Robotics 2

Midterm Test – April 19, 2023

Exercise #1

The end-effector of a 3R planar robot, having equal link lengths $l = 0.5$ [m], is executing a positional trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$ in the plane, commanded by joint accelerations $\ddot{\boldsymbol{q}}(t) \in \mathbb{R}^3$ that are updated every $T_c = 100$ ms. The robot is subject to the following hard bounds on joint velocities and accelerations:

$$|\dot{q}_i| \leq V_{max,i}, \qquad |\ddot{q}_i| \leq A_{max,i}, \qquad i = 1,2,3. \tag{1}$$

When the limits in (1) are

$$\boldsymbol{V}_{max} = \begin{pmatrix} 1.5 \\ 1.5 \\ 1 \end{pmatrix} \text{[rad/s]}, \qquad \boldsymbol{A}_{max} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \text{[rad/s}^2\text{]},$$

and the robot is in the configuration $\boldsymbol{q} = \begin{pmatrix} 0 & 0 & \pi/2 \end{pmatrix}^T$ [rad] with velocity $\dot{\boldsymbol{q}} = \begin{pmatrix} 0.8 & 0 & -0.8 \end{pmatrix}^T$ [rad/s], compute the acceleration command $\ddot{\boldsymbol{q}}$ of minimum norm that realizes the desired end-effector acceleration $\ddot{\boldsymbol{p}}_d = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ [m/s^2] while complying with the bounds imposed on robot motion.

Exercise #2

A 3R robot with Denavit-Hartenberg (DH) parameters $\alpha_i = 0$, $d_i = 0$, and $a_i = l_i > 0$, for $i = 1,2,3$, moves in a vertical plane. The i-th link has mass $m_i > 0$ and position of the center of mass (CoM) ${}^i\boldsymbol{r}_{c,i} = (r_{cx,i}, r_{cy,i}, 0)$ when expressed in the i-th DH frame, for $i = 1,2,3$. Define suitable relations between the link masses, lengths, and CoM positions of this robot such that the gravity term in the dynamic model takes the following expression:

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} m_1 g_0 \, r_{cy,1} \cos q_1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{with } g_0 = 9.81 \text{ [m/s}^2\text{]}. \tag{2}$$

Sketch a figure of a robot having the positions of the link CoMs consistent with (2).

Exercise #3

Compute the 4×4 inertia matrix of the 4P planar robot in Fig. 1. With the robot in a generic configuration $\boldsymbol{q}$, determine the joint velocity command $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that realizes a desired end-effector velocity $\boldsymbol{v}_d = \begin{pmatrix} v_{xd} & v_{yd} \end{pmatrix}^T$ while minimizing the kinetic energy T of the robot. Which would be the solution instead if the norm of the joint velocity $\|\dot{\boldsymbol{q}}\|$ is minimized?

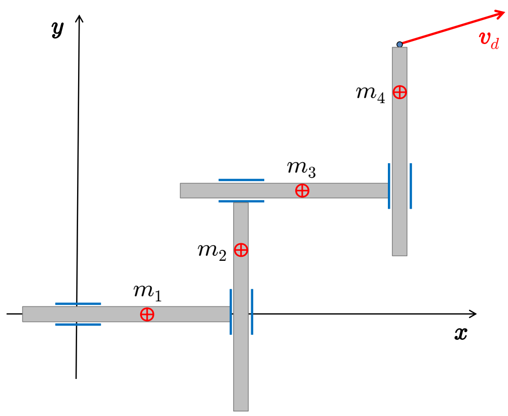


Figure 1: A 4P planar robot in a generic configuration.

Exercise #4

Consider the RPR spatial robot in Fig. 2. Based on the DH frames and joint variables defined therein, provide the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Assume that the three links have their center of mass, respectively along the $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{x}_3$ axes, and that the barycentric inertia matrix of the third link is diagonal and isotropic, i.e., ${}^3\boldsymbol{I}_{c3} = \text{diag}\,\{I_3, I_3, I_3\}$.

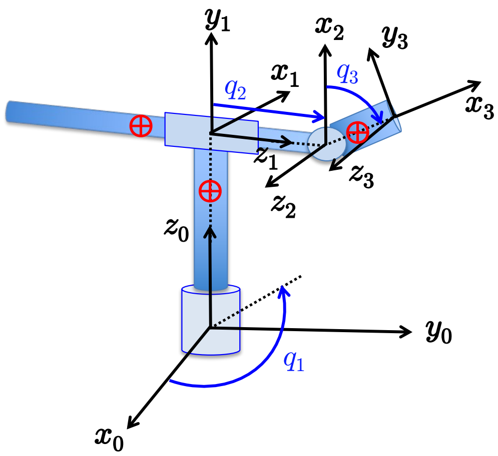


Figure 2: A spatial RPR robot, with DH frames assigned to each link.

