Robotics 2
Midterm test in classroom − April 26, 2018

Exercise 1

Consider the 4-dof planar robot in Fig. 1. The robot has the first two joints prismatic and the last two joints revolute, and moves in a vertical plane.

- Using the generalized coordinates $q \in \mathbb{R}^4$ and the dynamic parameters defined in Fig. 1, determine the symbolic expression of the dynamic model of this robot using a Lagrangian formulation and considering also the presence of viscous friction at the joints.

- Assume that all kinematic parameters as well as the acceleration of gravity ($g_0 = 9.81$) are known. Provide a linear parameterization of the obtained dynamic model in the form

$$Y(q, \dot{q}, \ddot{q})\, a = u, \tag{1}$$

where $u \in \mathbb{R}^4$ is the generalized force provided by the motors at the joints, and the vector of dynamic coefficients $a \in \mathbb{R}^p$ has the minimum possible dimension p.

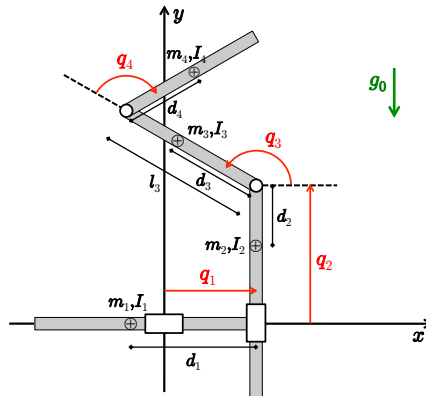Figure 1: A 4-dof planar 2P2R robot, with associated coordinates q and dynamic data.

Exercise 2

A 2-dof robot has the inertia matrix

$$B(q) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}. \tag{2}$$

- Find two matrices $S_1(q, \dot{q})$ and $S_2(q, \dot{q})$ factorizing the Coriolis and centrifugal terms $c(q.\dot{q})$ (i.e., $S_i(q, \dot{q})\dot{q} = c(q, \dot{q})$, for $i = 1, 2$), such that $\dot{B} - 2S_1$ is skew-symmetric, while $\dot{B} - 2S_2$ is not.

- Assume that the robot is on a horizontal plane. The second joint should be moved along a cubic trajectory by an angle Δq_2 from rest to rest in a given time $T > 0$, while the first link is kept still. Provide the symbolic expression of the torque $\tau(0) \in \mathbb{R}^2$ that needs to be applied at initial time $t = 0$ in order to start correctly the execution of this motion.

- Give the value of $\tau(0)$ for the following numerical data

$$a_1 = 17, \quad a_2 = 5, \quad a_3 = 3, \quad q_2(0) = -\frac{\pi}{2}, \quad \Delta q_2 = \frac{\pi}{2}, \quad T = 2 \text{ s},$$

and explain the physical meaning of the signs of its two components.

Exercise 3

Having defined a n-dimensional vector $\boldsymbol{q}$ of generalized coordinates, the dynamic model of a rigid robot in a Lagrangian formulation is obtained writing the Euler-Lagrange equations for $L = T - U$, where T is the kinetic energy and U is the potential energy of the robot. In the frictionless case, the model takes the usual second-order differential expression

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}. \tag{3}$$

On the other hand, one can also rely on the use of the total energy of a robot, $H = T + U$, the so-called Hamiltonian of the system. Using the generalized momentum $\boldsymbol{p} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} \in \mathbb{R}^n$, prove that the (first-order) dynamic equations of the robot in a state-space format can be written in the so-called Hamiltonian form

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \nabla_{\boldsymbol{q}} H \\ \nabla_{\boldsymbol{p}} H \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I}_n \end{pmatrix} \boldsymbol{u}, \tag{4}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and $\nabla_{\boldsymbol{x}} H = (\partial H / \partial \boldsymbol{x})^T$ is a n-dimensional (column) vector.

Exercise 4

Consider the PPR planar robot in Fig. 2 and the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ given therein. The length of the third link is $l > 0$. Assume that the Cartesian task to be performed by this robot is defined in terms of the position $\boldsymbol{p} \in \mathbb{R}^2$ of its end effector. Denote the associated task Jacobian as $\boldsymbol{J}(\boldsymbol{q})$.
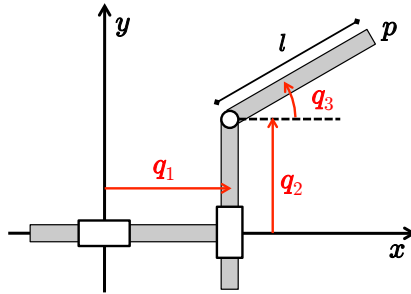

Figure 2: A 3-dof planar PPR robot, with associated coordinates $\boldsymbol{q}$.

