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Achieving Minimum Phase Behavior in a One-Link Flexible Arm

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Abstract

Inversion control techniques cannot be usually applied for end-effector trajectory tracking in flexible robotic structures, due to the non-minimum phase nature of the problem. However, considering finite-dimensional models, different choices of the actuation point and of the output location can lead to a minimum phase system for which inversion control is still feasible. These 'feasible' regions of input-output pairs are investigated for a linear model of a one-link slewing flexible arm. In the same framework, system passivity is studied and comparative conditions are given for a simple case. Simulation results show how the design of alternate input-output minimum phase pairs affects tracking performance and control effort.

Introduction

It is known that the task of reproducing a desired tip motion in a one flexible robot arm using an input joint torque is complicated by the presence of right-half-plane zeros in the transfer function *non-minimum phase.* This property, stated in linear terms [1], holds similarly also for nonlinear models of flexible arms [2], once the proper analogue of system zeros is introduced. As a result, standard application of inversion techniques for trajectory tracking becomes impossible in this case. In view of the potential advantages of an inversion control law, e.g. its straightforward extension to the nonlinear setting, it may be convenient to slightly modify problem specifications in order to recover minimum phase characteristics in the mechanical system under control. For this purpose, two different approaches can be devised in which either the controlled output or the actuation input are 'moved' to different locations along the arm.

The more natural choice of keeping the actuation point at the joint while varying the output location has been considered in [2]. It has been shown that one can select as output a particular point along the link, parameterized by λ, in such a way to guarantee stable zero dynamics in the resulting system. If a finite dimensional model of flexibility is assumed, an entire set of such points exists, including indeed the joint location. As an alternate approach, one may keep the output fixed at the tip and let the actuation point, parameterized by σ, vary along the link, as proposed in [3]. Again, for a finite number of deflection modes, a set of such actuation points exists, containing also the tip location, which convert the input-output mapping into a minimum phase one. As the number of modes tends to infinity, both cases degenerate leaving in the limit only the joint output point or the tip actuation point respectively as feasible strategies. However, due to the finite bandwidth of real control systems, these extreme cases are of limited significance and thus the two above approaches, aiming at the same goal, remain both of great practical interest.

In this paper conditions for achieving minimum phase behavior in a one-link flexible arm are further investigated in a fully linear setting, and the dual nature of the parameterized input and output strategies is illustrated in detail. In particular, it is shown how effective control strategies can be designed by combining input and output location modification. The main objective of this combined approach is to increase tracking accuracy of desired tip trajectories and to reduce at the same time torque demand of the remotely located actuator.

In the same framework it is possible to study system passivity, a convenient property that may lead to the design of adaptive and/or robust control structures for the flexible arm. However, the associated conditions on input-output locations are more restrictive than the ones for minimum phase. A simple necessary and sufficient condition for passivity to hold is derived for the case of two flexible modes.

Finally, simulations are reported illustrating the achieved control features.

Modeling of the flexible slewing link

Consider a one-link flexible slewing arm with a balanced tip payload. The arm moves on an horizontal plane and only transversal deflections are taken into account. The link is modeled as an Euler beam of uniform density ρ with length ℓ and inertia J_b with respect to the joint axis, while J_h is the hub inertia and the payload has mass M_p and inertia J_p. A linear dynamic model can be derived using the Lagrangian approach and choosing, for the description of deflection, a rotating frame with the x-axis intersecting the center of mass of the arm (pseudo-pinned condition), as in Fig. 1.

