Stabilization via Iterative State Steering with Application to Chained-Form Systems

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Abstract

A general approach is presented for the robust stabilization of controllable systems. The proposed strategy is viable whenever one can compute a finite-time control law that steers the state from any initial condition to a point closer to the desired equilibrium. Under suitable hypotheses, such control law can be applied in an iterative fashion, obtaining exponential convergence of the state to the equilibrium point. Moreover, small non-persistent perturbations are rejected, while small persistent perturbations induce limited errors. The proposed approach is applied to chained-form systems and simulations are presented for a unicycle.

1 Introduction

In this paper we address the asymptotic stabilization problem of equilibrium points for controllable systems. We propose a general control strategy that in some cases may have advantages over existing methods, allowing to solve difficult problems, such as the robust stabilization of the unicycle—a system representative of the class of controllable systems without drift.

For linear systems, controllability and exponential stabilizability of an equilibrium point by linear static or dynamic state feedback imply each other. This is not true for nonlinear systems [1]. Indeed, there exist classes of systems which are controllable but not asymptotically stabilizable by a continuous time-invariant feedback, e.g., nonholonomic systems. This fact has motivated the search for stabilizing time-varying [2, 3] and discontinuous [4, 5, 6] feedback control laws for particular classes of systems. Hence, there is a certain interest in control strategies different from the classic continuous state feedback, for they can allow to solve the stabilization problem in systems that are not controllable in the first approximation.

The control strategy herein proposed can be summarized as follows. Suppose that we can compute a control law, not necessarily dependent on the state variables, which steers the system in finite time from any point to another one closer to the desired equilibrium. By applying such control law in an iterative fashion one can, under some technical assumptions, prove exponential convergence to the desired equilibrium. The resulting control is a time-varying law whose expression depends on a sampled feedback action.

A similar control approach has been proposed in [7] for the design of discontinuous time-varying stabilizing feedback, resulting in a computationally intensive method applicable to systems of small dimension (≤ 3). Also, there are similarities between our strategy and receding horizon control methods [8].

A distinctive point of our approach is the possibility to guarantee certain robustness properties. In particular, non-persistent perturbations are rejected, while ultimate boundedness is guaranteed in the presence of persistent perturbations. Moreover, our method can be applied to systems with drift, whereas the aforementioned techniques work only for driftless systems.

The paper is organized as follows. The fundamental ideas on which iterative state steering is based are introduced in Sect. 2, while robustness of the approach is established in Sect. 3. The application to chained-form systems is presented in Sect. 4, together with simulation results for a wheeled mobile robot.

2 Iterative state steering

The stabilization strategy herein considered relies on the iterative application of a suitable control law. During each iteration, the state of the plant is steered from the current initial condition to one closer to the origin, which is assumed w.l.o.g. to be the desired equilibrium point. If the plant is controllable, and one is able to compute such a *contractive* steering control, the above strategy can be always pursued. In this section, we shall prove a theorem that formalizes this simple idea.

Consider the control system

$$\dot{x}(t) = f\left(x(t), u(t), t\right), \quad x(t_0) = x_0 \in B_0, \qquad (1)$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R^m$ is the control, and $B_0 = \{x : ||x|| \leq \rho\}$ is an open ball around the origin. Assume that

$$f(0, 0, t) = 0, \quad \forall\, t \in I\!R. \qquad (2)$$

Consider an indefinite sequence of time instants $\{t_k\}$, with $t_{k+1} = t_k + T_{k+1}$, and $0 < T_m \leq T_{k+1} \leq T_M < \infty$, for $k = 0, 1, 2, \ldots$. For compactness, we shall write in the following $x(t_k) = x_k$.

On the generic time interval $I_{k+1} = [t_k, t_{k+1}]$, let the steering control law be

$$u(t) = u_{k+1}(t) = \alpha(x(t), x_k, t),$$

with $\alpha(x(t), 0, t) = 0$, for all $(x, t) \in I\!R^n \times I_{k+1}$. Throughout the paper, we shall take the following

Assumption 1. *The steering control function α guarantees that*

$$|x_{k+1}| \leq \beta\, |x_k| < |x_k|, \qquad \forall x_k \in B_0. \tag{3}$$

In the simplest version, the control law within each iteration will be specified in open-loop, i.e., it will be $\alpha = \alpha(x_k, t)$ only. In general, however, one may resort to a feedback control law. Let

$$\dot{x} = \tilde{f}(x(t), x_k, t) = f(x(t), \alpha(x(t), x_k, t), t), \quad t \in I_{k+1}, \tag{4}$$

be the closed-loop dynamics of (1) in the $(k+1)$-th interval I_{k+1}. A basic result is established below.