Exercise #5

The inertia matrix of a 3-dof robot with coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ is given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2q_2 + a_3q_2^2 + 2a_4q_2 \sin q_3 + a_5 \sin^2 q_3 & 0 & 0 \\ 0 & a_3 & a_4 \cos q_3 \\ 0 & a_4 \cos q_3 & a_6 \end{pmatrix}, \tag{3}$$

where $\boldsymbol{a} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}^T$ is the vector of dynamic coefficients. Using (3), compute: *i)* the Coriolis and centrifugal term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in the robot dynamic model; *ii)* three **different** factorizations $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ and $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ are skew-symmetric matrices while $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not; *iii)* the unique 3×6 regressor matrix $\boldsymbol{Y}$ such that $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}$.

Exercise #6

A 2-dof robot has the axes of the first prismatic joint and of the second revolute joint coincident and vertical (i.e., aligned with the acceleration of gravity). The two joints should perform a displacement $\boldsymbol{\Delta q} = (\Delta q_1, \Delta q_2)$, by tracing a rest-to-rest cubic trajectory in the same motion time T. The input commands u_1 and u_2 at the joints (respectively, a force and a torque) are bounded as $|u_i| \leq U_{max,i}$, for $i = 1, 2$. Provide the minimum feasible motion time T^* to execute the task, as a function of the problem data and of the robot dynamics. Without loss of generality, assume that the actuators are at least strong enough to sustain statically the weight of the robot links.

[270 minutes (4.5 hours); open books]

Solution

April 19, 2023

Exercise #1

The task kinematics for the given 3R planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = l \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix},$$

with associated Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = l \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}.$$

The end-effector acceleration is then computed as

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

with

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\, \dot{\boldsymbol{q}} = -l \begin{pmatrix} \dot{q}_1^2\, c_1 + (\dot{q}_1 + \dot{q}_2)^2\, c_{12} + (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2\, c_{123} \\ \dot{q}_1^2\, s_1 + (\dot{q}_1 + \dot{q}_2)^2\, s_{12} + (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2\, s_{123} \end{pmatrix}.$$

When evaluating the terms at the given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ we obtain

$$\boldsymbol{J} = \begin{pmatrix} -0.5 & -0.5 & -0.5 \\ 1 & 0.5 & 0 \end{pmatrix}, \qquad \dot{\boldsymbol{J}} = \begin{pmatrix} -0.8 & -0.4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{n} = \begin{pmatrix} -0.64 \\ 0 \end{pmatrix}.$$

Therefore, the minimum norm joint acceleration realizing the desired task acceleration $\ddot{\boldsymbol{p}}_d$ in the absence of hard bounds on robot motion is

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#} \left(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}\right) = \begin{pmatrix} 0.3333 & 1 \\ -0.6667 & 0 \\ -1.6667 & -1 \end{pmatrix} \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -0.64 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1.88 \\ -1.76 \\ -5.40 \end{pmatrix} \ [\text{rad/s}^2]. \tag{4}$$

In order to verify if this command is feasible, we have to check both the direct limits on joint acceleration (i.e., whether $|\ddot{q}_i| \leq A_{max,i}$ is satisfied for all joints) and the indirect limits induced by the presence of joint velocity bounds. Since the acceleration command $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}\,(kT_c)$ at $t = kT_c$ is kept constant for an interval T_c, the joint velocity at the next sampling instant will be

$$\dot{\boldsymbol{q}}\,((k+1)T_c) = \dot{\boldsymbol{q}}\,(kT_c) + \ddot{\boldsymbol{q}}\,(kT_c)\, T_c.$$

Thus, the current joint acceleration should also satisfy the bounds (in vector format)

$$-\frac{\boldsymbol{V}_{max} + \dot{\boldsymbol{q}}\,(kT_c)}{T_c} \leq \ddot{\boldsymbol{q}}\,(kT_c) \leq \frac{\boldsymbol{V}_{max} - \dot{\boldsymbol{q}}\,(kT_c)}{T_c}.$$

As a result, we need to check componentwise (at the current instant) if

$$\ddot{Q}_{min,i} = \max\left\{-A_{max,i}, -\frac{V_{max,i} + \dot{q}_i}{T_c}\right\} \leq \ddot{q}_i \leq \min\left\{A_{max,i}, \frac{V_{max,i} - \dot{q}_i}{T_c}\right\} = \ddot{Q}_{max,i}, \qquad i = 1, 2, 3.$$

Plugging in the problem data, we have

$$\ddot{\boldsymbol{Q}}_{min} = \begin{pmatrix} -10 \\ -10 \\ -2 \end{pmatrix}, \qquad \ddot{\boldsymbol{Q}}_{max} = \begin{pmatrix} 7 \\ 10 \\ 10 \end{pmatrix}. \tag{5}$$

While the acceleration (4) is feasible at the first two joints, the third acceleration component $\ddot{q}_3 = -5.4$ exceeds the lower limit $\ddot{Q}_{min,3} = -2$. We apply thus a step of the SNS algorithm, as translated to the acceleration level.