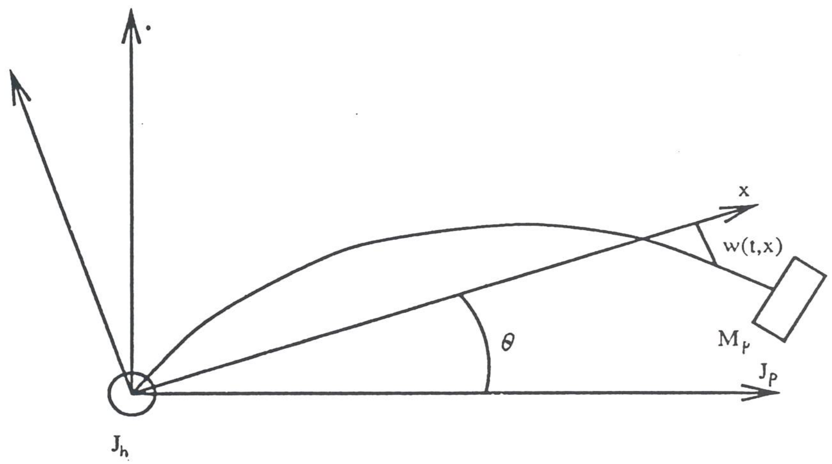

Fig. 1 — The one-link flexible arm

With reference to this frame, the eigenvalue problem associated to the link deformation can be solved easily and exact eigenfunctions $\psi_i(x)$ can be computed [4], including in the modal analysis the effects of tip payload and hub inertia. The choice of this reference frame is motivated by the simplicity of the associated dynamic model. Other reference frames in which to measure the link deflection and the rigid hub rotation can be defined and simple coordinate transformations can be found to show the equivalence between any of these choices (e.g. the pseudo-clamped reference frame in [4]).

The deflection $w(t,x)$ at a point x along the link is given by

$$w(t,x) = \sum_{i=1}^{\infty} \psi_i(x)\delta_i(t),$$

being $\delta_i(t)$ the time variable associated to the ith modal shape. Denote by $J = J_h + J_b + J_p + M_p\ell^2$ the total inertia seen at the joint and let the actuation input be located at a generic point $\sigma \in [0,\ell]$. Considering n flexible modes, the dynamic model becomes

$$B\ddot{q}(t) + Kq(t) = g(\sigma)\,u(t), \tag{1}$$

with

$$B = \begin{bmatrix} J & 0 \\ 0 & I_{n\times n} \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 \\ 0 & \Omega^2 \end{bmatrix}, \quad g(\sigma) = \begin{bmatrix} 1 \\ \frac{\psi(\sigma)}{\sigma} \end{bmatrix}, \quad \psi(\sigma) = [\psi_1(\sigma) \ \ \dots \ \ \psi_n(\sigma)]^T.$$

The generalized coordinates $q = (\theta, \delta) \in I\!R^{n+1}$ are the rigid rotation θ of the non-inertial frame with respect to an inertial one, and the vector of flexible variables $\delta = [\delta_1, \dots, \delta_n]^T$. Moreover Ω^2 is an $n \times n$ diagonal matrix with entries being the squared angular eigenfrequencies ω_i^2 associated to the modes. The obtained eigenfunctions, which naturally satisfy orthogonality conditions, have been properly normalized. The effect of input torque u on flexible variables is weighted by the n-column vector $\psi(\sigma)/\sigma$, whose components are the modal shapes, evaluated at the actuation point σ and scaled by this value. Modal damping can be easily included by adding $D\dot{q}$ to the right-hand-side of (1) being

$$D = \begin{bmatrix} 0 & 0 \\ 0 & D_\delta \end{bmatrix},$$

where D_δ is an $n \times n$ diagonal matrix with entries $d_i = 2\zeta_i\omega_i$.

As system output — the quantity to be controlled — one can take the angular position of a generic point along the link located at a distance $\lambda \in [0,\ell]$ from the joint. For small deformations, this output is

$$y(t,\lambda) = \theta(t) + \sum_{i=1}^{n} \frac{\psi_i(\lambda)}{\lambda}\delta_i(t) = c^T(\lambda)\,q(t). \tag{2}$$

Note that both the output function $y(t,\lambda)$ and the input vector $g(\sigma)$ are well defined also for $\lambda = 0$ and $\sigma = 0$ since, using De L'Hospital rule,

$$y(t,0) = \theta(t) + \sum_{i=1}^{n} \psi_i'(0)\delta(t), \qquad g(0) = [1 \quad \psi'(0)]^T,$$

where the $\psi_i'(0)$ match the boundary value at the hub for the pseudo-pinned case.

