Robotics 2

February 13, 2023

Exercise 1

The torque controlled 3R planar robot in Fig. 1 moves on a horizontal plane, performing a two-dimensional trajectory task with its end-effector. The links have equal length l and equal uniformly distributed mass m, with barycentric inertia $I_c = ml^2/12$. While at rest in the configuration $\bar{\boldsymbol{q}} = (\pi/4, -\pi/2, \pi/2)$ [rad], the end-effector should accelerate with $\ddot{\boldsymbol{p}}_d = (1, 0)$ [m/s^2] (as in figure).

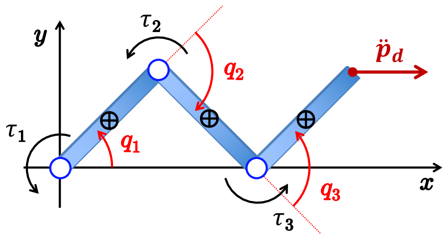

Figure 1: A 3R planar robot with equal links.

Determine, in a parametric way with respect to l and m, the torques $\boldsymbol{\tau}_i \in \mathbb{R}^3$, for $i = A, B, C$, that realize instantaneously the following objectives:

- $\boldsymbol{\tau}_A$ minimizes the squared norm of the joint accelerations $H_A = \frac{1}{2}\|\ddot{\boldsymbol{q}}\|^2$;
- $\boldsymbol{\tau}_B$ minimizes the squared norm of the absolute joint accelerations $H_B = \frac{1}{2}\|\ddot{\boldsymbol{q}}_a\|^2$, where

$$\ddot{q}_{a,i} = \sum_{j=1}^{i} \ddot{q}_j, \qquad i = 1, 2, 3;$$

- $\boldsymbol{\tau}_C$ minimizes the squared norm of the inertia-weighted joint accelerations $H_C = \frac{1}{2}\ddot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$.

Comment on the obtained results in terms of the control efforts at the joint level.

Exercise 2

Consider the single link under gravity in Fig. 2, with all dynamic parameters specified therein. The link should perform a rest-to-rest motion from $\theta(0) = 0$ to $\theta(T) = \pi$ (a swing-up maneuver, counterclockwise), by following a cubic polynomial interpolating trajectory $\theta_d(t)$ under the torque bound $|u| \leq u_{max}$. Suppose that the maximum available torque is large enough to sustain gravity in any configuration, typically with some extra torque left for dynamic motion.

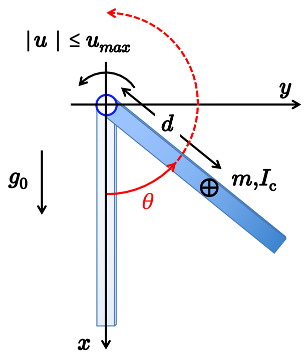

Figure 2: Swing-up maneuver of a single link under maximum torque bound.

Determine the expression of the minimum time $T = T_{min}$ for performing the task, discussing any assumption that you may introduce. Furthermore, suppose that, with a time $T = 1$ s, the motion is unfeasible for a given set of data. What will be the minimum uniform time scaling factor k of the original trajectory that allows to execute the task in a feasible way?

Exercise 3

Figure 3 shows a simplified one-dimensional model of two robots permanently interacting in a compliant mode at the level of their end effectors[1]. Compliance at the contact is modeled by a spring with stiffness $K > 0$. The two robots have equivalent masses m_1 and m_2 and are subject to control forces F_1 and F_2. Their positions are given by q_1 and q_2, with the zero reference for both variables corresponding to when the spring has no deformation (as shown in the figure).

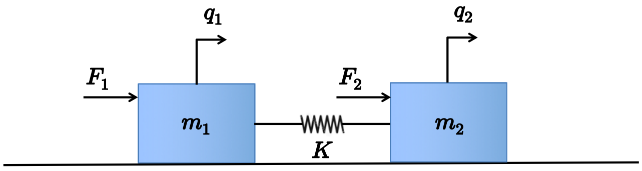

Figure 3: Two masses interacting through a spring.