Lemma 1 *Assume that f and u are such that the solution $\varphi(t, u, x_0, t_0)$ of eq. (1) exists and is unique. Then:*

1. *$x = 0$ is a geometrically stable equilibrium point of the discrete-time system*
$$x_{k+1} = \varphi(t_{k+1}, u_{k+1}, x_k, t_k). \tag{5}$$
2. *The state of the continuous-time system (1) asymptotically converges to zero.*

Proof

1. As $u_{k+1}(t) = \alpha(x(t), x_k, t)$ and $\alpha(x(t), 0, t) = 0$, it is $u_{k+1}(t) = 0$ if $x_k = 0$. Moreover, by eq. (2) one has $\varphi(t_{k+1}, 0, 0, t_k) = 0$, so that $x = 0$ is an equilibrium point of system (5). Its geometric stability is established by noting that, as $\beta < 1$,

$$|x_{k+1}| \leq \beta |x_k| \leq \beta^k |x_0|, \tag{6}$$

which shows that $|x_k| \to 0$ as $k \to \infty$.

2. The existence and uniqueness hypothesis guarantees that the solution $\varphi(t, u, x_0, t_0)$ of eq. (1) is continuous w.r.t. (t, x_0, t_0). Hence, the function

$$\tilde{\varphi}(t, x_0, t_0) = \varphi(t, u, x_0, t_0), \quad \tilde{\varphi}(t, 0, t_0) = 0,$$

is continuous w.r.t. x_0 for any t and t_0. In particular, one has

$$\lim_{x_0 \to 0} \tilde{\varphi}(t, x_0, t_0) = 0, \quad \forall t, t_0.$$

The continuity of $\varphi(t, u, x_0, t_0)$ implies the continuity of $\tilde{\varphi}(t, x_0, t_0)$, thus being a *sufficient* condition. For conditions on $\tilde{f}$ such that φ is continuous, see classical results on differential equations, e.g. [9, Sect. I.3]. ∎

As the current initial condition x_k approaches the origin, one would like the state to converge to zero at an *exponential* rate. We now establish conditions on the closed-loop dynamics $\tilde{f}$ so as to achieve this behavior.

Theorem 1 *Suppose that the following estimate holds:*

$$|\tilde{f}(x, x_k, t)| \leq \lambda |x| + \mu(|x_k|), \quad t \in I_{k+1},\, x_k \in B_0, \tag{7}$$

with $\mu : I\!R^+ \mapsto I\!R^+$ a continuous function at zero. Then,

1. *$x = 0$ is an asymptotically stable equilibrium of the continuous-time system (1).*
2. *More stringently, if eq. (7) is satisfied with*
$$\mu(|x_k|) \leq \bar{\mu} |x_k|^r, \quad \bar{\mu} > 0,\, r \geq 0, \tag{8}$$
$x = 0$ is an exponentially stable equilibrium of the continuous-time system (1).

Proof

1. Being $\tilde{f}(x, x_k, t)$ the closed-loop dynamics of system (1) within the k-th iteration, we have

$$x(t) = x_k + \int_{t_k}^{t} \tilde{f}(x(\tau), x_k, \tau) d\tau, \quad t \in I_{k+1}.$$

From this, we obtain

$$|x(t)| \leq |x_k| + \int_{t_k}^{t} \left(\lambda |x(\tau)| + \mu(|x_k|)\right) d\tau,$$

where eq. (7) has been used. By applying the generalized Gronwall inequality (e.g., see [9, p. 36]), one gets

$$\begin{aligned} |x(t)| \;&\leq\; |x_k| + \mu(|x_k|)(t - t_k) + \\ &\quad\; \lambda \int_{t_k}^{t} \left(|x_k| + \mu(|x_k|) s e^{\lambda(t-s)}\right) ds \\ &=\; |x_k| e^{\lambda(t-t_k)} + \mu(|x_k|)(t - t_k) + \\ &\quad\; \mu(|x_k|) \left(\frac{e^{\lambda(t-t_k)} - 1}{\lambda} - (t - t_k)\right), \end{aligned}$$

and finally

$$|x(t)| \leq |x_k| e^{\lambda(t-t_k)} + \mu(|x_k|) \frac{e^{\lambda(t-t_k)} - 1}{\lambda}.$$

Since μ is continuous at zero and $\mu(0) = 0$, the latter equation shows that, as x_k converges to zero, $x(t)$ asymptotically converges to zero for all t.

