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Robots with elastic joints
are linearizable via dynamic feedback

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Abstract

We prove that the complete dynamic model of robots with elastic joints can always be fully transformed into a linear, controllable, and input-output decoupled system through the use of nonlinear dynamic state feedback.

1. INTRODUCTION

The presence of flexibility concentrated at the joints is a common aspect in many current industrial robots, when motion transmission elements such as harmonic drives, transmission belts and long shafts are used [1]. The dynamic displacement between the position of the driving motors and that of the driven links requires doubling the number of generalized coordinates needed to describe the mechanical system w.r.t. the rigid case.

Several authors have considered the dynamic modeling and the control problem in the presence of joint elasticity [2]. There are mainly two different modelling assumptions for multi-link elastic joint robots. The *complete* model assumes that the rotors of the actuators have uniform mass distribution and center of mass on the rotation axis [3]. The *reduced* model assumes further that the kinetic energy of the rotors is due only to their spinning angular velocity [4].

In [4], it is shown that the reduced model always satisfies the conditions for full linearization via *static* state feedback. This allows to completely solve the trajectory tracking problem using a nonlinear control law, by taking advantage of the feedback transformation into a linear system.

On the contrary, it was first shown in [5] that the complete model of a 3R robot with elastic joints violates the necessary conditions for linearization via static state feedback. If the extra terms that characterize the complete model are neglected and the feedback linearizing control design is carried out based on the reduced model, significant errors may arise in the closed loop [6]. An attempt to solve the tracking problem using the complete model is provided in [7], but only a local result is found there.

In [8, 9], the use of the more general class of *dynamic* state feedback control laws has been first proposed for the complete model, in order to achieve full linearization and decoupling, useful for obtaining global tracking results. In particular, the overall design of a fourth-order nonlinear dynamic compensator is presented in [8] for a planar 2R arm (an eight-dimensional state-space system). The obtained closed-loop system is equivalent to two chains of six input-output integrators, for which the tracking problem

can be tackled with any desired linear control technique. In [9], a case-by-case study was performed on several kinematic structures with joint elasticity.

Up to now, there is no rigorous proof that the complete model can *always* be transformed into a linear and decoupled system via dynamic state feedback. In this note, we show that this property holds in general. For this purpose, we take advantage of some insights in the structure of the model, as shown in [3], and of an easy-to-check sufficient condition for full linearization via dynamic state feedback introduced in [10].

2. DYNAMIC MODEL OF ELASTIC JOINT ROBOTS

We consider the complete dynamic model of a multi-link robot with elastic joints given in [3]:

$$
\begin{aligned}
B_1(q_\ell)\ddot{q}_\ell &+ B_2(q_\ell)\ddot{q}_m + [C_{A1}(q_\ell, \dot{q}_m) + C_{B1}(q_\ell, \dot{q}_\ell)]\,\dot{q}_\ell \\
&+ C_{B2}(q_\ell, \dot{q}_\ell)\dot{q}_m + K_e(q_\ell - q_m) + g(q_\ell) = 0 \qquad (1)
\end{aligned}
$$
$$
B_2^T(q_\ell)\ddot{q}_\ell + B_3\ddot{q}_m + C_{B3}(q_\ell, \dot{q}_\ell)\dot{q}_\ell + K_e(q_m - q_\ell) = u,
$$

where $q_\ell \in I\!\!R^n$ and $q_m \in I\!\!R^n$ denote the link and motor angular positions, respectively. The positive definite inertia matrix is partitioned in $n \times n$ blocks as

$$
B(q_\ell) = \begin{bmatrix} B_1(q_\ell) & B_2(q_\ell) \\ B_2^T(q_\ell) & B_3 \end{bmatrix},
$$

while the C-terms denote the centrifugal and Coriolis forces, $g(q_\ell)$ is the gravity force acting on the links, $K_e = \mathrm{diag}\{k_1, \ldots, k_n\}$ is the joint stiffness matrix, and u is the motor input torque.