Define the two control laws

$$F_1 = K_{P1} (q_{1d} - q_1) - K_{D1}\dot{q}_1, \qquad F_2 = K_{P2} (q_{2d} - q_2) - K_{D2}\dot{q}_2, \tag{1}$$

with all gains strictly positive, and where the target positions q_{1d} and q_{2d} for the two masses are generic but different (i.e., $q_{1d} \neq q_{2d}$). These control laws have a decentralized structure, since they both use feedback information only local to the controlled mass, i.e., F_i is function only of $(q_i, \dot{q}_i)$, for $i = 1, 2$.

- Find the unique equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\bar{\boldsymbol{q}}, \boldsymbol{0})$ for the closed-loop system under the control (1).
- Prove the global asymptotic stability of this equilibrium state by a Lyapunov/LaSalle argument.
- Is the equilibrium configuration $\bar{\boldsymbol{q}}$ such that $\bar{q}_1 = q_{1d}$ and $\bar{q}_2 = q_{2d}$? If not, how would you modify the controllers (1), possibly keeping the decentralized structure, for the same previous target positions so that $\bar{\boldsymbol{q}} = \boldsymbol{q}_d$ becomes the unique asymptotically stable equilibrium configuration?

[180 minutes; open books]

[1] This ideal situation is not unrealistic. In fact, it can be obtained by applying a preliminary feedback linearizing and decoupling control law in the Cartesian space to two articulated robot manipulators.

Solution

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Exercise 1

For the considered instantaneous situation, we need to compute only the 2×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ and the 3×3 inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the robot. In fact, since the robot moves on a horizontal plane ($\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$) and is currently at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), its dynamic model simplifies to

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{\tau}, \tag{2}$$

while the second-order differential kinematics for the positional task becomes

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}. \tag{3}$$

Moreover, the absolute joint coordinates $\boldsymbol{q}_a$ are related to $\boldsymbol{q}$ by a constant matrix:

$$\boldsymbol{q}_a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{q} = \boldsymbol{T}\boldsymbol{q} \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}}_a = \boldsymbol{T}\ddot{\boldsymbol{q}}. \tag{4}$$

For a desired end-effector acceleration $\ddot{\boldsymbol{p}}_d$ at the current state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\bar{\boldsymbol{q}}, \boldsymbol{0})$, being $\bar{\boldsymbol{q}}$ a nonsingular configuration for the Jacobian, the three schemes that are locally using the robot redundancy are obtained as particular solutions of the general LQ (Linear Quadratic) optimization problem by the following torques:

- minimization of the squared norm of the joint accelerations

  $$H_A = \frac{1}{2}\,\|\ddot{\boldsymbol{q}}\|^2 = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\ddot{\boldsymbol{q}}$$

  gives

  $$\boldsymbol{\tau}_A = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^{\dagger}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d, \qquad \text{with } \boldsymbol{J}^{\dagger}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}; \tag{5}$$

- minimization of the squared norm of the absolute joint accelerations

  $$H_B = \frac{1}{2}\,\|\ddot{\boldsymbol{q}}_a\|^2 = \frac{1}{2}\,\|\boldsymbol{T}\ddot{\boldsymbol{q}}\|^2 = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{W}\ddot{\boldsymbol{q}}$$

  gives

  $$\boldsymbol{\tau}_B = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^{\dagger}_{\boldsymbol{W}}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d, \qquad \text{with } \boldsymbol{J}^{\dagger}_{\boldsymbol{W}}(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{6}$$

  where, from (4), $\boldsymbol{W}$ is the symmetric, positive definite weighting matrix

  $$\boldsymbol{W} = \boldsymbol{T}^T\boldsymbol{T} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} > 0\,;$$

- minimization of the squared norm of the inertia-weighted joint accelerations

  $$H_C = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$$

gives

$$\boldsymbol{\tau}_C = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^{\dagger}_{\boldsymbol{M}}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d = \boldsymbol{J}^T(\bar{\boldsymbol{q}})\left(\boldsymbol{J}(\bar{\boldsymbol{q}})\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\boldsymbol{J}^T(\bar{\boldsymbol{q}})\right)^{-1}\ddot{\boldsymbol{p}}_d. \tag{7}$$

Note that, in view of (2), this case is equivalent to the minimization of the squared norm of the inverse inertia-weighted torques:

$$H_C = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \frac{1}{2}\,\left(\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}\right)^T\boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}\right) = \frac{1}{2}\,\boldsymbol{\tau}^T\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}.$$

We proceed then by computing the required matrices. Note that, because of the uniform nature of the links, the symbolic factors l and $m\,l^2$ can be isolated in the computations of kinematic and, respectively, dynamic terms.