- Find the joint velocity $\dot{\boldsymbol{q}}$ that minimizes $\frac{1}{2}\|\dot{\boldsymbol{q}}\|^2$ while realizing a desired task velocity $\dot{\boldsymbol{p}}$. When the linear quantities in the problem are first expressed in [m] and [m/s] units and then in [cm] and [cm/s] units, the solution does not remain the same. Why? Illustrate this by computing the numerical solution with the following data: $l = 0.5$ [m] $= 50$ [cm], $q_3 = \pi/6$ [rad], $\dot{\boldsymbol{p}} = (-1 \ \ 1)^T$ [m/s] $= (-100 \ \ 100)^T$ [cm/s].

- With a weighted pseudoinverse solution, may the above issue be resolved using a suitable diagonal weighting matrix $\boldsymbol{W} > 0$? Illustrate your conclusion by reconsidering the same numerical example of the previous item. Provide also an interpretation of the role of weighting.

- Using again another weighted pseudoinverse, show than it is possible to realize a generic Cartesian task velocity $\dot{\boldsymbol{p}}$ by moving almost only the first two joints of the PPR robot. Illustrate this on the same previous numerical example.

[270 minutes (4.5 hours); open books, but no computer or smartphone]

Exercise 1

Kinetic energy

First two links:

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2 \right).$$

Third link:

$$\boldsymbol{p}_{c3} = \begin{pmatrix} p_{c3,x} \\ p_{c3,y} \end{pmatrix} = \begin{pmatrix} q_1 + d_3 \cos q_3 \\ q_2 + d_3 \sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_1 - d_3 \sin q_3 \, \dot{q}_3 \\ \dot{q}_2 + d_3 \cos q_3 \, \dot{q}_3 \end{pmatrix}, \qquad \omega_3 = \dot{q}_3$$

$$\Rightarrow \quad T_3 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_3^2 \dot{q}_3^2 + 2 d_3 \dot{q}_3 \left(\dot{q}_2 \cos q_3 - \dot{q}_1 \sin q_3 \right) \right) + \frac{1}{2} I_3 \, \dot{q}_3^2.$$

Fourth link:

$$\boldsymbol{p}_{c4} = \begin{pmatrix} p_{c4,x} \\ p_{c4,y} \end{pmatrix} = \begin{pmatrix} q_1 + l_3 \cos q_3 + d_4 \cos(q_3 + q_4) \\ q_2 + l_3 \sin q_3 + d_4 \sin(q_3 + q_4) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c4} = \dot{\boldsymbol{p}}_{c4} = \begin{pmatrix} \dot{q}_1 - l_3 \sin q_3 \, \dot{q}_3 - d_4 \sin(q_3 + q_4)(\dot{q}_3 + \dot{q}_4) \\ \dot{q}_2 + l_3 \cos q_3 \, \dot{q}_3 + d_4 \cos(q_3 + q_4)(\dot{q}_3 + \dot{q}_4) \end{pmatrix}, \qquad \omega_4 = \dot{q}_3 + \dot{q}_4$$

$$\Rightarrow \quad T_4 = \frac{1}{2} m_4 \Big(\dot{q}_1^2 + \dot{q}_2^2 + l_3^2 \dot{q}_3^2 + d_4^2 \left(\dot{q}_3 + \dot{q}_4 \right)^2 + 2 l_3 \dot{q}_3 \left(\dot{q}_2 \cos q_3 - \dot{q}_1 \sin q_3 \right) + 2 l_3 d_4 \cos q_4 \, \dot{q}_3 \left(\dot{q}_3 + \dot{q}_4 \right)$$
$$+ 2 d_4 \left(\dot{q}_3 + \dot{q}_4 \right) \left(\dot{q}_2 \cos(q_3 + q_4) - \dot{q}_1 \sin(q_3 + q_4) \right) \Big) + \tfrac{1}{2} I_4 \left(\dot{q}_3 + \dot{q}_4 \right)^2.$$