The inertia matrix and the gravity terms are independent from q_m. The submatrix $B_2(q_\ell)$ has a strictly upper triangular structure and its non-zero elements have the special dependence $B_{2,ij} = B_{2,ij}(q_{\ell,i}, \ldots, q_{\ell,j-1})$ $(j > i)$. Moreover, B_3 is a diagonal matrix containing the inertia of the rotors around their rotation axis. Finally, one has also

$$
C_{A1,ij}(q_\ell, \dot{q}_m) = \frac{1}{2}\left(\frac{\partial B_2^i}{\partial q_{\ell,j}} - \frac{\partial B_2^j}{\partial q_{\ell,i}} \right)\dot{q}_m
$$
$$
C_{B1,ij}(q_\ell, \dot{q}_\ell) = \frac{1}{2}\left[\dot{q}_\ell^T \frac{\partial B_{1,ij}}{\partial q_\ell} + \left(\frac{\partial B_1^i}{\partial q_{\ell,j}} - \frac{\partial B_1^j}{\partial q_{\ell,i}} \right)\dot{q}_\ell \right]
$$
$$
C_{B2,ij}(q_\ell, \dot{q}_\ell) = \frac{1}{2}\left(\dot{q}_\ell^T \frac{\partial B_{2,ij}}{\partial q_\ell} - \frac{\partial \left(B_2^T\right)^j}{\partial q_{\ell,i}}\dot{q}_\ell \right) \qquad (2)
$$
$$
C_{B3,ij}(q_\ell, \dot{q}_\ell) = \frac{1}{2}\left(\dot{q}_\ell^T \frac{\partial B_{2,ji}}{\partial q_\ell} - \frac{\partial \left(B_2^T\right)^i}{\partial q_{\ell,j}}\dot{q}_\ell \right),
$$

where A^i denotes the ith *row* of a matrix A.

We note that in the reduced dynamic model of [4], the off-diagonal block B_2 of the inertia matrix is *always* zero. Matrix B_2 is found to be zero also for particular kinematic arrangements, even with the complete modeling. Whenever $B_2 = 0$, the model (1) is linearizable via static state feedback.

For tracking purposes, we define the link position vector q_ℓ as the controlled output.

3. Full linearization via dynamic state feedback

We recall the general sufficient condition of [10] for full linearization via dynamic state feedback of square nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y &= h(x), \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^\mu$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, f and the m columns g_i are analytic vector fields, and h is an analytic mapping. For the robot model in eq. (1), $\mu = 4n$.

The goal is to find a *dynamic state-feedback compensator*

$$\begin{aligned} \dot{\zeta} &= a(\zeta, x) + b(\zeta, x)v \\ u &= c(\zeta, x) + d(\zeta, x)v, \end{aligned} \tag{4}$$

with compensator state $\zeta \in \mathbb{R}^\nu$ and $v \in \mathbb{R}^m$, such that the closed-loop system (3)-(4) becomes, after a suitable change of coordinates $z = \phi(\zeta, x)$,

$$\begin{aligned} \dot{z} &= Az + Bv \\ y &= Cz, \end{aligned} \tag{5}$$

with (A, B) controllable and (A, C) observable.

We make use also of the concept of zero-dynamics of a system in the form (3). This is the internal dynamics left in the system when the output is forced to be zero (or constant) at all times, under the action of a proper input. Then, the following result holds:

Theorem 1. [10] Suppose that the system (3):

i) is invertible;

ii) has no zero-dynamics.

Then, the system (3) can be fully linearized via dynamic state feedback of the form (4).

The two assumptions of Theorem 1 guarantee that, when the decoupling procedure of [11] is applied, the repeated use of invertible static state feedback plus dynamic extension by input integrators does not introduce zero-dynamics when the original system has none. The invertibility assumption guarantees that, after a finite number of steps, the decoupling procedure ends up with an extended system that can be decoupled via static feedback. Moreover, the absence of zero-dynamics of the final extended system implies that the sum of the associate relative degrees equals the dimension $\mu + \nu$ of the extended state space. Therefore, the system can be converted by a feedback transformation into a completely controllable and observable linear system. Moreover, due to the procedure followed, the linear system (5) will also be input-output decoupled.

4. Dynamic feedback linearization of elastic joint robots

In order to apply Theorem 1, we need to verify the validity of its two assumptions. For, we first show that the robot model (1), with output $y = q_\ell$, has no zero dynamics.

Lemma 1. If $y(t) = \bar{q}_\ell$ (constant) for all $t \geq 0$, then the state $(q_\ell(t), q_m(t), \dot{q}_\ell(t), \dot{q}_m(t))$ of the robot is constant for all $t \geq 0$, together with the input $u(t)$.