Jacobian

$$\boldsymbol{p} = l\begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial\boldsymbol{p}}{\partial\boldsymbol{q}} = l\begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}.$$

Therefore, at $\bar{\boldsymbol{q}} = (\,\pi/4, -\pi/2, \pi/2\,)$ we have

$$\boldsymbol{J}(\bar{\boldsymbol{q}}) = l\begin{pmatrix} -0.7071 & 0 & -0.7071 \\ 2.1213 & 1.4142 & 0.7071 \end{pmatrix}.$$

Jacobian pseudoinverse

$$\boldsymbol{J}^{\dagger}(\bar{\boldsymbol{q}}) = \frac{1}{l}\begin{pmatrix} -0.2357 & 0.2357 \\ 0.9428 & 0.4714 \\ -1.1785 & -0.2357 \end{pmatrix}.$$

Jacobian weighted pseudoinverse

Being

$$\boldsymbol{W}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

we obtain

$$\boldsymbol{J}^{\dagger}_{\boldsymbol{W}}(\bar{\boldsymbol{q}}) = \frac{1}{l}\begin{pmatrix} -0.3536 & 0.3536 \\ 1.0607 & 0.3536 \\ -1.0607 & -0.3536 \end{pmatrix}.$$

Kinetic energy

By the uniform mass distribution of the link, which is also considered as a thin rod, we have

$$T_1 = \frac{1}{2}\,m\left(\frac{l}{2}\right)^2\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12}\,m\,l^2\right)\dot{q}_1^2 = \frac{1}{6}\,m\,l^2\dot{q}_1^2.$$

For the second link, being

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\left(l\begin{pmatrix} c_1 \\ s_1 \end{pmatrix} + \frac{l}{2}\begin{pmatrix} c_{12} \\ s_{12} \end{pmatrix}\right) = l\begin{pmatrix} -s_1\dot{q}_1 - \frac{1}{2}\,s_{12}\,(\dot{q}_1 + \dot{q}_2) \\ c_1\dot{q}_1 + \frac{1}{2}\,c_{12}\,(\dot{q}_1 + \dot{q}_2) \end{pmatrix},$$

it is

$$\|\boldsymbol{v}_{c2}\|^2 = l^2 \left(\dot{q}_1^2 + \frac{1}{4} (\dot{q}_1 + \dot{q}_2)^2 + c_2 \, \dot{q}_1(\dot{q}_1 + \dot{q}_2) \right),$$

and thus

$$\begin{aligned} T_2 &= \frac{1}{2} m \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} \left(\frac{1}{12} m \, l^2 \right) (\dot{q}_1 + \dot{q}_2)^2 \\ &= \frac{1}{6} m \, l^2 \left(4 \, \dot{q}_1^2 + \dot{q}_2^2 + 2 \, \dot{q}_1 \dot{q}_2 + 3 \, c_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \right). \end{aligned}$$

Similarly, for the third link it is

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \frac{d}{dt} \left(l \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix} + \frac{l}{2} \begin{pmatrix} c_{123} \\ s_{123} \end{pmatrix} \right) = l \begin{pmatrix} -s_1 \dot{q}_1 - s_{12} \, (\dot{q}_1 + \dot{q}_2) - \frac{1}{2} \, s_{123} \, (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ c_1 \dot{q}_1 + c_{12} \, (\dot{q}_1 + \dot{q}_2) + \frac{1}{2} \, c_{123} \, (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