Robot inertia matrix

From

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} b_{ij}(\boldsymbol{q}) \, \dot{q}_i \dot{q}_j,$$

we obtain the elements $b_{ij} = b_{ji}$ of the symmetric inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$
\begin{aligned}
b_{11} &= m_1 + m_2 + m_3 + m_4 \\
b_{12} &= 0 \\
b_{13} &= -\left(m_3 d_3 + m_4 l_3 \right) \sin q_3 - m_4 d_4 \sin(q_3 + q_4) \\
b_{14} &= -m_4 d_4 \sin(q_3 + q_4) \\
b_{22} &= m_2 + m_3 + m_4 \\
b_{23} &= \left(m_3 d_3 + m_4 l_3 \right) \cos q_3 + m_4 d_4 \cos(q_3 + q_4) \\
b_{24} &= m_4 d_4 \cos(q_3 + q_4) \\
b_{33} &= I_3 + m_3 d_3^2 + I_4 + m_4 d_4^2 + m_4 l_3^2 + 2 m_4 d_4 l_3 \cos q_4 \\
b_{34} &= I_4 + m_4 d_4^2 + m_4 d_4 l_3 \cos q_4 \\
b_{44} &= I_4 + m_4 d_4^2.
\end{aligned}
$$

Note that the inertia matrix is only a function of q_3 and q_4.

Minimal parametrization of the inertia matrix

We can collect the dynamic parameters appearing in the robot inertia matrix into 6 dynamic coefficients, which are defined as follows:

$$
\begin{aligned}
a_1 &= m_1 + m_2 + m_3 + m_4 \\
a_2 &= m_2 + m_3 + m_4 \\
a_3 &= I_3 + m_3 d_3^2 + I_4 + m_4 d_4^2 + m_4 l_3^2 \\
a_4 &= I_4 + m_4 d_4^2 \\
a_5 &= m_4 d_4 \\
a_6 &= m_3 d_3 + m_4 l_3.
\end{aligned}
\tag{5}
$$

As a result, the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ takes the more compact, linearly parametrized form

$$
\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix}
a_1 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\
0 & a_2 & a_6 c_3 + a_5 c_{34} & a_5 c_{34} \\
-a_6 s_3 - a_5 s_{34} & a_6 c_3 + a_5 c_{34} & a_3 + 2a_5 l_3 c_4 & a_4 + a_5 l_3 c_4 \\
-a_5 s_{34} & a_5 c_{34} & a_4 + a_5 l_3 c_4 & a_4
\end{pmatrix},
\tag{6}
$$

where the shorthand notation for trigonometric functions was used (e.g., $s_{34} = \sin(q_3 + q_4)$). Indeed, l_3 could also be incorporated into the coefficient a_6, but this would lead, as we shall see, to an additional dynamic coefficient in the definition of the gravity terms.

Coriolis and centrifugal terms

Defining by $\boldsymbol{b}_i$ the ith column of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$, we compute the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ using the Christoffel symbols:

$$
c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{B}}{\partial q_i} \right), \qquad i = 1, \ldots, 4.
$$

This is tedious, but straightforward. For the four components we obtain:

$$
\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2} &\left(\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\
0 & 0 & -a_5 c_{34} & -a_5 c_{34}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\
0 & 0 & -a_5 c_{34} & -a_5 c_{34}
\end{pmatrix} - \boldsymbol{O} \right) \\
= &\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\
0 & 0 & -a_5 c_{34} & -a_5 c_{34}
\end{pmatrix} \quad \Rightarrow \quad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5 c_{34} (\dot{q}_3 + \dot{q}_4)^2 - a_6 c_3 \dot{q}_3^2
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2} &\left(\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\
0 & 0 & -a_5 s_{34} & -a_5 s_{34}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\
0 & 0 & -a_5 s_{34} & -a_5 s_{34}
\end{pmatrix} - \boldsymbol{O} \right) \\
= &\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\
0 & 0 & -a_5 s_{34} & -a_5 s_{34}
\end{pmatrix} \quad \Rightarrow \quad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5 s_{34} (\dot{q}_3 + \dot{q}_4)^2 - a_6 s_3 \dot{q}_3^2
\end{aligned}
$$