Proof. If $y \equiv \bar{q}_\ell$, then $\dot{q}_\ell = \ddot{q}_\ell = 0$. Under these conditions, the first matrix equation in (1) reduces to

$$B_2(\bar{q}_\ell)\ddot{q}_m + K_e(\bar{q}_\ell - q_m) + g(\bar{q}_\ell) = 0. \tag{6}$$

Starting from the last row of (6), using the upper triangular structure of B_2 and the diagonality of K_e, and proceeding backward, one has recursively

$$q_{m,n} = \bar{q}_{\ell,n} + \frac{g_n(\bar{q}_\ell)}{k_n} \Rightarrow \dot{q}_{m,n} = \ddot{q}_{m,n} = 0$$

$$q_{m,n-1} = \bar{q}_{\ell,n-1} + \frac{g_{n-1}(\bar{q}_\ell)}{k_{n-1}} \Rightarrow \dot{q}_{m,n-1} = \ddot{q}_{m,n-1} = 0$$

$$\vdots \qquad \vdots$$

$$q_{m,1} = \bar{q}_{\ell,1} + \frac{g_1(\bar{q}_\ell)}{k_1} \Rightarrow \dot{q}_{m,1} = \ddot{q}_{m,1} = 0.$$

Thus, $(q_\ell(t), q_m(t), \dot{q}_\ell(t), \dot{q}_m(t)) = (\bar{q}_\ell, \bar{q}_m, 0, 0) = const$, with $\bar{q}_m = \bar{q}_\ell + K_e^{-1}g(\bar{q}_\ell)$. From the second matrix equation in (1), it follows that $u(t) = K_e(\bar{q}_m - \bar{q}_\ell)$, constant for all $t \geq 0$. $\quad\square$

In order to show that the robot system (1) is invertible, we provide directly the unique input associated with a desired evolution of the output.

Lemma 2. Given $y_d(t) = q_\ell^d(t)$, a smooth reference link trajectory for all $t \geq 0$, then the motor trajectory, together with its derivatives, and the control input are uniquely determined for all $t \geq 0$.

Proof. We will first show how, from a given $q_\ell^d(t)$ together with its derivatives, it is possible to compute in closed form the associated trajectory $q_m^d(t)$ for q_m.

Consider the nth row of the first equation in (1), with $(q_\ell, \dot{q}_\ell, \ddot{q}_\ell)$ substituted by $(q_\ell^d, \dot{q}_\ell^d, \ddot{q}_\ell^d)$,

$$B_1^n(q_\ell^d)\ddot{q}_\ell^d + \left[C_{A1}^n(q_\ell^d, \dot{q}_m) + C_{B1}^n(q_\ell^d, \dot{q}_\ell^d) \right] \dot{q}_\ell^d \tag{7}$$
$$+ C_{B2}^n(q_\ell^d, \dot{q}_\ell^d)\dot{q}_m + k_n(q_{\ell,n}^d - q_{m,n}) + g_n(q_\ell^d) = 0.$$

In this equation, the terms in the l.h.s. are known functions of time, except for the second, the fourth, and the sixth one. However, since $B_2^n = 0$ and B_2 is independent from $q_{\ell,n}$, using the expressions (2), one has

$$C_{A1}^n(q_\ell^d, \dot{q}_m) = C_{B2}^n(q_\ell^d, \dot{q}_\ell^d) = 0.$$

Therefore, eq. (7) becomes linear in the unknown $q_{m,n}$ and we can solve for it as

$$q_{m,n} = q_{\ell,n}^d(t) + \frac{1}{k_n} f_n(t) = q_{m,n}^d(t),$$

where

$$f_n(t) = B_1^n(q_\ell^d)\ddot{q}_\ell^d + C_{B1}^n(q_\ell^d, \dot{q}_\ell^d)\dot{q}_\ell^d + g_n(q_\ell^d).$$

Thus, $q_{m,n}^d(t)$ and its derivatives are actually known functions of time.

