{b}, \mathbf{c}, \mathbf{b} \times \mathbf{c})$. In such frame, the dependence on s_K would be eliminated from $\mathbf{T}_K$ and $\mathbf{Y}_R$, recovering the classical description with *selection matrices* [11,12,13].

There are situations in which, due to the structure of the task, a decoupled description of translational and rotational quantities is *never* possible. To this purpose, consider the case of a crank with a *fixed knob* at the web extremity (Fig. 5). Only the first kinematic parameter, relabeled s_K, is needed now for specifying end-effector velocity

$$ {}^0\mathbf{v} = \begin{bmatrix} 0 \\ -r \sin s_K \\ r \cos s_K \\ 1 \\ 0 \\ 0 \end{bmatrix} \dot{s}_K = \mathbf{T}_K(s_K)\dot{s}_K. \qquad (36) $$

The column $\mathbf{T}_K$ represents the only generalized direction of admissible motion. Equation (36) implies that there cannot be a translational velocity without an angular one and viceversa. Similarly, all reaction forces are in the image of

$$ \mathbf{Y}_R(s_K) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\sin s_K & \cos s_K & 0 & 0 \\ 0 & \cos s_K & \sin s_K & 0 & 0 \\ 0 & -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (37) $$

The second column in (37) identifies a generalized direction in which the environment reaction balances a force along $\mathbf{b} \times \mathbf{c}$ *together with* a torque about $\mathbf{b}$. This case is not treated by the selection matrix approach because, even using a task-frame, there cannot be a single non-zero element in all columns.

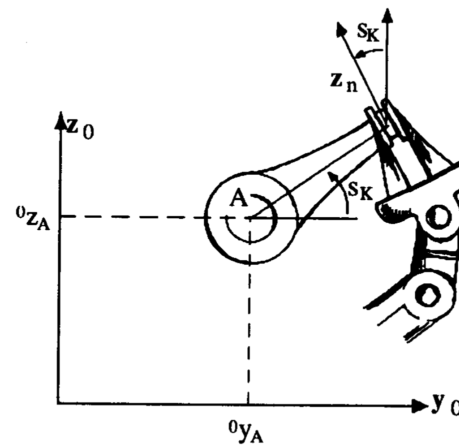

Fig. 5 – Robot turning a crank with a fixed knob

To introduce the concept of dynamic directions, assume now that the crank has non-negligible mass and inertia. Only the *free knob* set-up will be analyzed. The dynamic model of the environment can be written using the angle between the crank web and the absolute axis $\mathbf{y}_0$ as the only dynamic variable s_D. This is the same angle previously denoted as $s_{K,1}$ in Fig. 2; the remaining kinematic variable $s_{K,2}$ becomes here s_K. Admissible end-effector velocities are still given by (34), after proper substitution, as ${}^0\mathbf{v} = \mathbf{T}_D \dot{s}_D + \mathbf{T}_K \dot{s}_K$. Differently from the pure kinematic case, a contact force component appears that does not lie in *span* $[\mathbf{Y}_R]$. Active and reaction forces can then be parametrized by

$$ \mathbf{Y}_A = \begin{bmatrix} 0 \\ -\sin s_D \\ \cos s_D \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{Y}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos s_D & 0 & 0 \\ 0 & \sin s_D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (38) $$

This choice satisfies (19) and (20). The overall dynamic model of the system is given by

$$\begin{aligned} \mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) &= \mathbf{u} - \mathbf{J}^{\mathrm{T}}(\mathbf{q})^{0}\mathbf{F}, \\ I_c \ddot{s}_D + D_c \dot{s}_D - m_c g l_c \cos s_D &= \mathbf{T}_D^{\mathrm{T}}(\mathbf{s})^{0}\mathbf{F}, \end{aligned} \tag{39}$$

where m_c is the crank mass, l_c is the distance from center of mass to axis $\mathbf{a}$, I_c is the moment of inertia of the crank w.r.t. $\mathbf{a}$, and D_c is the viscous friction coefficient at the same axis. The generalized force $\mathbf{T}_D^{\mathrm{T}}(\mathbf{s})^{0}\mathbf{F}$ acting on the environment is a torque about $\mathbf{a}$ whose value is $r\lambda_A$. Note that the ranges of $\mathbf{T}_D$ and of $\mathbf{Y}_A$ coincide in this case, but if the knob is fixed then

$$\mathbf{T}_D = \begin{bmatrix} 0 \\ -r \sin s_D \\ r \cos s_D \\ 1 \\ 0 \\ 0 \end{bmatrix} \not\parallel \begin{bmatrix} 0 \\ -\sin s_D \\ \cos s_D \\ -r \\ 0 \\ 0 \end{bmatrix} = \mathbf{Y}_A. \tag{40}$$

V. CONCLUSIONS

A framework for modeling motion of robots in contact with a possibly dynamic environment has been presented. A minimal parametrization is introduced, describing environment and contact interaction through dynamic and kinematic variables. The following generalized directions are identified:

- admissible directions of kinematic end-effector motion, along which displacements do not produce work;
- admissible directions of dynamic end-effector motion;
- directions of active forces, in which an energy transfer occurs between robot and environment;
- directions of reaction forces, balanced by the environment.