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$$\boldsymbol{C}_3(\boldsymbol{q}) = \frac{1}{2} \left(\left(\begin{pmatrix} 0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ 0 & 0 & 0 & -2a_5 l_3 s_4 \\ 0 & 0 & 0 & -a_5 l_3 s_4 \end{pmatrix} \right. \right.$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_6 c_3 - a_5 c_{34} & -a_6 s_3 - a_5 s_{34} & 0 & 0 \\ -a_5 c_{34} & -a_5 s_{34} & -2a_5 l_3 s_4 & -a_5 l_3 s_4 \end{pmatrix}$$

$$\left. \left. - \begin{pmatrix} 0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ -a_6 c_3 - a_5 c_{34} & -a_6 s_3 - a_5 s_{34} & 0 & 0 \\ -a_5 c_{34} & -a_5 s_{34} & 0 & 0 \end{pmatrix} \right) \right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5 l_3 s_4 \\ 0 & 0 & -a_5 l_3 s_4 & -a_5 l_3 s_4 \end{pmatrix} \quad \Rightarrow \quad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5 l_3 s_4 \, \dot{q}_4 \, (2\dot{q}_3 + \dot{q}_4)$$

$$\boldsymbol{C}_4(\boldsymbol{q}) = \frac{1}{2} \left(\left(\begin{pmatrix} 0 & 0 & -a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_5 s_{34} & -a_5 s_{34} \\ 0 & 0 & 0 & -a_5 l_3 s_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_5 c_{34} & -a_5 s_{34} & 0 & 0 \\ -a_5 c_{34} & -a_5 s_{34} & -a_5 l_3 s_4 & 0 \end{pmatrix} \right. \right.$$

$$\left. \left. - \begin{pmatrix} 0 & 0 & -a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_5 s_{34} & -a_5 s_{34} \\ -a_5 c_{34} & -a_5 s_{34} & -2a_5 l_3 s_4 & -a_5 l_3 s_4 \\ -a_5 c_{34} & -a_5 s_{34} & -a_5 l_3 s_4 & 0 \end{pmatrix} \right) \right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 l_3 s_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad c_4(\boldsymbol{q}, \dot{\boldsymbol{q}}) = a_5 l_3 s_4 \, \dot{q}_3^2.$$

Summarizing, we have a very short final expression:

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_5 c_{34} \, (\dot{q}_3 + \dot{q}_4)^2 - a_6 c_3 \, \dot{q}_3^2 \\ -a_5 s_{34} \, (\dot{q}_3 + \dot{q}_4)^2 - a_6 s_3 \, \dot{q}_3^2 \\ -a_5 l_3 s_4 \, \dot{q}_4 \, (2\dot{q}_3 + \dot{q}_4) \\ a_5 l_3 s_4 \, \dot{q}_3^2 \end{pmatrix}. \tag{7}$$

Potential energy and gravity terms

$$U_1 = 0, \qquad U_2 = m_2 g_0 (q_2 - d_2), \qquad U_3 = m_3 g_0 \, p_{c3,y} = m_3 g_0 \, (q_2 + d_3 \sin q_3),$$

$$U_4 = m_4 g_0 \, p_{c4,y} = m_4 g_0 \, (q_2 + l_3 \sin q_3 + d_4 \sin(q_3 + q_4)) \qquad \text{[all defined up to a constant]}.$$

From $U = \sum_1^4 U_i$, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} 0 \\ (m_2 + m_3 + m_4)\, g_0 \\ (m_3 d_3 + m_4 l_3)\, g_0 c_3 + m_4 d_4 g_0 c_{34} \\ m_4 d_4 g_0 c_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ a_2 g_0 \\ a_6 g_0 c_3 + a_5 g_0 c_{34} \\ a_5 g_0 c_{34} \end{pmatrix}, \qquad (8)$$

where the three previously defined dynamic coefficents a_2, a_5, and a_6 have been used.

Complete dynamic model with viscous friction

Using the expressions eqs. (6), (7), and (8), and considering the presence of viscous friction acting at the individual joints, we have

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \begin{pmatrix} f_{v1}\dot{q}_1 \\ f_{v2}\dot{q}_2 \\ f_{v3}\dot{q}_3 \\ f_{v4}\dot{q}_4 \end{pmatrix} = \boldsymbol{u}. \qquad (9)$$

By defining the four more dynamic coefficients

$$a_7 = f_{v1}, \qquad a_8 = f_{v2}, \qquad a_9 = f_{v3}, \qquad a_{10} = f_{v4},$$

we conclude that the robot dynamic model depends linearly on a total of $p = 10$ dynamic coefficients, which can be organized in the vector $\boldsymbol{a} \in \mathbb{R}^{10}$. This is also the minimal number of coefficients in the present case.