Study of minimum phase characteristics

For the undamped case, the transfer function from actuation input $u_\sigma(s)$ at σ to link output $y_\lambda(s)$ at λ is of the form

$$W(s) = \frac{y_\lambda(s)}{u_\sigma(s)} = \frac{1}{Js^2} + \sum_{i=1}^{n} \frac{\psi_i(\sigma)\psi_i(\lambda)}{\sigma\lambda(s^2+\omega_i^2)} = \frac{a_n s^{2n} + a_{n-1} s^{2(n-1)} + \dots + a_1 s^2 + a_0}{s^2(s^2+\omega_1^2)\dots(s^2+\omega_n^2)}.$$

The coefficients have the following explicit expressions:

$$a_n = \frac{1}{J}\left(1 + J\sum_{i=1}^{n} \frac{\psi_i(\lambda)}{\lambda}\frac{\psi_i(\sigma)}{\sigma}\right),$$

$$a_{n-1} = \frac{1}{J}\left[\sum_{i=1}^{n} \omega_i^2\left(1 + J\sum_{j\neq i}^{n} \frac{\psi_j(\lambda)}{\lambda}\frac{\psi_j(\sigma)}{\sigma}\right)\right],$$

$$a_{n-2} = \frac{1}{J}\left[\sum_{i=1}^{n}\sum_{j>i}^{n} \omega_i^2\omega_j^2\left(1 + J\sum_{k\neq i,j}^{n} \frac{\psi_k(\lambda)}{\lambda}\frac{\psi_k(\sigma)}{\sigma}\right)\right],$$

$$\vdots$$

$$a_0 = \frac{1}{J}\prod_{i=1}^{n} \omega_i^2.$$

For the damped case, the numerator of the transfer function is generically a full $2n$-th order polynomial in s

$$W_d(s) = \frac{\tilde{a}_{2n} s^{2n} + \tilde{a}_{2n-1} s^{2n-1} + \tilde{a}_{2n-2} s^{2n-2} + \dots + \tilde{a}_1 s + \tilde{a}_0}{s^2(s^2+d_1 s+\omega_1^2)\dots(s^2+d_n s+\omega_n^2)}.$$

with the coefficients $\tilde{a}_i$ reported in Appendix.

Necessary and sufficient conditions for the system to be minimum phase, i.e. for the zeros of the transfer function to be with negative real part, can be found by directly applying the Routh criterion to the numerator of $W(s)$ or of $W_d(s)$.

From a structural design point of view, the main objective at this stage would be to find a combination of mechanical parameters of the arm, such as inertia and mass, so to achieve a suitable behavior of the chosen input-output pair, at least for a given number of modes. On the other hand, once the mechanical parameters of the flexible beam are assigned, the degrees of freedom left in the control design can be exploited with the purpose of achieving minimum phase. This accounts in the proper selection of λ and σ appearing in the terms $\psi_i(\lambda)\psi_i(\sigma)/\lambda\sigma$. Therefore, from a control point of view:

- for a given output, i.e. for a λ^*, one can choose σ, the actuation point along the link;
- for a given actuation point, i.e. for a σ^*, one can choose λ, the output point along the link.

It is interesting to characterize regions in the plane $[\sigma, \lambda]$ which guarantee the minimum phase property, i.e. the stability of the numerator of the transfer function, for σ and λ varying in $[0, \ell]$. Several methods, mostly inspired by Kharitonov's result, are known in order to investigate the behavior of the roots of a polynomial with respect to parametric variations. The most useful results deal with coefficients which can either be independently perturbed or in which perturbated parameters enter multilinearly. Unfortunately, these cannot be applied in a straightforward way since the coefficients of the numerator do not vary independently. In fact, parameters λ and σ appear nonlinearly in the dynamic model, and thus in the numerator of the transfer function, through eigenfunctions and eigenfrequencies. Also, the mechanical parameters affect directly the model through J as well as indirectly just as before.

One could still attempt a conservative estimation of feasible bounds by choosing as independent parameters the products $\psi_i(\lambda)\psi_i(\sigma)/\lambda\sigma$ entering linearly in the coefficients of the polynomial. The problem is then reformulated as follows. For an arm with n flexible modes, the numerator of the transfer function can be put in the form

$$P(s,p) = \sum_{i=0}^{n} a_i(p)s^{2i}$$

with $p = [p_1, \ldots, p_n]$ and $p_i = \psi_i(\lambda)\psi_i(\sigma)/\lambda\sigma$. Suppose that these parameters are restrained to the intervals $[p_i^-, p_i^+]$ and define

$$p^i = [p_1^i \quad \ldots \quad p_n^i]^T, \quad \text{with} \quad p_j^i = p_j^- \quad \text{or} \quad p_j^+.$$

Since the parameters p_i enter linearly in the coefficients a_i of $P(s,p)$, the set of all polynomials can be expressed as the convex hull of a finite number of generating polynomials $P^i(s)$, i.e.