2. From eq. (8) we get

$$\begin{aligned} |x(t)| &\leq |x_k| e^{\lambda(t_{k+1}-t_k)} + \mu(|x_k|) \frac{e^{\lambda(t_{k+1}-t_k)} - 1}{\lambda} \\ &\leq |x_k| e^{\lambda(t_{k+1}-t_k)} + \bar{\mu} |x_k|^r \frac{e^{\lambda(t_{k+1}-t_k)} - 1}{\lambda}, \end{aligned}$$

for $t \in I_{k+1}$. Hence,

$$\begin{aligned} |x_k| \leq 1, r < 1 \quad &\Rightarrow \quad |x(t)| \leq \theta_1 |x_k|^r, \\ |x_k| \leq 1, r \geq 1 \quad &\Rightarrow \quad |x(t)| \leq \theta_2 |x_k|, \end{aligned}$$

where θ_1 and θ_2 are bounded quantities whose expressions are easily obtained. Finally, eq. (6) yields $|x_k| \leq \beta^k x_0$, and we conclude that

$$|x_k| \leq 1, r < 1 \quad \Rightarrow \quad |x(t)| \leq \theta_1 \beta^{kr} |x_0|^r, \tag{9}$$

$$|x_k| \leq 1, r \geq 1 \quad \Rightarrow \quad |x(t)| \leq \theta_2 \beta^k |x_0|, \tag{10}$$

i.e., also $x(t)$ converges exponentially to zero, with convergence rate $r \frac{\log \beta}{T_M}$ if $r < 1$ or $\frac{\log \beta}{T_M}$ if $r \geq 1$. ∎

Remark 1 Note that the Lipschitz condition w.r.t. x in eq. (7) is generally satisfied, while the continuity requirement w.r.t. x_k is specific to the proposed strategy.

Remark 2 In the problem formulation, we have assumed that the duration T_{k+1} of the $(k+1)$-th iteration is not fixed, and may take values within $[T_m, T_M]$. In fact, as the current initial condition x_k approaches the origin, a smaller time T will be sufficient for achieving condition (3). Besides, only the upper bound T_M has influence on the convergence rate estimate.

Remark 3 Note that the point $x = 0$ is not an equilibrium point of the continuous-time closed loop system, but only of the discrete-time system (5). In fact, once the control $u_{k+1}(\cdot)$ is applied from time t_k, the state may go through the origin without stopping.

3 Robustness analysis

Suppose the control system (1) is perturbed as

$$\dot{y}(t) = f(y(t), u(t), t) + \epsilon g(y(t), u(t), t), \qquad (11)$$

being y the state of the perturbed system. As before, existence and uniqueness of the solution of eq. (11) is assumed. During the $(k+1)$-th iteration, the closed-loop system (4) becomes

$$\dot{y} = \tilde{f}(y(t), y_k, t) + \epsilon \tilde{g}(y(t), y_k, t), \quad t \in I_{k+1}. \qquad (12)$$

We shall distinguish the perturbations between

$$\begin{cases} \textit{persistent} & \tilde{g}(0,0,t) = 0 \\ \textit{non-persistent} & \tilde{g}(0,0,t) \neq 0. \end{cases}$$

For continuous-time dynamical systems, classic converse Lyapunov theorems can be used in order to show that exponential stability of an equilibrium point guarantees complete rejection of small non-persistent perturbations, while small persistent perturbations give rise to limited errors [10, Th. 56.2 and 56.4]. Such a property is often referred to as *total stability.* Unfortunately, this kind of result does not apply in our case, for system (4) is not a dynamical system. In fact, the system state may converge to zero in finite time and thus backward uniqueness of the solution is not guaranteed. In addition, we are not dealing with a true equilibrium point of the continuous-time system (see Remark 3).

However, a characterization similar to total stability can be given for the continuous-time system (1). Before, we give a result valid for geometrically stable discrete-time systems—in particular, for system (5). The proof is immediate and left to the reader.