Set first the third joint acceleration to its lower limit, $\ddot{q}_{SNS,3} = -2$. Then, recompute the solution for the other two joints by using the reduced 2×2 Jacobian $\boldsymbol{J}_{-3}$, obtained by removing the third column $\boldsymbol{J}_3$ from the task Jacobian $\boldsymbol{J}$; the desired task acceleration $\ddot{\boldsymbol{p}}_d$ should be modified accordingly to account for the saturated contribution of the third joint. We have

$$\ddot{\boldsymbol{p}}_{SNS,d} = \ddot{\boldsymbol{p}}_d - \boldsymbol{J}_3\, \ddot{q}_{SNS,3} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 0 \end{pmatrix} \cdot (-2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus, as unique possible solution, we obtain

$$\begin{pmatrix} \ddot{q}_{SNS,1} \\ \ddot{q}_{SNS,2} \end{pmatrix} = (\boldsymbol{J}_{-3})^{-1} \left(\ddot{\boldsymbol{p}}_{SNS,d} - \boldsymbol{n} \right) = \begin{pmatrix} -0.5 & -0.5 \\ 1 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1.64 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.28 \\ -8.56 \end{pmatrix}.$$

The solution

$$\ddot{\boldsymbol{q}}_{SNS} = \begin{pmatrix} 5.28 \\ -8.56 \\ -2 \end{pmatrix} \ [\text{rad/s}^2].$$

is now feasible, i.e., it stays within the limits (5) and, by the property of the SNS algorithm, its has also the minimum norm property among all feasible acceleration solutions.

Exercise #2

Since the DH twist angles α_i are all zero, the 3R robot is planar. Choose the axis $\boldsymbol{x}_0$ pointing downward in the vertical plane[1], so that the gravity acceleration vector is $\boldsymbol{g}_0 = \begin{pmatrix} g_0 & 0 & 0 \end{pmatrix}^T$ (with $g_0 = 9.81$ [m/s^2]). The potential energy of each link is given by

$$U_i = -m_i\, \boldsymbol{g}_0^T \boldsymbol{r}_{ci}, \qquad i = 1, 2, 3. \tag{6}$$

In order to use the constant expressions of the CoMs in the local frames, we have

$$\boldsymbol{r}_{c,i}^{hom} = \begin{pmatrix} \boldsymbol{r}_{c,i} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_i(q_1, ..., q_i) \begin{pmatrix} {}^i\boldsymbol{r}_{c,i} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \dots {}^{i-1}\boldsymbol{A}_i(q_i) \begin{pmatrix} r_{cx,i} \\ r_{cy,i} \\ 0 \\ 1 \end{pmatrix}, \qquad i = 1, 2, 3.$$

where the homogeneous transformation matrices are computed from the DH parameters as

$${}^{i-1}\boldsymbol{A}_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 & l_i c_i \\ s_i & c_i & 0 & l_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad i = 1, 2, 3.$$

Performing the computations in (6), one obtains

$$\begin{aligned} U_1 &= m_1 g_0 \left(r_{cy,1}\, s_1 - (l_1 + r_{cx,1})\, c_1 \right) \\ U_2 &= m_2 g_0 \left(r_{cy,2}\, s_{12} - (l_2 + r_{cx,2})\, c_{12} - l_1\, c_1 \right) \\ U_3 &= m_3 g_0 \left(r_{cy,3}\, s_{123} - (l_3 + r_{cx,3})\, c_{123} - l_2\, c_{12} - l_1\, c_1 \right). \end{aligned}$$

[1] A different choice for the direction of $\boldsymbol{x}_0$ (e.g., horizontal or upward) would not affect the conditions that impose $g_2 = g_3 = 0$ in the gravity term of the dynamic model, but only the actual trigonometric function appearing in $g_1(\boldsymbol{q})$, i.e., $\pm \sin q_1$ or $\pm \cos q_1$.