$$\mathcal{P} = \text{conv}\,\{P^1(s), \ldots, P^n(s)\} \qquad \text{with} \qquad P^i(s) = \sum_{j=0}^{n} a_j(p^i)s^{2j}.$$

Two different techniques of analysis could be applied to $\mathcal{P}$: the Edge Theorem [5] and the Stability Testing Function [6]. However, both methods require at least an estimate on the bounds of the perturbed parameters and no direct way is known to derive these quantities. Nevertheless, a more accurate investigation on the use of robust design techniques would be surely of great interest. Indeed, since the number of independent parameters in the case under study is two (σ and λ), use of both numerical and graphical techniques turns out to be of great practical help.

The case of two flexible modes ($n = 2$) is simple enough to be treated analytically. In this case the transfer function is

$$W(s) = \frac{a_2 s^4 + a_1 s^2 + a_0}{s^2(s^2 + \omega_1^2)(s^2 + \omega_2^2)},$$

with

$$a_2 = \frac{1}{J} + \frac{\psi_1(\lambda)\psi_1(\sigma) + \psi_2(\lambda)\psi_2(\sigma)}{\lambda\sigma},$$

$$a_1 = \frac{1}{J}(\omega_1^2 + \omega_2^2) + \frac{\psi_1(\lambda)\psi_1(\sigma)\omega_2^2 + \psi_2(\lambda)\psi_2(\sigma)\omega_1^2}{\lambda\sigma},$$

$$a_0 = \frac{1}{J}\omega_1^2\omega_2^2.$$

The necessary and sufficient conditions derived from Routh criterion are

$$a_2 > 0, \qquad \text{and} \qquad a_1^2 - 4a_2a_0 > 0. \tag{3}$$

For a given set of arm parameters, all possible combinations of λ and σ resulting in a minimum phase behavior can be visualized on a 3-D plot. Figure 2 refers to the following numerical data: $\ell = 0.7$ m, $\rho = 2.64$ kg m, $EI = 2.45$ N m^2, $J_b = 3.02 \cdot 10^{-1}$ kg m^{-2}, $J_h = 1.3 \cdot 10^{-3}$ kg m^2 and $J_p = M_p = 0$. The sign function is properly used for indicating regions where both inequalities (3) hold.

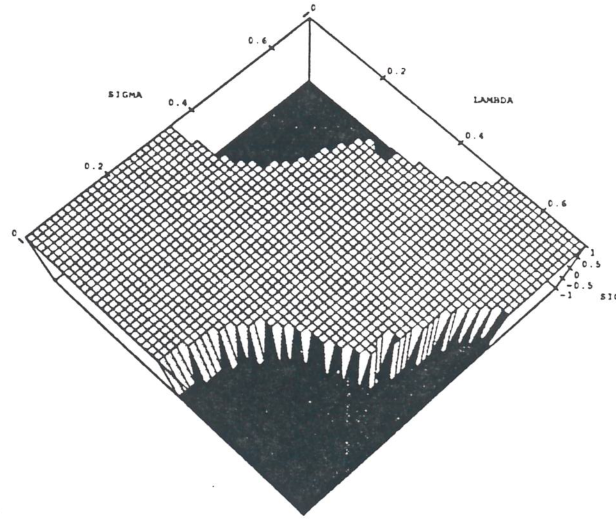

Fig. 2 — Minimum phase region obtained by varying σ and λ: $n = 2$ flexible modes

This plot deserves some comments. The existence of critical values for the minimum phase property was pointed out in [2] and [3], namely of a highest value λ_{crit} for $\sigma = 0$, and of a lowest value σ_{crit} for $\lambda = \ell$, respectively. It is apparent that these critical values are not unique when both parameters are left free to vary, and situations may arise in which admissible regions are created while other disappear. Further, all points (x, x) with $x \in [0, \ell]$ in the (σ, λ) plane are always admissible, as a result of input-output collocation. This can also be checked analytically by substituting the coefficients expressions, with $\lambda = \sigma$, in (3). The plot symmetry confirms the inherent duality of the two strategies. Therefore, since the joint location becomes itself a non-minimum phase output when actuation is moved sufficiently far along the link, a reverse situation occurs for the tip output moving the actuation location closer to the link base.