Lemma 2 *Consider the system*

$$\xi_{k+1} = \phi(\xi_k, k),$$

for which

$$|\phi(\xi_k, k)| \leq \beta |\xi_k|, \quad \beta < 1,$$

and the perturbed system

$$\xi_{k+1} = \phi(\xi_k, k) + \gamma(\xi_k, k). \qquad (13)$$

1. *Assume that the perturbation γ is non-persistent, and in particular satisfies the following estimate:*

   $$|\gamma(\xi_k, k)| \leq \epsilon |\xi_k|^r, \quad r \geq 1. \qquad (14)$$

   Then, the contraction (hence, the geometric stability) is locally preserved for sufficiently small ϵ (namely, for $|\xi_k| \leq 1$ and $\epsilon < 1 - \beta$).
2. *If the perturbation is persistent and bounded as*

   $$|\gamma(\xi_k, k)| \leq \epsilon,$$

   ξ is ultimately confined to the ball

   $$\{\xi : |\xi| \leq \frac{\epsilon}{1-\beta}\}.$$

Thus, in order to prove ultimate boundedness or complete rejection of the perturbation for the continuous-time system (4), it is necessary to establish conditions on $f(x(t), u(t), t)$ and $g(x(t), u(t), t)$ (or equivalently on $\tilde{f}(x(t), x_k, t)$ and $\tilde{g}(x(t), x_k, t)$) such that the corresponding perturbation on the discrete-time system (5) satisfies the appropriate hypotheses of Lemma 2.

Theorem 2 *Assume that the iterative stabilization strategy is applied to the perturbed system (11) and that, for $t \in I_{k+1}$, $x_k \in B_0$, it is*

$$|\tilde{f}(x, x_k, t)| \leq \lambda |x| + \bar{\mu} |x_k|^r, \quad \bar{\mu} > 0, \ r \geq 0, \qquad (15)$$

$$|\tilde{g}(x, x_k, t)| \leq \hat{\lambda} |x| + \hat{\mu}(|x_k|), \qquad (16)$$

with $\hat{\mu} : I\!R^+ \mapsto I\!R^+$. Then, for sufficiently small ϵ:

1. *The state of the perturbed system (11) is ultimately bounded.*
2. *More stringently, if*

   $$\hat{\mu}(|x_k|) \leq \tilde{\mu} |x_k|^r, \quad r \geq 1, \qquad (17)$$

   the state of the perturbed system (11) locally exponentially converges to zero.

Proof

1. By virtue of condition (15), Theorem 1 applies (in particular, point 2), and we conclude that the solution of the unperturbed system (1) is exponentially converging to zero:

$$|x(t)| \leq m e^{-ht} |x_0|,$$

with the actual expressions of m and h depending on the value of r (see eqs. (9–10)).

For $t \in I_{k+1}$, we have $x(t) = x^k + \int_{t_k}^t \tilde{f}(x, x_k, \tau) d\tau$ and $y(t) = x_k + \int_{t_k}^t (\tilde{f}(y, x_k, \tau) + \epsilon \tilde{g}(y, x_k, \tau)) d\tau$, with $y(t) \in B_0$ for sufficiently small ϵ. Letting $d(t) = y(t) - x(t)$, it is

$$|d(t)| \leq \int_{t_k}^t \left(\lambda |d(\tau)| + \epsilon \hat{\lambda} |y| + \epsilon \hat{\mu}(|y_k|) \right) d\tau.$$

Being

$$|y(t)| \leq |d(t)| + |x(t)| \leq |d(t)| + m e^{-ht} |x_0|,$$

one has

$$|d(t)| \leq \int_{t_k}^{t} \Big(\lambda |d(\tau)| + \epsilon\hat{\lambda}(|d(\tau)| + m' e^{-h(\tau - t_k)}|x_k|) + \epsilon\hat{\mu}(|x_k|)\Big)\, d\tau,$$

with $m' = m|x_0|/|x_k|$. Hence

$$\begin{aligned} |d(t)| &\leq \int_{t_k}^{t} \Big((\lambda + \epsilon\hat{\lambda})|d(\tau)| + \\ &\quad \epsilon(\hat{\lambda} m' e^{-h(\tau - t_k)}|x_k|) + \hat{\mu}(|x_k|))\Big)\, d\tau, \\ &= (\lambda + \epsilon\hat{\lambda}) \int_{t_k}^{t} |d(\tau)| d\tau + \\ &\quad \epsilon \left(\hat{\lambda} m' |x_k| \frac{1 - e^{-h(t-t_k)}}{h} + \hat{\mu}(|x_k|)(t - t_k)\right), \end{aligned}$$