Linear parametrization

The robot dynamic model (9) can be expressed in the linearly parametrized form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}.\ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u}$$

where the 4×10 matrix $\boldsymbol{Y}$ is given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 0 & 0 & 0 & & & \dot{q}_1 & 0 & 0 & 0 \\ 0 & \ddot{q}_2 + g_0 & 0 & 0 & & & 0 & \dot{q}_2 & 0 & 0 \\ 0 & 0 & \ddot{q}_3 & \ddot{q}_4 & \boldsymbol{Y}_5 & \boldsymbol{Y}_6 & 0 & 0 & \dot{q}_3 & 0 \\ 0 & 0 & 0 & \ddot{q}_3 + \ddot{q}_4 & & & 0 & 0 & 0 & \dot{q}_4 \end{pmatrix}. \qquad (10)$$

The expressions of the fifth and sixth columns of this matrix are

$$\boldsymbol{Y}_5 = \begin{pmatrix} -s_{34}\,(\ddot{q}_3 + \ddot{q}_4) - c_{34}\,(\dot{q}_3 + \dot{q}_4)^2 \\ c_{34}\,(\ddot{q}_3 + \ddot{q}_4) - s_{34}\,(\dot{q}_3 + \dot{q}_4)^2 \\ l_3 c_4\,(2\ddot{q}_3 + \ddot{q}_4) + c_{34}\,(\ddot{q}_2 + g_0) - s_{34}\ddot{q}_1 - l_3 s_4 \dot{q}_4\,(2\dot{q}_3 + \dot{q}_4) \\ l_3 c_4 \ddot{q}_3 + c_{34}\,(\ddot{q}_2 + g_0) - s_{34}\ddot{q}_1 + l_3 s_4 \dot{q}_3^2 \end{pmatrix}$$

and

$$\boldsymbol{Y}_6 = \begin{pmatrix} -s_3 \ddot{q}_3 - c_3 \dot{q}_3^2 \\ c_3 \ddot{q}_3 - s_3 \dot{q}_3^2 \\ c_3\,(\ddot{q}_2 + g_0) - s_3 \ddot{q}_1 \\ 0 \end{pmatrix}.$$

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Exercise 2

From the inertia matrix (2) of the robot, the matrices of Christoffel symbols are computed as

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & -a_2 \sin q_2 \\ -a_2 \sin q_2 & -a_2 \sin q_2 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} a_2 \sin q_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{11}$$

leading to the Coriolis and centrifugal terms

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \, \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \, \dot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} -a_2 \sin q_2 \left(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2 \right) \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{pmatrix}. \tag{12}$$

Moreover, the time derivative of the inertia matrix is

$$\dot{\boldsymbol{B}} = \begin{pmatrix} -2a_2 \sin q_2 \, \dot{q}_2 & -a_2 \sin q_2 \, \dot{q}_2 \\ -a_2 \sin q_2 \, \dot{q}_2 & 0 \end{pmatrix}. \tag{13}$$

A factorization $\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ of (12) that satisfies the skew-symmetry of $\dot{\boldsymbol{B}} - 2\boldsymbol{S}$ is found using directly the matrices (11) of Christoffel symbols. It is easy to check that the matrix

$$\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -a_2 \sin q_2 \, \dot{q}_2 & -a_2 \sin q_2 \left(\dot{q}_1 + \dot{q}_2 \right) \\ a_2 \sin q_2 \, \dot{q}_1 & 0 \end{pmatrix} \tag{14}$$

provides

$$\dot{\boldsymbol{B}} - 2\boldsymbol{S}_1 = \begin{pmatrix} 0 & a_2 \sin q_2 (2\dot{q}_1 + \dot{q}_2) \\ -a_2 \sin q_2 \left(2\dot{q}_1 + \dot{q}_2 \right) & 0 \end{pmatrix},$$

satisfying the desired skew-symmetric property. On the other hand, a feasible factorization that uses the matrix

$$\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2a_2 \sin q_2 \, \dot{q}_2 & -a_2 \sin q_2 \, \dot{q}_2 \\ a_2 \sin q_2 \, \dot{q}_1 & 0 \end{pmatrix} \tag{15}$$

provides

$$\dot{\boldsymbol{B}} - 2\boldsymbol{S}_2 = \begin{pmatrix} 2a_2 \sin q_2 \, \dot{q}_2 & a_2 \sin q_2 \, \dot{q}_2 \\ -a_2 \sin q_2 \left(2\dot{q}_1 + \dot{q}_2 \right) & 0 \end{pmatrix},$$

which is clearly not a skew-symmetric matrix.