As the number of flexible modes included in the dynamic model increases, the 'feasible' region shrinks and will eventually include only the collocated situations (on the 'diagonal'). The undamped case with three flexible modes is shown in Fig. 3.

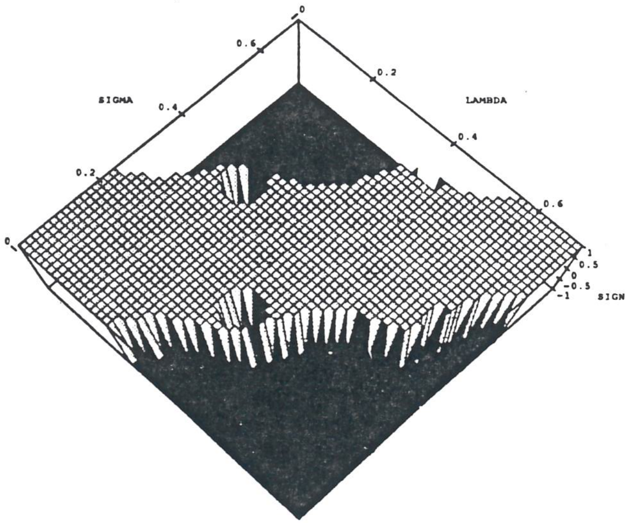

Fig. 3 — Minimum phase region obtained by varying σ and λ: $n = 3$ flexible modes

As already noted in [7], modification of operative conditions for the flexible structure can lead to a minimum phase behavior, for a fixed number of modes, in the joint-tip transfer function. As an example, increasing the previous values to $J_h = 7.8 \cdot 10^{-3}$ kg m^2, $J_p = 5J_h$, and $M_p = 0.4$ kg, the Routh conditions (3), i.e. for $n = 2$, will be satisfied at every point in the (σ, λ) plane.

For any input-output combination (σ, λ) that belongs to the minimum phase region, use of inversion control guarantees exact trajectory reproduction. Nonetheless, if the chosen output differs from the tip point (i.e. $\lambda \neq \ell$), a tracking error still results with respect to the desired tip motion — the ultimate objective of control. This error is expected to become smaller when the distance of the chosen output point from the tip decreases. On the other hand, when applying feedback laws based on inversion, control effort increases with λ approaching λ_{crit}, and becomes unbounded at the crossing of this critical value. These considerations hold for a fixed value of σ, in particular zero. At this stage, a proper modulation of σ (e.g. a sufficient increase) achieves positive effects on torque demand while keeping low the trajectory error at the tip. The effects of different admissible parameter choices on control performance will be assessed by simulation.

A more restrictive design approach: passivity

The general requirement for design procedures yielding simple control laws has recently motivated the study of passivity properties for the flexible link [8]. In particular, it is known that a strictly passive output feedback controller with finite gain stabilizes (in a L_2 sense) any passive system (Passivity Theorem). A scalar input-output system is *passive* if

$$\int_0^T y(t)u(t)dt \geq 0, \qquad \forall\, T, u \quad \text{where} \quad y(s) = W_p(s)u(s)$$

As mentioned in [8], there exists some particular set of mechanical parameters for which a certain transfer function is passive. In particular, the transfer function of concern will relate input torque to output velocity, i.e. for the $n = 2$ undamped case

$$W_p(s) = \frac{a_2 s^4 + a_1 s^2 + a_0}{s(s^2 + \omega_1^2)(s^2 + \omega_2^2)}. \qquad (4)$$

Recall that a real rational transfer function is *reactive* if all its poles and zeros lie on the $j\omega$-axis, are distinct and alternate each other. A reactive function is always passive. Moreover, a passive system is necessarily minimum phase. On the other hand, in the minimum phase case, all the zeros of the undamped transfer function lie on the $j\omega$-axis (as well as the poles), due to the absence of odd powers of s. Therefore a design requirement which guarantees passivity, at least for the undamped case, is to 'place' the zeros of $W_p(s)$ so to achieve the interlacing property with the poles.