Relations among the above subspaces, and in particular orthogonality, have been redefined in the context of dynamic environments. As a result, a number of issues involved in compliant tasks are naturally understood:

- additional variables are needed, depending on the contact or grasp type, to complete the end-effector pose description as seen from the environment;
- there exist tasks that cannot be represented without coupling translational and rotational quantities;
- velocity and force singularities may arise during task execution, due to the interaction of robot and environment kinematics;
- dynamic directions can be determined where active forces and end-effector motions exist at the same time.

In deriving the overall dynamics of the robot-environment system, a way to eliminate dependent variables and reduce dynamic equations were presented. The proposed task-oriented dynamic modeling approach lends itself to the definition and realization of new interesting hybrid control schemes (see [14]).

ACKNOWLEDGMENTS

We would like to thank Prof. Fernando Nicolò for his promoting support.

REFERENCES

[1] B. Shimano and B. Roth, "On force sensing information and its use in controlling manipulators," *8th Int. Symp. on Industrial Robots*, pp. 119–126, 1978.

[2] M.T. Mason, "Compliance and force control for computer controlled manipulators," *IEEE Trans. on Systems, Man, and Cybernetics*, vol. SMC-11, no. 6, pp. 418–432, 1981.

[3] D.E. Whitney, "Historical perspective and state of the art in robot force control," *Int. J. of Robotic Research*, vol. 6, no. 1, pp. 3–14, 1987.

[4] T. Yoshikawa, "Dynamic hybrid position/force control of robot manipulators — Description of hand constraints and calculation of joint driving force," *IEEE J. of Robotics and Automation*, vol. RA-3, no. 5, pp. 386–392, 1987.

[5] N.H. McClamroch and D. Wang, "Feedback stabilization and tracking in constrained robots," *IEEE Trans. on Automatic Control*, vol. AC-33, no. 5, pp. 419–426, 1988.

[6] N. Hogan, "Impedance control: An approach to manipulation: Part I – Theory," "Part II – Implementation," "Part III – Applications," *Trans. ASME J. of Dynamic Systems, Measurement, and Control*, vol. 107, no. 3, pp. 1–24, 1985.

[7] H. Kazerooni, T.B. Sheridan, and P.K. Houpt, "Robust compliant motion for manipulators, Part I: The fundamental concepts of compliant motion," *IEEE J. of Robotics and Automation*, vol. RA-2, no. 2, pp. 83–92, 1986.

[8] N. Hogan, "On the stability of manipulators performing contact tasks," *IEEE J. of Robotics and Automation*, vol. 4, no. 6, pp. 677–686, 1988.

[9] J. Wen, "Position and force control of robot arms," *4th IEEE Int. Symp. on Intelligent Control*, Albany, NY, pp. 251–256, Sept. 1989.

[10] J. Duffy, "The fallacy of modern hybrid control theory that is based on 'orthogonal complements' of twist and wrench spaces," *J. of Robotic Systems*, vol. 7, no. 2, pp. 139–144, 1990.

[11] M.H. Raibert and J.J. Craig, "Hybrid position/force control of manipulators," *Trans. ASME J. of Dynamic Systems, Measurement, and Control*, vol. 102, no. 3, pp. 126–133, 1981.

[12] O. Khatib, "A unified approach to motion and force control of robot manipulators: the operational space formulation," *IEEE J. of Robotics and Automation*, vol. RA-3, pp. 43–53, 1987.

[13] A. De Luca, C. Manes, and F. Nicolò, "A task space decoupling approach to hybrid control of manipulators," *2nd IFAC Symp. on Robot Control (SYROCO'88)*, Karlsruhe, D, pp. 157–162, Oct. 1988.

[14] A. De Luca and C. Manes, "Hybrid force-position control for robots in contact with a general class of passive environments," *3rd IFAC Symp. on Robot Control (SYROCO'91)*, Vienna, A, Sep. 1991.