For the second part of the exercise, we need to solve a simple inverse dynamics problem. Since the robot moves on a horizontal plane, we have $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$. Moreover, the desired motion starts at $t = 0$ with zero joint velocity, so that $\boldsymbol{c}(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = \boldsymbol{0}$ as well. Finally, the first link should not move, so that $\ddot{q}_1 \equiv 0$. The dynamic equations at the initial instant become then

$$\boldsymbol{B}(\boldsymbol{q}(0)) \begin{pmatrix} 0 \\ \ddot{q}_2(0) \end{pmatrix} = \boldsymbol{\tau}(0).$$

Since the second joint should perform a rest-to-rest motion of Δq_2 in time T with a cubic trajectory, we have

$$q_2(t) = q_2(0) + \Delta q_2 \left(3 \left(\frac{t}{T} \right)^2 - 2 \left(\frac{t}{T} \right)^3 \right) \quad \Rightarrow \quad \ddot{q}_2(0) = \frac{6\Delta q_2}{T^2}.$$

Therefore, the required initial torque at the two joints is

$$\boldsymbol{\tau}(0) = \begin{pmatrix} a_3 + a_2 \cos q_2(0) \\ a_3 \end{pmatrix} \frac{6\Delta q_2}{T^2}. \tag{16}$$

Replacing the given numerical data in (16), we obtain

$$\boldsymbol{\tau}(0) = \begin{pmatrix} 7.0686 \\ 7.0686 \end{pmatrix} \text{ [Nm]}.$$

Since $a_3 > 0$ (always), the torque at the second joint will have the same sign of the desired displacement/acceleration of that joint. In the present case, also the first torque is positive (and equal to the second one, being $\cos q_2(0) = 0$). This positive (counterclockwise) torque is needed to contrast the clockwise motion that the first link would otherwise have, due to the inertial coupling with the instantaneous acceleration of the second link.

Exercise 3

In a Lagrangian formulation, we have

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - U(\boldsymbol{q})$$

and thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\text{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\dot{\boldsymbol{q}}\right\} + \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u},$$
(17)

where

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\text{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\dot{\boldsymbol{q}}\right\} = \frac{1}{2}\text{col}\left\{\dot{\boldsymbol{q}}^T \left[\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}}{\partial q_i}\right]\dot{\boldsymbol{q}}\right\}, \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T,$$

being $\boldsymbol{x} = \text{col}\{x_i\}$ a n-dimensional column vector with x_i as the ith component, for $i = 1, \ldots, n$.

Since $\dot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p}$, it is easy to see that the Hamiltonian can be rewritten as a function of the generalized coordinates $\boldsymbol{q}$ and the generalized momentum $\boldsymbol{p}$ only:

$$H(\boldsymbol{q}, \boldsymbol{p}) = T(\boldsymbol{q}, \boldsymbol{p}) + U(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p} + U(\boldsymbol{q}).$$

We compute then the partial derivatives

$$\nabla_{\boldsymbol{p}} H = \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T = \nabla_{\boldsymbol{p}} T = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p},$$

$$\nabla_{\boldsymbol{q}} H = \left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T = \nabla_{\boldsymbol{q}} T + \nabla_{\boldsymbol{q}} U = \frac{1}{2}\text{col}\left\{\boldsymbol{p}^T \frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\boldsymbol{p}\right\} + \boldsymbol{g}(\boldsymbol{q}).$$
(18)

Differentiating w.r.t. time $\boldsymbol{p} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}$ and using (17), it is

$$\dot{\boldsymbol{p}} = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{u} + \frac{1}{2}\text{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\dot{\boldsymbol{q}}\right\} - \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \frac{1}{2}\text{col}\left\{\boldsymbol{p}^T \boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial q_i}\boldsymbol{B}^{-1}\boldsymbol{p}\right\} - \boldsymbol{g}(\boldsymbol{q}). \quad (19)$$