For two flexible modes, a necessary and sufficient condition under which the zeros and poles of the transfer function $W_p(s)$ alternate each other can be given in terms of the eigenfunctions as

$$\psi_1(\lambda)\psi_1(\sigma) > 0 \qquad \text{and} \qquad \psi_2(\lambda)\psi_2(\sigma) > 0. \qquad (5)$$

The proof is very simple and relies on a basic property of second-order polynomial equations: if equation $N(z) = az^2 + bz + c = 0$ has two real roots z_1 and z_2, a point z^* belongs to the interval $[z_1, z_2]$ if and only if $\text{sign}\{N(z^*)\} = -\text{sign}\{a\}$. Setting $z = s^2$, $N(z) =$ Numerator of $W_p(s)$ and noting that the second Routh condition in (3) implies real roots for $N(z) = 0$, it is possible to show that the first pole $j\omega_1$ lies between the two zeros while the second pole lies beyond. Therefore, being $0 < \omega_1 < \omega_2$, the only possible pattern is the alternation of pole and zeros. Inequalities (5) should be compared with (3) after substitution of the proper expressions of the coefficients.

With the same parameter values for the flexible link as in Fig. 2, the passivity conditions (5) are represented in Fig. 4, using properly the sign function. As expected, the passive region is a subset of the minimum phase one.

Note that a slight modification of the output parameterization (2) is necessary to apply all the previous results to the outputs considered in [8], that is

$$\bar{y}(t, \lambda) = \theta(t) + \sum_{i=1}^{n} \frac{\psi_i(|\lambda|)}{\lambda} \delta_i(t)$$

so that one can let λ vary in $[-\ell, 0]$. $\bar{y}(t, \lambda)$ is the specular angular position with respect to the undeflected arm configuration. In [8] it is shown how this choice of the output leads, for some parameter values, to a passive system.

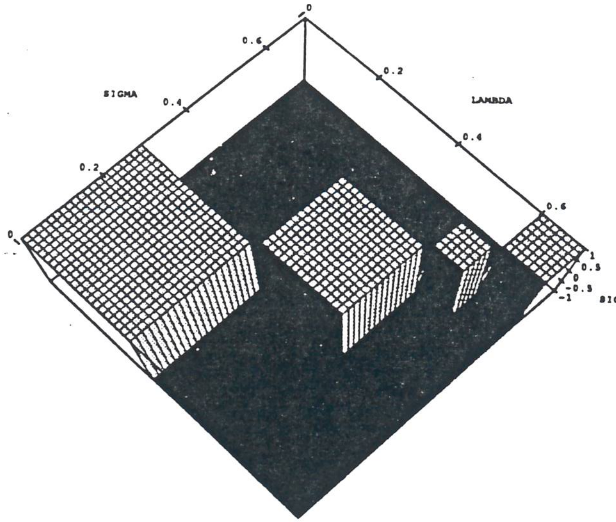

Fig. 4 — Passivity regions obtained by varying σ and λ: $n = 2$ flexible modes

Simulation results

The proposed control approach was tested by simulation on a flexible link considering two modes and, unless specified, with the values previously reported (no payload). As a desired trajectory $y_d(t)$, a sixth order polynomial has been chosen with assigned zero initial and final velocities and accelerations, starting at 0° and ending at 90° in 2 sec.

For a minimum phase system, trajectory tracking is achievable using standard inversion control law [7]. For system (1) and (2) this takes on the form:

$$u_{inv}(t) = \frac{1}{c^T B^{-1} g}(c^T B^{-1} K q(t) + v(t)).$$

In both the linear and nonlinear case the system after inversion consists of a double integrator relating the new input $v(t)$ to the output $y(t, \lambda$ plus an induced unobservable part whose dimension is given by the number of considered modes. When the original system is minimum phase, these closed-loop dynamics is stable. In the following, since initial conditions are met and the true parameter values are used in the control, $v(t) = \ddot{y}_d(t)$.

The joint actuation strategy ($\sigma = 0$) is illustrated in Figs. 5–7 with two different output locations, both belonging to the feasible region of Fig. 2: $\lambda_1 = 0.4\ell$ and $\lambda_2 = 0.15\ell$. In Fig. 5 the desired trajectory, which is exactly tracked by both chosen outputs, is shown together with the tip behavior corresponding to the two cases denoted 'tip1' and 'tip2' respectively. An end-effector error is present (Fig. 6) since the controlled output is not at the tip location. Results

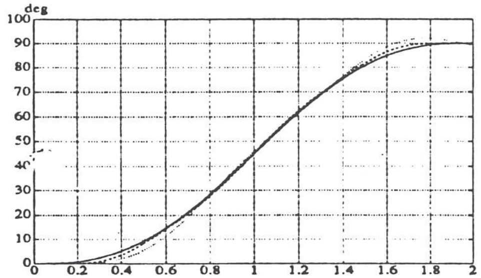

Fig.5 - Joint actuation: output(-) tip1(--) tip2(...)