and by using the generalized Gronwall inequality

$$\begin{aligned} |d(t)| &\leq \epsilon \left(\hat{\lambda} m' |x_k| \frac{1 - e^{-h(t-t_k)}}{h} + \hat{\mu}(|x_k|)(t - t_k)\right) + \\ &\quad \epsilon(\lambda + \epsilon\hat{\lambda}) \cdot \int_{t_k}^{t} \left(\hat{\lambda} m' |x_k| \frac{1 - e^{-h(s-t_k)}}{h} + \right. \\ &\quad \left. \hat{\mu}(|x_k|)(s - t_k) e^{(\lambda + \epsilon\hat{\lambda})(t - (s - t_k))}\right) ds, \end{aligned} \tag{18}$$

showing that the perturbation is arbitrarily small at a finite time $t \in I_{k+1}$. Thus, Lemma 2 indicates that the state of the perturbed system (5) is ultimately bounded. Due to the continuity of system (13) w.r.t. y_k, also its state is ultimately bounded.

2. If the additional condition (17) holds, eq. (18) yields directly an estimate in the form (14) for the perturbation of the discrete-time system. Therefore, Lemma 2 indicates that exponential convergence is preserved for the continuous-time system (13). ∎

Remark 4 In condition (16), $\hat{\mu}(\cdot)$ may also be a constant, thus covering the *persistent* perturbation case.

4 Application to chained-form systems

In principle, the controllability assumption guarantees that a steering control exists for system (1), so that our proposed strategy is always applicable. In practice, however, an effective method for computing the steering control within each iteration must be available. In this section, we shall consider a particular class of systems for which a steering control can be computed with a reasonable effort, namely driftless controllable systems that can be put in *chained form*, and present an application to the case of a wheeled mobile cart.

Consider the driftless control system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1, \end{aligned} \tag{19}$$

with 2 inputs and n states. System (19) is called a $(2, n)$ *chained form* [11]. The controllability rank condition [12] is readily verified by using the two input vector fields, and their repeated Lie brackets.

Murray [13] has established conditions for converting a two-input driftless system in chained form by means of a feedback transformation. In particular, controllable driftless systems with two inputs and $n \leq 4$ state variables can always be put in chained form.

Among the available techniques for open-loop control of chained systems, we cite sinusoidal steering [11], piecewise-constant inputs [14, 15], and mixed piecewise-constant/polynomial inputs [16]. Hereafter, we show how to design a simple open-loop control that satisfies the hypotheses of Theorems 1 and 2.

4.1 The (2,3) chained form

Consider the chained-form system (19) with $n = 3$, and let $x(t_0) = x_0 = (x_{10}, x_{20}, x_{30})$ and $t_0 = 0$. Let

$$u_1 = \begin{cases} -\psi - 2x_{10}, & t \in [0, 1/2], \\ \psi, & t \in [1/2, 1], \end{cases} \tag{20}$$

$$u_2 = \begin{cases} 0, & t \in [0, 1/2], \\ c_1 + c_2\,(t - 1/2)\,, & t \in [1/2, 1], \end{cases} \tag{21}$$

with ψ, c_1 and c_2 constant values to be determined as a function of the initial conditions. In particular, it must be $u_1 = u_2 = 0$ for $x_0 = 0$.

By forward integration, it is easily verified that

$$x_1(1) = x_{10} - \frac{\psi + 2x_{10}}{2} + \frac{\psi}{2} = 0,$$

and

$$x_2(t) = x_{20} + c_1 \left(t - \frac{1}{2}\right) + \frac{c_2}{2}\left(t - \frac{1}{2}\right)^2, \quad t \in [1/2, 1].$$

Imposing $x_2(1) = 0$, one obtains

$$c_1 = -\left(2x_{20} + \frac{c_2}{4}\right). \tag{22}$$

Similarly, it is

$$x_3\left(\frac{1}{2}\right) = x_{30} - (\psi + 2x_{10})\frac{x_{20}}{2},$$

and

$$x_3(1) = x_3\left(\frac{1}{2}\right) + \psi\left(\frac{x_{20}}{2} + \frac{c_1}{8} + \frac{c_2}{48}\right).$$