Moreover, from the identity

$$\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q}) = \boldsymbol{I} \quad \Rightarrow \quad \frac{\partial}{\partial q_i}\left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\right) = \left(\frac{\partial}{\partial q_i}\boldsymbol{B}(\boldsymbol{q})\right)\boldsymbol{B}^{-1}(\boldsymbol{q}) + \boldsymbol{B}(\boldsymbol{q})\left(\frac{\partial}{\partial q_i}\boldsymbol{B}^{-1}(\boldsymbol{q})\right) = \boldsymbol{O},$$

we obtain the general property

$$\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i} = -\boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial q_i}\boldsymbol{B}^{-1}. \tag{20}$$

Replacing (20) in (19) leads to

$$\dot{\boldsymbol{p}} = \boldsymbol{u} - \frac{1}{2}\,\mathrm{col}\left\{\boldsymbol{p}^T\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\,\boldsymbol{p}\right\} - \boldsymbol{g}(\boldsymbol{q}).$$

As a result, using also eq. (18) one can write

$$\dot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p} = \nabla_{\boldsymbol{p}}H$$

$$\dot{\boldsymbol{p}} = -\left(\frac{1}{2}\,\mathrm{col}\left\{\boldsymbol{p}^T\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\,\boldsymbol{p}\right\} + \boldsymbol{g}(\boldsymbol{q})\right) + \boldsymbol{u} = -\nabla_{\boldsymbol{q}}H + \boldsymbol{u},$$

which can be immediately rearranged in the form of the stated equations (4).

Exercise 4

The task Jacobian of the planar PPR robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & -l\sin q_3 \\ 0 & 1 & l\cos q_3 \end{pmatrix}, \tag{21}$$

with

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{p}} \in \mathbb{R}^2, \qquad \dot{\boldsymbol{q}} \in \mathbb{R}^3.$$

Clearly, the Jacobian (21) is always of maximum rank $\rho = 2$. Therefore, the minimum norm joint velocity that realizes a desired $\dot{\boldsymbol{p}}$ is provided by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}, \tag{22}$$

where the pseudoinverse matrix takes the explicit expression

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} = \frac{1}{1+l^2}\begin{pmatrix} 1+l^2\cos^2 q_3 & l^2\sin q_3\cos q_3 \\ l^2\sin q_3\cos q_3 & 1+l^2\sin^2 q_3 \\ -l\sin q_3 & l\cos q_3 \end{pmatrix}, \tag{23}$$

being $\det(\boldsymbol{J}\boldsymbol{J}^T) = 1 + l^2$. When looking at the elements of the matrix in (23), it is easy to realize that there is a problem of unit inconsistency. In particular, the expression of the determinant of $\boldsymbol{J}\boldsymbol{J}^T$ consists of the sum of a non-dimensional quantity and of one having dimension of length to the square. As a matter of fact, the robot has joints of different nature. Therefore, the norm $\|\dot{\boldsymbol{q}}\|$ has mixed expressions and its minimization leads in general to different solutions, depending on the units chosen for the linear and the angular quantities.

For illustration, if we evaluate the solution (22) with the given numerical data using [m] as linear unit,

$$l = 0.5\ [\text{m}], \qquad q_3 = \pi/6\ [\text{rad}], \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\ [\text{m/s}],$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.95 & 0.0866 \\ 0.0866 & 0.85 \\ -0.2 & 0.3464 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.8634 \\ 0.7634 \\ 0.5464 \end{pmatrix}\begin{matrix} [\text{m/s}] \\ [\text{m/s}] \\ [\text{rad/s}] \end{matrix}\ . \tag{24}$$

Instead, when using [cm] as the linear unit,

$$l = 50 \text{ [cm]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -100 \\ 100 \end{pmatrix} \text{ [cm/s]},$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.7501 & 0.4328 \\ 0.4328 & 0.2503 \\ -0.01 & 0.0173 \end{pmatrix} \begin{pmatrix} -100 \\ 100 \end{pmatrix} = \begin{pmatrix} -31.72 \\ -18.25 \\ 2.731 \end{pmatrix} \begin{matrix} \text{[cm/s]} \\ \text{[cm/s]} \\ \text{[rad/s]} \end{matrix} = \begin{pmatrix} -0.3172 \\ -0.1825 \\ 2.731 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix} , \qquad (25)$$

which is completely different (in all components) with respect to (24).