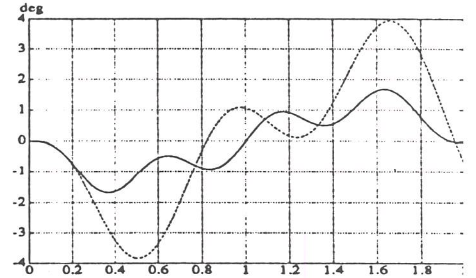

Fig.6 - Joint actuation: errors tip1(-) tip2(--)

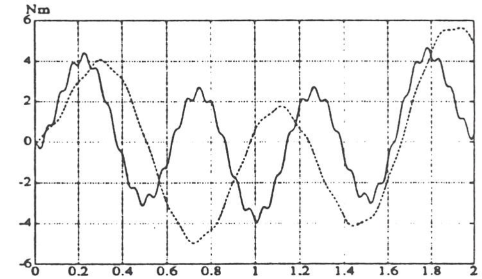

Fig.7 - Joint actuation: torque input1(-) input2(--)

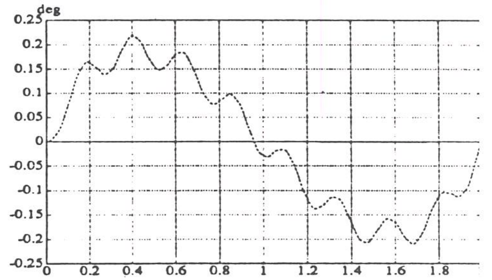

Fig.8 - Tip actuation: output(-) and tip(--) error

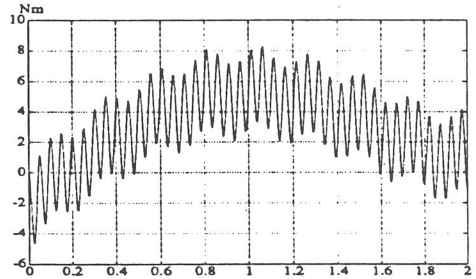

Fig.9 - Tip actuation: input torque

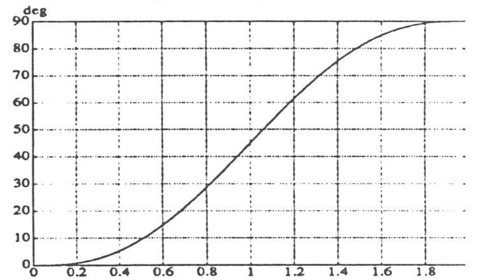

Fig.10 - Joint actuation: desired and tip trajectories

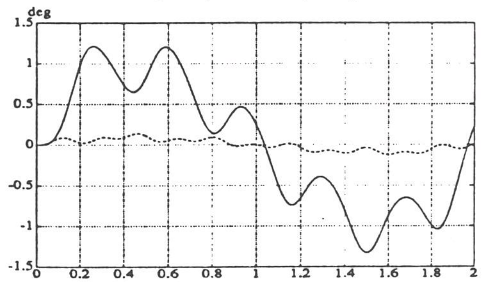

Fig.11 - Joint actuation: flexible variables

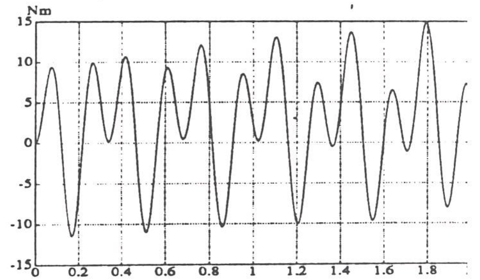

Fig.12 - Joint actuation: input torque

are clearly better for the value $\lambda_1 = 0.4\ell$, since this is closer to $\lambda_{crit} = 0.5\ell$ but yet not so to have an unacceptable torque requirement (Fig. 7).

The tip actuation strategy ($\sigma = \ell$) has been tested with an output chosen at $\lambda = 0.77\ell$. Figure 8 shows the output and the tip error in this case. The torque actually supplied by the actuation remotely located at the joint is reported in Fig. 9. It is interesting to note that the 'dual' choice $\sigma = 0.77\ell$ and $\lambda = l$ would require the same torque but would lead of course to zero error at the tip. The limit of these choices is evident from the oscillatory behavior of the required torque. This aspect should be carefully considered in order to avoid undesired excitation of higher order unmodeled dynamics (spill-over effect).