Finally, imposing $x_3(1) = 0$ we get

$$c_2 = \frac{96}{\psi}\left(x_{30} - x_{10}x_{20} - \frac{x_{20}\psi}{4}\right). \tag{23}$$

At this point, it is necessary to choose the expression of ψ so as to satisfy condition (8). Letting

$$\psi(x_0) = ||x_0||^{\frac{1}{1+\kappa}}, \quad \kappa > 0, \tag{24}$$

one obtains a continuous function at zero such that $\psi(0) = 0$, and guarantees the same properties for c_1 and c_2 as given by eqs. (22) and (23), respectively.

Therefore, Theorem 1 guarantees that the $(2,3)$ chained form is exponentially stable under iterative application of the above open-loop controller. Moreover, Theorem 2 indicates that sufficiently small non-persistent perturbations satisfying (17) are rejected, while persistent perturbations produce limited errors.

4.2 Case study: Unicycle

Many types of mobile robots with multiple wheels have a kinematic model equivalent to that of a unicycle (see Fig. 1). The generalized coordinates are (p_x, p_y, θ, ϕ), where (p_x, p_y) are the Cartesian coordinates of the ideal contact point, θ is the orientation of the vehicle with respect to the x axis and ϕ is the angular position of the wheel. Without loss of generality, we shall assume that the radius of the wheel is $r = 1$.

The kinematic equations of the system are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 1 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v_2, \tag{25}$$

where v_1 is the angular velocity of the wheel (*driving* velocity) and v_2 is the *steering* velocity input.

If the wheel angular position is not of interest, the mobile robot configuration vector is $q = (p_x, p_y, \theta)$, and the kinematic model is given by the first three rows of eq. (25). The resulting system is put in (2,3) chained form by using the change of coordinates

$$\begin{aligned} x_1 &= \theta \\ x_2 &= p_x \cos\theta + p_y \sin\theta \\ x_3 &= p_x \sin\theta - p_y \cos\theta, \end{aligned}$$

together with the input transformation

$$v = \begin{bmatrix} x_3 & 1 \\ 1 & 0 \end{bmatrix} u.$$

In order to test the robustness of the proposed stabilization strategy, we have introduced a perturbation on the wheel radius, whose value is assumed to be $1 + \Delta r$. One may easily verify that, with the above change of coordinates and input transformation, the (2,3) chained form is perturbed as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 + \Delta r \cdot x_3 u_1 \\ \dot{x}_3 &= x_2 u_1. \end{aligned}$$