From $U = U_1 + U_2 + U_3$, we get

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} m_1 r_{cy,1}\, c_1 + (m_1(l_1 + r_{cx,1}) + (m_2 + m_3) l_1)\, s_1 \\ + m_2 r_{cy,2}\, c_{12} + (m_2(l_2 + r_{cx,2}) + m_3 l_2)\, s_{12} \\ + m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \\ m_2 r_{cy,2}\, c_{12} + (m_2(l_2 + r_{cx,2}) + m_3 l_2)\, s_{12} \\ + m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \\ m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \end{pmatrix} = \begin{pmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \\ g_3(\boldsymbol{q}) \end{pmatrix}.$$

Proceeding backward from the last component, in order to obtain the desired structure (2) of the gravity term, we have to set first

$$r_{cx,3} = -l_3, \quad r_{cy,3} = 0 \qquad \Rightarrow \qquad g_3 \equiv 0,$$

and then also

$$m_2\, r_{cx,2} = -\,(m_2 + m_3)\, l_2, \quad r_{cy,2} = 0 \qquad \Rightarrow \qquad g_2 \equiv 0,$$

and finally

$$m_1 r_{cx,1} = -\,(m_1 + m_2 + m_3)\, l_1 \qquad \Rightarrow \qquad g_1(q_1) = m_1 g_0\, r_{cy,1}\, c_1.$$

Figure 3 shows a sketch of a possible 3R planar robot satisfying the conditions for having the desired gravity term (2) in its dynamic model. We have chosen here $l_1 = l_2 = l_3 = l$ [m], $m_1 = 4\, m_2 = 16\, m_3 = 10$ [kg], and $r_{cy,1} = 0.2\, l$ [m].

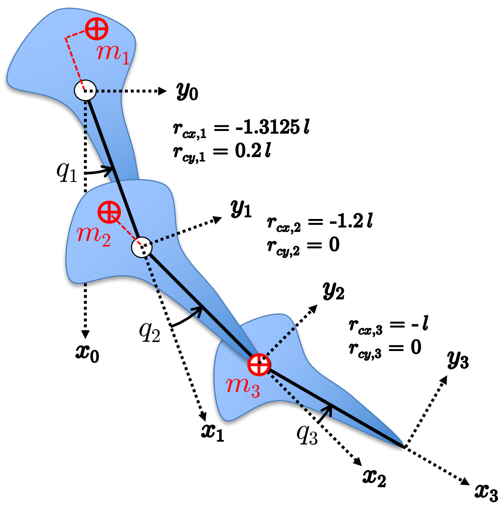


Figure 3: Localization of the CoMs of a 3R planar robot having the dynamic term $\boldsymbol{g}(\boldsymbol{q})$ as in eq. (2).

Exercise #3

Since there is no angular motion, the kinetic energy of the 4P planar robot is simply computed as

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2} \sum_{i=1}^{4} m_i \|\boldsymbol{v}_{ci}\|^2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}},$$

with velocity vectors (conveniently written in $\mathbb{R}^2$)

$$\boldsymbol{v}_{c1} = \begin{pmatrix} \dot{q}_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}, \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 + \dot{q}_3 \\ \dot{q}_2 \end{pmatrix}, \quad \boldsymbol{v}_{c4} = \begin{pmatrix} \dot{q}_1 + \dot{q}_3 \\ \dot{q}_2 + \dot{q}_4 \end{pmatrix}.$$

As a result, the robot inertia matrix is constant and is given by

$$\boldsymbol{M} = \begin{pmatrix} m_1+m_2+m_3+m_4 & 0 & m_3+m_4 & 0 \\ 0 & m_2+m_3+m_4 & 0 & m_4 \\ m_3+m_4 & 0 & m_3+m_4 & 0 \\ 0 & m_4 & 0 & m_4 \end{pmatrix}.$$

The end-effector Jacobian for the linear velocity $\boldsymbol{v}$ in the plane (x, y) is also constant:

$$\boldsymbol{J} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The joint velocity that produces the desired end-effector velocity $\boldsymbol{v}_d$ while minimizing T is obtained by using the inertia-weighted pseudoinverse of $\boldsymbol{J}$:

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_{\boldsymbol{M}}^{\#}\,\boldsymbol{v}_d = \boldsymbol{M}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1}\boldsymbol{v}_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_{xd} \\ v_{yd} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_{xd} \\ v_{yd} \end{pmatrix}. \tag{7}$$