so that

$$\begin{aligned} \|\boldsymbol{v}_{c3}\|^2 = l^2 \Big(& \dot{q}_1^2 + (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2} (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ & + 2 \, c_2 \, \dot{q}_1 (\dot{q}_1 + \dot{q}_2) + c_{23} \, \dot{q}_1 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) + c_3 \, (\dot{q}_1 + \dot{q}_2)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \Big), \end{aligned}$$

and thus

$$\begin{aligned} T_3 &= \frac{1}{2} m \|\boldsymbol{v}_{c3}\|^2 + \frac{1}{2} \left(\frac{1}{12} m \, l^2 \right) (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ &= \frac{1}{6} m \, l^2 \big(7 \, \dot{q}_1^2 + 4 \, \dot{q}_2^2 + \dot{q}_3^2 + 8 \, \dot{q}_1 \dot{q}_2 + 2 \, \dot{q}_1 \dot{q}_2 + 2 \, \dot{q}_2 \dot{q}_3 \\ &\qquad + 6 \, c_2 \, \dot{q}_1 (\dot{q}_1 + \dot{q}_2) + 3 \, c_3 \left((\dot{q}_1 + \dot{q}_2)^2 + \dot{q}_3 (\dot{q}_1 + \dot{q}_2) \right) + 3 \, c_{23} \, \dot{q}_1 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \big). \end{aligned}$$

Finally,

$$T = T_1 + T_2 + T_3 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}.$$

Inertia matrix

Rather than rewriting the lengthy terms contained in the contributions T_i of the kinetic energy, we directly evaluate numerically inertia matrix at $\bar{\boldsymbol{q}}$. Factoring out the common symbolic factor $m \, l^2$, one has

$$\boldsymbol{M}(\bar{\boldsymbol{q}}) = m \, l^2 \begin{pmatrix} 5.0000 & 2.1667 & 0.8333 \\ 2.1667 & 1.1667 & 0.3333 \\ 0.8333 & 0.3333 & 0.3333 \end{pmatrix},$$

with inverse

$$\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}}) = \frac{1}{m \, l^2} \begin{pmatrix} 0.6316 & -0.6316 & -0.9474 \\ -0.6316 & 1.3816 & 0.1974 \\ -0.9474 & 0.1974 & 5.1711 \end{pmatrix}.$$

At this stage, we can evaluate the three solutions (5)–(7) obtaining

$$\boldsymbol{\tau}_A = ml \begin{pmatrix} -0.1179 \\ 0.6678 \\ -0.2750 \end{pmatrix}, \qquad \boldsymbol{\tau}_B = ml \begin{pmatrix} -0.3536 \\ 0.6482 \\ -0.2946 \end{pmatrix}, \qquad \boldsymbol{\tau}_C = ml \begin{pmatrix} 0.4950 \\ 0.7189 \\ -0.2239 \end{pmatrix}. \tag{8}$$

The associated accelerations, computed as $\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\boldsymbol{\tau}$, are

$$\ddot{\boldsymbol{q}}_A = \frac{1}{l}\begin{pmatrix} -0.2357 \\ 0.9428 \\ -1.1785 \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_B = \frac{1}{l}\begin{pmatrix} -0.3536 \\ 1.0607 \\ -1.0607 \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_C = \frac{1}{l}\begin{pmatrix} 0.0707 \\ 0.6364 \\ -1.4849 \end{pmatrix}. \tag{9}$$

There are no major differences between the three results in (8) and (9), except for the fact that the torque and the acceleration of the first joint in the inertia-weighted case C have an opposite sign with respect to the other two cases. Moreover, solution C has lower acceleration at joint 2 and (even less) at joint 1, due to the fact that the inertia of the robot has been taken into account. This result is consistent with the intuitive idea that in a serial manipulator it is more convenient, in terms of torque/acceleration efforts, to move distal joints in the chain rather than proximal ones.