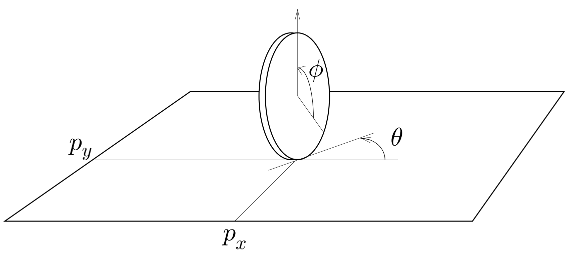

Figure 1: Generalized coordinates for the unicycle

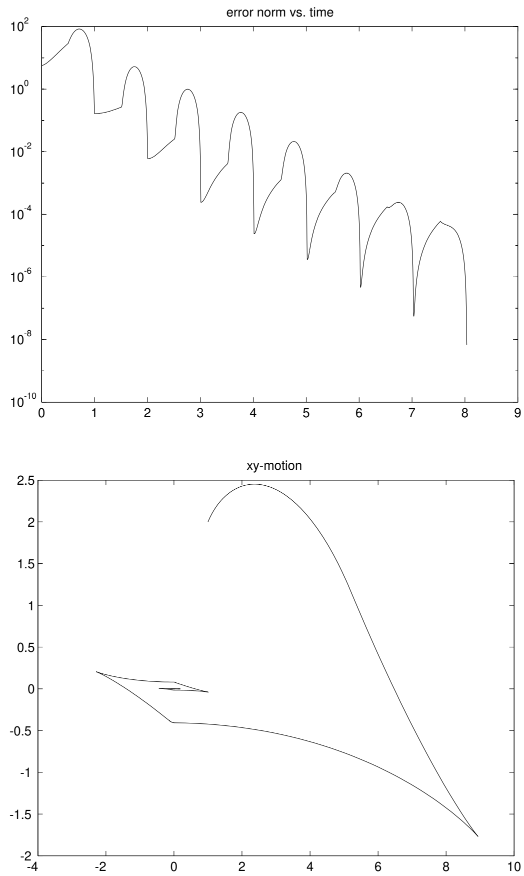

Figure 2: Error norm vs. time and Cartesian motion

Conditions (16) and (17) are satisfied with $\hat{\mu}(|x_k|) \equiv 0$, showing that the perturbation is non-persistent. Thus, the iterative controller of Sect. 4.1 designed on the *nominal* system (i.e., for $\Delta r = 0$) guarantees robust exponential stability, for sufficiently small Δr.

The simulation results reported in Fig. 2 are in accordance with this expectation. Here, the robot must move from $(1, 2, \pi/4)$ to $(0, 0, 0)$. The control law is given by eqs. (20) and (21), having set $\kappa = 1$ in eq. (24). A 20 % perturbation on the wheel radius was included in simulation. The logarithmic plot of the squared error norm after eight iterations shows the exponential convergence rate. Note also the quite natural cartesian motion of the unicycle.

4.3 Extension to the $(2, n)$ chained form

We shall now outline the extension of the iterative steering technique of Sect. 4.1 to the case of the $(2, n)$ chained-form system (19).

Let

$$\begin{aligned} u_1 &= \begin{cases} -\psi - 2x_{10}, & t \in [0, 1/2], \\ \psi, & t \in [1/2, 1], \end{cases} \\ u_2 &= \begin{cases} 0, & t \in [0, 1/2], \\ c_1 + \ldots + c_{n-1}\left(t - \frac{1}{2}\right)^{n-2}, & t \in [1/2, 1], \end{cases} \end{aligned}$$

with ψ, $c_1, \ldots, c_{n-1}$ constant values to be determined

as a function of the initial conditions. As before, it must be $u_1 = u_2 = 0$ for $x_0 = 0$.
As in Sect. 4.1, by performing the forward integration of the dynamic equations and imposing the final desired conditions one obtains

$$c_{n-1} = \frac{1}{\psi} \left(a_{n-1}(x_0) + b_{n-1}(x_0, \psi)\right),$$

in which $a_{n-1}(x_0)$ collects pure polynomial terms in $x_{10}, \ldots, x_{n0}$ while $b_{n-1}(x_0, \psi)$ accounts for product terms containing ψ and $x_{10}, \ldots, x_{n0}$ (e.g., see (23)). Similarly, we get

$$c_{n-2} = \frac{1}{\psi} \left(a_{n-2}(x_0) + b_{n-2}(x_0, \psi)\right) + h_{n-2} c_{n-1},$$

with h_{n-2} a constant coefficient. Substituting the expression of c_{n-1} in the latter equation, we have

$$c_{n-2} = \frac{1}{\psi} \left(a'_{n-2}(x_0) + b'_{n-2}(x_0, \psi)\right)$$

with obvious meaning of the symbols $a'_{n-2}(x_0)$ and $b'_{n-2}(x_0, \psi)$. Repeating this procedure for the other control coefficients, we finally obtain

$$c_1 = \frac{1}{\psi} \left(a'_1(x_0) + b'_1(x_0, \psi)\right).$$

At this point, it is necessary to choose the expression of ψ so as to satisfy condition (8). Any continuous function ψ such that

$$\psi(x_0) = o((a'_1(x_0) + b'_1(x_0, \psi)),$$

is feasible, for one obtains $\psi(0) = 0$ and guarantees the same properties for $c_1, \ldots, c_{n-2}$. In particular, the stabilization strategy obtained by the iterated application of this open-loop controller shares the same robustness properties of the $(2, 3)$ case.

5 Conclusions

We have presented a general approach for the robust stabilization of controllable systems. The main ingredient is an open-loop control that steers the system closer to the desired equilibrium point x_d in a finite time T. Whenever the computation of such control is viable, its iterated application (i.e., from the state attained at the end of the previous iteration) yields exponential convergence to x_d under the assumption that the open-loop control is continuous with respect to the initial conditions. The resulting stabilization scheme can reject small non-persistent perturbations, while small persistent perturbations induce limited errors. For illustration, we have applied the proposed method to chained-form systems, a particular class of driftless controllable systems, and presented simulation results for a unicycle.

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