This result is rather intuitive: moving only the last two joints, each by the corresponding component of the end-effector desired velocity, involves the displacement of the minimum amount of mass, and is thus the minimum kinetic energy solution. By this observation, the use of the following intermediate matrix computations is really unnecessary:

$$\boldsymbol{M}^{-1} = \begin{pmatrix} \frac{1}{m_1+m_2} & 0 & -\frac{1}{m_1+m_2} & 0 \\ 0 & \frac{1}{m_2+m_3} & 0 & -\frac{1}{m_2+m_3} \\ -\frac{1}{m_1+m_2} & 0 & \frac{m_1+m_2+m_3+m_4}{(m_1+m_2)(m_3+m_4)} & 0 \\ 0 & -\frac{1}{m_2+m_3} & 0 & \frac{m_2+m_3+m_4}{m_4(m_2+m_3)} \end{pmatrix}, \qquad \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T = \begin{pmatrix} \frac{1}{m_3+m_4} & 0 \\ 0 & \frac{1}{m_4} \end{pmatrix}.$$

In comparison with (7), the minimum velocity norm solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\boldsymbol{v}_d = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} v_{xd} \\ v_{yd} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v_{xd} \\ v_{yd} \\ v_{xd} \\ v_{yd} \end{pmatrix}$$

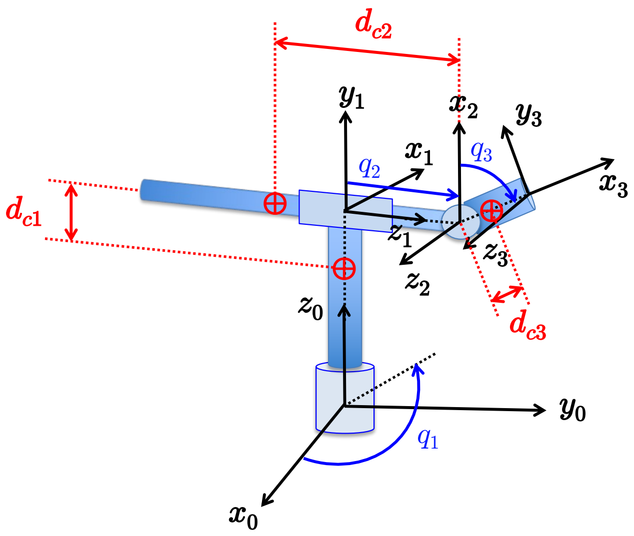
equally distributes the desired Cartesian velocity between the pairs of robot joints that move, respectively, along the x and the y directions.

Exercise #4

We need to compute the kinetic energy of the three links. For $i = 1, 2, 3$, denote by m_i the mass of link i, by l_i its kinematic length (i.e., the parameter d_i or a_i of the DH convention), and by ${}^i\boldsymbol{I}_{ci}$ and ${}^i\boldsymbol{r}_{ci} \in \mathbb{R}^3$, respectively its 3×3 barycentric inertia matrix (for the third short link, this matrix is assumed to be diagonal and uniform) and the constant position vector of its center of mass (CoM), both expressed in the local DH frame. Because of the assumptions on the location of the CoMs of the links, only one component of each ${}^i\boldsymbol{r}_{ci}$ will be different from zero. With reference to Fig. 4, we have

$${}^1\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ -d_{c1} \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{r}_{c2} = \begin{pmatrix} 0 \\ d_{c2} \\ 0 \end{pmatrix}, \qquad {}^3\boldsymbol{r}_{c3} = \begin{pmatrix} -l_3 + d_{c3} \\ 0 \\ 0 \end{pmatrix}.$$

where $d_{ci} > 0$, for $i = 1, 2, 3$. For the first two links, computation of the kinetic energy is rather straightforward. For the third link, it is convenient to use the moving frame algorithm mainly to obtain ${}^3\omega_3$.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	l_1	q_1
2	$-\pi/2$	0	q_2	$\pi/2$
3	0	l_3	0	q_3

Figure 4: Localization of the link CoMs and DH table for the spatial RPR robot.

Link 1

$$T_1 = \frac{1}{2}\, I_{c1,yy}\, \dot{q}_1^2 = \frac{1}{2}\, I_1 \dot{q}_1^2,$$

where we set $I_1 = I_{c1,yy}$ for compactness. Note that the actual position of the CoM of link 1 along the axis $\boldsymbol{z}_0 = \boldsymbol{y}_1$ is irrelevant.

Link 2

$$T_2 = \frac{1}{2}\, m_2 \left((r_{c2,y} - q_2)^2\, \dot{q}_1^2 + \dot{q}_2^2 \right) + \frac{1}{2}\, I_{c2,xx}\, \dot{q}_1^2 = \frac{1}{2} \left(m_2 \left(q_2 - d_{c2} \right)^2 + I_2 \right) \dot{q}_1^2 + \frac{1}{2}\, m_2\, \dot{q}_2^2,$$

where $r_{c2,y} = d_{c2} > 0$ is the distance of the CoM of link 2 from the axis of joint 3 and we set $I_{c2,xx} = I_2$ for compactness.