For comparison, consider also a fourth case in which the solution $\boldsymbol{\tau}$ minimizes the squared norm of the torques:

$$\min H_D = \frac{1}{2}\|\boldsymbol{\tau}\|^2 \quad \Rightarrow \quad \boldsymbol{\tau}_D = \left(\boldsymbol{J}(\bar{\boldsymbol{q}})\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\right)^{\dagger}\ddot{\boldsymbol{p}}_d = ml\begin{pmatrix} -0.0339 \\ 0.6748 \\ -0.2680 \end{pmatrix} \quad \Rightarrow \quad \ddot{\boldsymbol{q}}_D = \frac{1}{l}\begin{pmatrix} -0.1937 \\ 0.9008 \\ -1.2205 \end{pmatrix}.$$

This solution has, by construction, the minimum norm of the torque and, compared to the other control torques, also by far the lowest value of torque at the first joint.

Exercise 2

The dynamic model of the actuated pendulum in Fig. 2 is

$$\left(I_c + md^2\right)\ddot{\theta} + mg_0 d\sin\theta = u. \tag{10}$$

In the following, let $I = I_c + md^2$. The assigned cubic trajectory for performing the swing-up maneuver in time T can be written (in normalized time) as

$$\theta_d(t) = \pi\left(-2\tau^3 + 3\tau^2\right), \qquad \tau = \frac{t}{T} \in [0,1],$$

with acceleration

$$\ddot{\theta}_d(t) = \frac{6\pi}{T^2}\left(1 - 2\tau\right).$$

By inverse dynamics, the torque needed to execute this trajectory is then

$$u_d(t) = \frac{6\pi I}{T^2}\left(1 - 2\tau\right) + mg_0 d\sin\left(\pi\left(-2\tau^3 + 3\tau^2\right)\right), \qquad \tau \in [0,1]. \tag{11}$$

The torque (11) is the sum of two terms: a linear contribution $u_a(t)$ due to acceleration, which is maximum in absolute value at the start and end of the trajectory, with $u_a(0) = -u_a(T) = 6\pi I/T^2$, and zero at the midpoint $t = T/2$; and a sinusoidal contribution $u_g(t)$ due to gravity, which is zero at the start and end of the trajectory, always positive otherwise, and maximum at the midpoint, with $u_g(T/2) = mg_0 d$. It is easy to show that the superposition of the two torques will have a maximum value which occurs certainly in the first half of the motion (where both terms are positive), but not necessarily at $t = 0$ or $t = T/2$ (i.e., $\tau = 0.5$). Moreover, it is also clear that the faster will be the assigned trajectory (i.e., the smaller the total motion time T), the more will the acceleration term grow and dominate the gravitatonal term, which does not change in fact its profile being dependent only on the configuration θ.

In Fig. 4, using the numerical data

$$I = 1.5 \text{ [Nm s}^2\text{]}, \qquad mg_0 d = 14.715 \text{ [Nm]}, \tag{12}$$

we report for illustration two typical situations, the first for a slow trajectory having $T_s = 1.5$ s, the second for a fast trajectory with $T_f = 0.8$ s. The exchanged roles of the two contributions in assessing the maximum absolute value of the torque is clear.

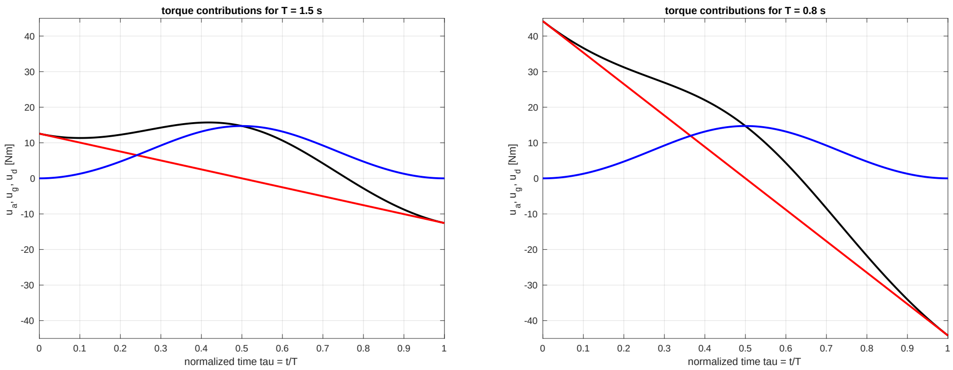

Figure 4: The two contributions $u_a(t)$ (in red) and $u_g(t)$ (in blue) to the total driving torque $u_d(t)$ (in black): slow trajectory with $T_s = 1.5$ s [left]; fast trajectory with $T_f = 0.8$ s [right].