Next, consider row $(n-1)$,

$$B_1^{n-1}(q_\ell^d)\ddot{q}_\ell^d + B_2^{n-1}(q_\ell^d)\ddot{q}_m^d$$
$$+ \left[C_{A1}^{n-1}(q_\ell^d, \dot{q}_m) + C_{B1}^{n-1}(q_\ell^d, \dot{q}_\ell^d)\right]\dot{q}_\ell^d + C_{B2}^{n-1}(q_\ell^d, \dot{q}_\ell^d)\dot{q}_m \quad (8)$$
$$+ k_{n-1}(q_{\ell,n-1}^d - q_{m,n-1}) + g_{n-1}(q_\ell^d) = 0.$$

Due to the structure of B_2, one has also that

$$B_2^{n-1}(q_\ell^d)\ddot{q}_m^d = B_{2,(n-1)n}(q_{\ell,n-1}^d)\ddot{q}_{m,n},$$

and

$$C_{A1}^{n-1}(q_\ell^d, \dot{q}_m) = C_{A1}^{n-1}(q_\ell^d, \dot{q}_{m,n})$$
$$C_{B2}^{n-1}(q_\ell^d, \dot{q}_\ell^d) = (0 \quad 0 \quad \dots \quad 0 \quad *).$$

Therefore, the only term in eq. (8) depending from an unknown motor variable is the linear term in $q_{m,n-1}$. Thus, from eq. (8) we can solve for

$$q_{m,n-1} = q_{\ell,n-1}^d(t) + \frac{1}{k_{n-1}} f_{n-1}(t) = q_{m,n-1}^d(t),$$

where $f_{n-1}(t)$ is again a known function of time, depending on the quantities $(q_\ell^d, \dot{q}_\ell^d, \ddot{q}_\ell^d, \dot{q}_{m,n}^d, \ddot{q}_{m,n}^d)$.
From row $(n-2)$ of the first matrix equation in (1), one can proceed in the same way and recursively deduce the whole q_m^d and, by symbolic differentiation, $\dot{q}_m^d$ and $\ddot{q}_m^d$. Finally, from the second matrix equation in (1), one has

$$u^d(t) = B_2^T(q_\ell^d)\ddot{q}_\ell^d + B_3\ddot{q}_m^d + C_{B3}(q_\ell^d, \dot{q}_\ell^d)\dot{q}_\ell^d + K_e(q_\ell^d - q_m^d).$$

$$\square$$

Remark. In the above proof we have used the fact that C_{B2} turns out to be strictly upper triangular and that C_{A1}^i does not depend on any $\dot{q}_{m,j}$, for $j \le i$, $i = 1, \dots, n$.

We can state now our main result.

Theorem 2. Elastic joint robots modeled by eq. (1), with output $y = q_\ell$, are always globally feedback linearizable via dynamic state feedback.

Proof. From Lemma 1, the dynamic model (1) has no zero-dynamics since there is no internal dynamics consistent with the constraint $y(t) \equiv const$. From Lemma 2, a unique state evolution $x^d(t) = (q_\ell^d(t), q_m^d(t), \dot{q}_\ell^d(t), \dot{q}_m^d(t))$ and input $u^d(t)$ are associated with a given output trajectory $y^d(t)$ so that the robot system is left invertible and, being square, invertible. Therefore, Theorem 1 applies. $\square$

As a consequence of Theorem 2, it is possible to achieve global and stable output tracking via state feedback for the complete model of robots with elastic joints. Once eq. (1) is rendered, via dynamic extension, feedback equivalent to a controllable and observable linear system, it is easy to design a controller with arbitrary linear error dynamics that asymptotically tracks any smooth desired output trajectory q_ℓ^d, starting from any initial state. We note that the desired state trajectory $(q_\ell^d(t), q_m^d(t), \dot{q}_\ell^d(t), \dot{q}_m^d(t))$ can be computed off line, as shown in the proof of Lemma 2.

5. CONCLUSIONS

We have shown that the complete dynamic model of robots with elastic joints can always be fully transformed into a linear and decoupled system through the use of dynamic state feedback. The closed-loop system is diffeomorphic to chains of input-output integrators of proper length, whose stabilization is easy. As a result, the trajectory tracking problem is globally solvable also for the complete model of robots with elastic joints, at least in the nominal case.

The actual construction of the dynamic compensator and the associated coordinate transformations are still a difficult task to be defined in a general way. This is because the dimension ν of the compensator depends on the particular kinematic arrangement of the arm. An example of such a controller can be found in [8].

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