Link 3

Since $d_{c3} > 0$ is the distance of the CoM of link 3 from the axis of joint 3, we have

$$\boldsymbol{p}_{c3} = \begin{pmatrix} (q_2 - d_{c3}s_3)\, s_1 \\ -\,(q_2 - d_{c3}s_3)\, c_1 \\ l_1 + d_{c3}c_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} (q_2 - d_{c3}s_3)\, c_1 \dot{q}_1 + (\dot{q}_2 - d_{c3}c_3\, \dot{q}_3)\, s_1 \\ (q_2 - d_{c3}s_3)\, s_1 \dot{q}_1 - (\dot{q}_2 - d_{c3}c_3\, \dot{q}_3)\, c_1 \\ -d_{c3}s_3\, \dot{q}_3 \end{pmatrix}.$$

Moreover,

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad {}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T\, {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ 0 \\ 0 \end{pmatrix},$$

and so

$${}^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3) \left({}^2\boldsymbol{\omega}_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_3 \end{pmatrix} \right) = \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ 0 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} c_3\, \dot{q}_1 \\ -s_3\, \dot{q}_1 \\ \dot{q}_3 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} T_3 &= \frac{1}{2}\, m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} + \frac{1}{2}\, {}^3\boldsymbol{\omega}_3^T\, {}^3\boldsymbol{I}_{c3}\, {}^3\boldsymbol{\omega}_3 \\ &= \frac{1}{2}\, m_3 \left((q_2 - d_{c3}s_3)^2\, \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \dot{q}_3^2 - 2d_{c3}c_3\dot{q}_2\dot{q}_3 \right) + \frac{1}{2}\, I_3 \left(\dot{q}_1^2 + \dot{q}_3^2 \right). \end{aligned}$$

As a result, the total kinetic energy is

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the robot inertia matrix given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11}(q_2, q_3) & 0 & 0 \\ 0 & m_2 + m_3 & -m_3 d_{c3}\, c_3 \\ 0 & -m_3 d_{c3}\, c_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}, \tag{8}$$

where

$$m_{11}(q_2, q_3) = I_1 + I_2 + m_2 d_{c2}^2 + I_3 - 2m_2 d_{c2} q_2 + (m_2 + m_3) q_2^2 - 2m_3 d_{c3} q_2 s_3 + m_3 d_{c3}^2 s_3^2.$$

Note finally that by defining the six dynamic coefficients

$$\begin{aligned} a_1 &= I_1 + I_2 + m_2 d_{c2}^2 + I_3 \\ a_2 &= -m_2 d_{c2} \\ a_3 &= m_2 + m_3 \\ a_4 &= -m_3 d_{c3} \\ a_5 &= m_3 d_{c3}^2 \\ a_6 &= I_3 + m_3 d_{c3}^2, \end{aligned}$$

the inertia matrix (8) is exactly the same input matrix (3) of the next exercise.

Exercise #5

This exercise is solved by the following symbolic code of MATLAB.

```
syms q1 q2 q3 dq1 dq2 dq3 ddq1 ddq2 ddq3 a1 a2 a3 a4 a5 a6 real

disp('the given robot inertia matrix')
M=[a1+2*a2*q2+a3*q2^2+2*a4*q2*sin(q3)+a5*(sin(q3))^2  0  0;
   0    a3          a4*cos(q3);
   0    a4*cos(q3)  a6]

disp('Christoffel matrices')
q=[q1;q2;q3];
M1=M(:,1);
C1=(1/2)*(jacobian(M1,q)+jacobian(M1,q)'-diff(M,q1))
M2=M(:,2);
C2=(1/2)*(jacobian(M2,q)+jacobian(M2,q)'-diff(M,q2))
M3=M(:,3);
C3=(1/2)*(jacobian(M3,q)+jacobian(M3,q)'-diff(M,q3))

disp('robot centrifugal and Coriolis terms')
dq=[dq1;dq2;dq3];
c1=dq'*C1*dq;
c2=dq'*C2*dq;
c3=dq'*C3*dq;
c=[c1;c2;c3]

disp('time derivative of the inertia matrix')
dM=diff(M,q1)*dq1+diff(M,q2)*dq2+diff(M,q3)*dq3
```

```
disp('skew-symmetric factorization of velocity terms')
S1=dq'*C1;
S2=dq'*C2;
S3=dq'*C3;
S=[S1;S2;S3]

disp('check skew-symmetry of N=dM-2*S')
N=simplify(dM-2*S)
N_plus_NT=simplify(N+N')

disp('a second, different factorization of velocity terms (yet with skew-symmetry)')
SS=[0 -dq3 dq2;dq3 0 -dq1;-dq2 dq1 0]
Sprime=S+SS
%namely, obtained by adding to S a skew symmetric matrix SS such that SS*dq=0

disp('check validity of Sprime and skew-symmetry of N=dM-2*Sprime')
checkzero=simplify(c-Sprime*dq)
Nprime=simplify(dM-2*Sprime)
Nprime_plus_NprimeT=simplify(Nprime+Nprime')

disp('a third factorization of velocity terms (without skew-symmetry)')
S2prime=S+[0 -dq3 dq2;dq3 0 -dq1;0 0 0]

disp('check validity of S2prime and absence of skew-symmetry of N=dM-2*S2prime')
checkzero=simplify(c-S2prime*dq)
N2prime=simplify(dM-2*S2prime)
N2prime_plus_N2primeT=simplify(N2prime+N2prime')

disp('regressor Y in linear parametrization Y(q,dq,ddq)*a=tau')
ddq=[ddq1;ddq2;ddq3];
tau=M*ddq+c;
a=[a1;a2;a3;a4;a5;a6];
Y=simplify(jacobian(tau,a))
```