It has been noted that the modification of the flexible beam boundary values for the mass (at the tip) and inertia (at the tip and at the hub) may induce a minimum phase behavior from joint to tip. Using $J_h = 7.8 \cdot 10^{-3}$ kg m^2, $J_p = 5J_h$, and $M_p = 0.4$ kg, Fig. 10 shows that the tip tracks exactly the desired trajectory under inversion control even with joint actuation. The internal stable behavior of the closed-loop system is described in Fig. 11 through the flexible deformation variables $\delta_1(t)$ and $\delta_2(t)$. The applied torque shown in Fig. 12 has higher peak values (due to the increased payload) but much less high-frequency content.

Conclusions

For a linear finite-dimensional model of the flexible beam, it is always possible to achieve a minimum phase behavior by suitably considering input-output locations along the link. This allows to use inversion control techniques and to obtain satisfactory tip behavior with acceptably low torque effort. Similarly, passive characteristics can be induced and robust controllers could be designed for regulation and tracking.

These results can be extended to nonlinear models of one-link flexible arms as well as to the multi-link case, moving to the nonlinear notion of stable zero dynamics.

Acknowledgments

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Appendix

Explicit expressions for the coefficients of the transfer function $W_p(s)$, corresponding to the damped case, are given here:

$$\tilde{a}_{2n} = a_n ,$$

$$\tilde{a}_{2n-1} = \frac{1}{J}\Big[\sum_{i=1}^{n} d_i\Big(1 + J\sum_{j\neq i}^{n} \frac{\psi_j(\lambda)}{\lambda}\frac{\psi_j(\sigma)}{\sigma}\Big)\Big],$$

$$\tilde{a}_{2n-2} = a_{n-1} + \frac{1}{J}\Big[\sum_{i=1}^{n}\sum_{j>i}^{n} d_i d_j\Big(1 + J\sum_{k\neq i,j}^{n} \frac{\psi_k(\lambda)}{\lambda}\frac{\psi_k(\sigma)}{\sigma}\Big)\Big],$$

$$\tilde{a}_{2n-3} = \frac{1}{J}\Big[\sum_{i=1}^{n}\sum_{j\neq i}^{n} d_i\omega_j^2\Big(1 + J\sum_{k\neq i,j}^{n} \frac{\psi_k(\lambda)}{\lambda}\frac{\psi_k(\sigma)}{\sigma}\Big) + \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n} d_i d_j d_k\Big(1 + J\sum_{l\neq i,j,k}^{n} \frac{\psi_l(\lambda)}{\lambda}\frac{\psi_l(\sigma)}{\sigma}\Big)\Big],$$

$$\tilde{a}_{2n-4} = a_{n-2} + \frac{1}{J}\Big[\sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k\neq i,j}^{n} d_i d_j\omega_k^2\Big(1 + J\sum_{l\neq i,j,k}^{n} \frac{\psi_l(\lambda)}{\lambda}\frac{\psi_l(\sigma)}{\sigma}\Big) + \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n}\sum_{l>k}^{n} d_i d_j d_k d_l\Big(1 + J\sum_{m\neq i,j,k,l}^{n} \frac{\psi_m(\lambda)}{\lambda}\frac{\psi_m(\sigma)}{\sigma}\Big)\Big],$$

$$\tilde{a}_{2n-5} = \frac{1}{J}\Big[\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{k\neq i,j}^{n} d_i\omega_j^2\omega_k^2\Big(1 + J\sum_{l\neq i,j,k}^{n} \frac{\psi_l(\lambda)}{\lambda}\frac{\psi_l(\sigma)}{\sigma}\Big) + \sum_{i=1}^{n}\sum_{j>i}^{n}\sum_{k>j}^{n}\sum_{l\neq i,j,k}^{n} d_i d_j d_k\omega_l^2\Big(1 + J\sum_{m\neq i,j,k,l}^{n} \frac{\psi_m(\lambda)}{\lambda}\frac{\psi_m(\sigma)}{\sigma}\Big)\Big],$$

$$\vdots$$

$$\tilde{a}_0 = a_0 .$$