Consider now the use of a weighted pseudoinverse to define a solution as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_{\boldsymbol{W}}^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}, \qquad (26)$$

where, being the task Jacobian always of full rank, we can use the explicit expression

$$\boldsymbol{J}_{\boldsymbol{W}}^{\#}(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^{T}(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^{T}(\boldsymbol{q}) \right)^{-1}. \qquad (27)$$

Since the issue here is the different nature of the first two joints (prismatic) with respect to the last one (revolute), it makes sense to focus our attention to a diagonal weighting matrix of the form

$$\boldsymbol{W} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & w \end{pmatrix}, \qquad \text{with } w > 0. \qquad (28)$$

When computing the inverse matrix in (27), one gets

$$\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^{T}(\boldsymbol{q}) \right)^{-1} = \begin{pmatrix} 1 + \dfrac{l^2}{w}\sin^2 q_3 & \dfrac{l^2}{w}\sin q_3 \cos q_3 \\ \dfrac{l^2}{w}\sin q_3 \cos q_3 & 1 + \dfrac{l^2}{w}\cos^2 q_3 \end{pmatrix} \quad \text{and} \quad \det\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^{T} \right) = 1 + \dfrac{l^2}{w}.$$

Therefore, the choice $w = l^2$ in (28) would eliminate the presence of terms with dimensional units in this key expression (in particular, the determinant would become equal to 2). In fact, with this choice the final expression of such a properly weighted pseudoinverse is

$$\boldsymbol{J}_{\boldsymbol{W}}^{\#}(\boldsymbol{q}) = \frac{1}{2} \begin{pmatrix} 1 + \cos^2 q_3 & \sin q_3 \cos q_3 \\ \sin q_3 \cos q_3 & 1 + \sin^2 q_3 \\ -\dfrac{\sin q_3}{l} & \dfrac{\cos q_3}{l} \end{pmatrix}. \qquad (29)$$

Evaluating the solution (26) with $w = l^2$ in the previous numerical example, and using [m] as linear unit,

$$l = 0.5 \text{ [m]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ [m/s]},$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.875 & 0.2165 \\ 0.2165 & 0.625 \\ -0.5 & 0.866 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6585 \\ 0.4085 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix} . \qquad (30)$$

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When using [cm] as the linear unit,

$$ l = 50 \text{ [cm]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -100 \\ 100 \end{pmatrix} \text{ [cm/s]}, $$

we obtain in this case

$$ \dot{\boldsymbol{q}} = \begin{pmatrix} 0.875 & 0.2165 \\ 0.2165 & 0.625 \\ -0.005 & 0.0087 \end{pmatrix} \begin{pmatrix} -100 \\ 100 \end{pmatrix} = \begin{pmatrix} -65.8484 \\ -40.8494 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[cm/s]} \\ \text{[cm/s]} \\ \text{[rad/s]} \end{matrix} = \begin{pmatrix} -0.6585 \\ 0.4085 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix} \quad , \quad (31) $$

which is exactly the solution (30) already obtained with the other linear units.

The role of the suitable weighting $w = l^2$ on the velocity of the third joint can be given the following interpretation. The general solution (26) minimizes the objective

$$ H = \frac{1}{2}\|\dot{\boldsymbol{q}}\|_{\boldsymbol{W}}^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W}\dot{\boldsymbol{q}} = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + w\dot{q}_3^2\right) = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + (l\dot{q}_3)^2\right), $$

subject to the task constraint $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$. Therefore, this (unique!) weight will equalize the cost for a Cartesian displacement of the robot end-effector of a given length, no matter if this is achieved by one of the two prismatic joints or by the third revolute joint. Stated differently, a linear path achieved by translating along the first or second prismatic joint at 1 [m/s] for 1 second will cost in the objective function H the same as an arc of a circle of radius l achieved by rotating the third joint at $1/l$ [rad/s] for 1 second.

On the other hand, selecting a very large weight $w \gg 1$ in (28) will penalize the motion of the third (revolute) joint with respect to the prismatic ones. The PPR robot will then try to achieve the desired Cartesian velocity moving mainly the first two joints. For instance, using $w = 1000$ in the previous numerical example (and using [m] for the linear units) provides

$$ \dot{\boldsymbol{q}} = \begin{pmatrix} 0.9999 & 0.0001 \\ 0.0001 & 0.9998 \\ -0.0002 & 0.0004 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.9998 \\ 0.9997 \\ 0.0007 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix} \quad , \qquad (32) $$

namely a velocity that involves in practice only the first two prismatic joints.

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