The output of this code yields the matrices of Christoffel symbols

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & a_2 + a_3 q_2 + a_4 s_3 & (a_4 q_2 + a_5 s_3)\, c_3 \\ a_2 + a_3 q_2 + a_4 s_3 & 0 & 0 \\ (a_4 q_2 + a_5 s_3)\, c_3 & 0 & 0 \end{pmatrix} \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} -(a_2 + a_3 q_2 + a_4 s_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_4 s_3 \end{pmatrix} \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \begin{pmatrix} -(a_4 q_2 + a_5 s_3)\, c_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{9}$$

from which the Coriolis and centrifugal terms are obtained (with $c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}$, for $i = 1, 2, 3$):

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2\,(a_2 + a_3 q_2 + a_4 s_3)\, \dot{q}_1 \dot{q}_2 + 2\,(a_4 q_2 + a_5 s_3)\, c_3\, \dot{q}_1 \dot{q}_3 \\ -(a_2 + a_3 q_2 + a_4 s_3)\, \dot{q}_1^2 - a_4 s_3\, \dot{q}_3^2 \\ -(a_4 q_2 + a_5 s_3)\, c_3\, \dot{q}_1^2 \end{pmatrix}. \tag{10}$$

The time derivative of the inertia matrix is

$$\dot{\boldsymbol{M}} = \begin{pmatrix} 2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_2 + 2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_3 & 0 & 0 \\ 0 & 0 & -a_4 s_3\,\dot{q}_3 \\ 0 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}.$$

The standard factorization of (10) yielding the skew-symmetric property is given by the matrix having its rows $\boldsymbol{S}_i^T$ built with the Christoffel matrices ($\boldsymbol{S}_i^T(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})$, for $i = 1,2,3$):

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} (a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_2 + (a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_3 & (a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 & (a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 \\ -\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 & 0 & -a_4 s_3\,\dot{q}_3 \\ -\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 & 0 & 0 \end{pmatrix}.$$

A different factorization yielding again the skew-symmetric property is obtained by adding a skew-symmetric matrix $\mathrm{Skew}(\dot{\boldsymbol{q}})$ built with the components of $\dot{\boldsymbol{q}}$,

$$\boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathrm{Skew}(\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & 0 \end{pmatrix},$$

which is certainly another valid factorization of $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$, being $\mathrm{Skew}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \times \dot{\boldsymbol{q}} = \boldsymbol{0}$. Both choices lead in fact to the skew-symmetry, respectively of

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S} = \begin{pmatrix} 0 & -2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 & -2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 \\ 2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 & 0 & a_4 s_3\,\dot{q}_3 \\ 2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}$$

and of

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \begin{pmatrix} 0 & -2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 + 2\dot{q}_3 & -2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 - 2\dot{q}_2 \\ 2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 - 2\dot{q}_3 & 0 & 2\dot{q}_1 + a_4 s_3\,\dot{q}_3 \\ 2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 + 2\dot{q}_2 & -2\dot{q}_1 - a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}.$$

On the other hand, the choice

$$\boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ 0 & 0 & 0 \end{pmatrix}$$

is still a feasible factorization, being $\boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$, but leads to a matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}'' = \begin{pmatrix} 0 & -2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 + 2\dot{q}_3 & -2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 - 2\dot{q}_2 \\ 2\,(a_2 + a_3 q_2 + a_4 s_3)\,\dot{q}_1 - 2\dot{q}_3 & 0 & 2\dot{q}_1 + a_4 s_3\,\dot{q}_3 \\ 2\,(a_4 q_2 + a_5 s_3)\,c_3\,\dot{q}_1 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}$$

which is not skew-symmetric.

Finally, the regressor matrix $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ that linearly parametrizes the robot dynamics (in the absence of gravity), i.e.,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a},$$

is obtained from (3) and (10) as

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 2\,(q_2\,\ddot{q}_1 + \dot{q}_1\dot{q}_2) & q_2^2\,\ddot{q}_1 + 2q_2\,\dot{q}_1\dot{q}_2 & 2\,(q_2 s_3\,\ddot{q}_1 + (s_3\,\dot{q}_2 + q_2 c_3\,\dot{q}_3)\dot{q}_1) & s_3^2\,\ddot{q}_1 + 2s_3 c_3\,\dot{q}_1\dot{q}_3 & 0 \\ 0 & -\dot{q}_1^2 & \ddot{q}_2 - q_2\,\dot{q}_1^2 & c_3\,\ddot{q}_3 - s_3\left(\dot{q}_1^2 + \dot{q}_3^2\right) & 0 & 0 \\ 0 & 0 & 0 & c_3\left(\ddot{q}_2 - q_2\,\dot{q}_1^2\right) & -s_3 c_3\,\dot{q}_1^2 & \ddot{q}_3 \end{pmatrix}.$$

Exercise #6

The 2-dof system under consideration is a PR robot, as sketched in Fig. 5 together with its relevant dynamic parameters. The kinetic energy of this robot is

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2\, \dot{q}_2^2\right) + \frac{1}{2} I_{c2} \dot{q}_2^2 \qquad \Rightarrow \qquad T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

while the potential energy due to gravity and the corresponding gravity term are

$$U_1 = m_1 g_0 q_1, \quad U_2 = m_2 g_0 q_1 \quad \Rightarrow \quad U = U_1 + U_2 \quad \Rightarrow \qquad \boldsymbol{g} = \frac{\partial U}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 + m_2)\, g_0 \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ 0 \end{pmatrix}$$

with $g_0 = 9.81$ [m/s^2]. As a result, the dynamic model of this PR robot is given by two linear and decoupled differential equations:

$$\begin{aligned} M_1 \ddot{q}_1 + g_1 &= u_1 \\ M_2 \ddot{q}_2 &= u_2. \end{aligned} \tag{11}$$

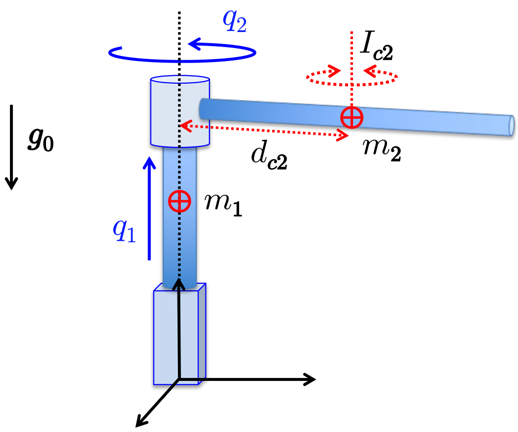


Figure 5: The PR robot with first and second axis coincident and vertical.

The desired rest-to-rest joint trajectory is the cubic polynomial

$$\boldsymbol{q}_d(t) = \boldsymbol{\Delta q} \left(3 \left(\frac{t}{T}\right)^2 - 2 \left(\frac{t}{T}\right)^3 \right), \qquad t \in [0, T],$$

where T is the (coordinated) motion time. The associated acceleration has a linear profile in time

$$\ddot{\boldsymbol{q}}_d(t) = \frac{6 \boldsymbol{\Delta q}}{T^2} \left(1 - 2 \left(\frac{t}{T}\right) \right).$$

From the bounds $|u_i| \leq U_{max,i}$, $i = 1, 2$, and from eqs. (11) it follows that the maximum absolute value of the acceleration, which is reached at $t = 0$ and $t = T$,

$$|\ddot{\boldsymbol{q}}_d(0)| = |\ddot{\boldsymbol{q}}_d(T)| = \frac{6\, |\boldsymbol{\Delta q}|}{T^2},$$

should satisfy componentwise

$$M_1 \frac{6\, |\Delta q_1|}{T^2} \leq U_{max,1} - g_1, \qquad M_2 \frac{6\, |\Delta q_2|}{T^2} \leq U_{max,2}.$$

Therefore, the minimum feasible motion time is given by

$$T^* = \max\left\{\sqrt{\frac{6|\Delta q_1|M_1}{U_{max,1} - g_1}}, \sqrt{\frac{6|\Delta q_2|M_2}{U_{max,2}}}\right\},$$

which is well defined since $U_{max,1} - g_1 > 0$ by assumption.

* * * * *