In practice, the available torque u_{max} will not only be larger that $mg_0 d$ (the maximum gravity load on the link), but also capable of providing a sufficient acceleration at the time instants $t = 0$ and $t = T$ (where $u_d(t) = u_a(t)$), so as to quickly start and stop motion. Therefore, given a sufficiently large maximum torque u_{max}, the minimum time T_{min} will be specified by the value of the acceleration component at $t = 0$. Thus,

$$u_d(0) = u_a(0) = \frac{6\pi I}{T^2} = u_{max} \qquad \Rightarrow \qquad T_{min} = \sqrt{\frac{6\pi I}{u_{max}}}. \tag{13}$$

Using the same data as in (12) and setting $u_{max} = 20$ [Nm] leads to the optimal solution of Fig. 5, with minimum time $T_{min} = 1.1890$ s, as evaluated from (13).

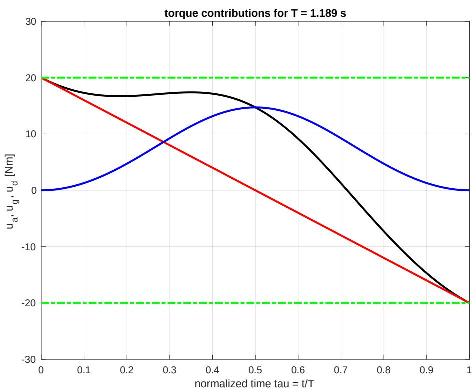

Figure 5: The two contributions $u_a(t)$ (in red) and $u_g(t)$ (in blue) in the time-optimal driving torque $u_d(t)$ (in black).

Finally, consider again the same pendulum with $u_{max} = 18$ [Nm] and set the motion time to

$T = 1$ s. The corresponding torque profile will be unfeasible since

$$u_d(0) = \frac{6I\pi}{T^2} = 28.27 > 18 = u_{max}.$$

Therefore, the minimum uniform time scaling factor to recover feasibility is computed as

$$k = \sqrt{\frac{6\pi I}{u_{max}}} = 1.2533 > 1. \tag{14}$$

The scaled trajectory is slower, with a longer duration $T_s = kT = 1.2533$ s. Feasibility is automatically recovered, with the bound being saturated only at instants with the largest unfeasible torque (in absolute value). Note that, when computing the scaling factor, gravity needs not to be removed because the maximum violating torque already occurs at an instant with zero gravity contribution. The effect of uniform time scaling on the unfeasible trajectory is illustrated in Fig. 6.

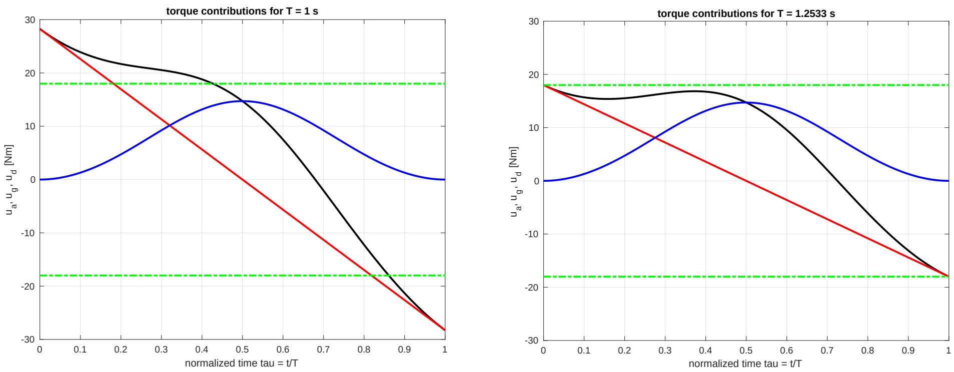

Figure 6: The contributing and total torques for an unfeasible trajectory with $T = 1$ s [left] and after uniform time scaling by $k = 1.2533$ [right].

Exercise 3

The dynamic model of the system of two masses with a spring in between is

$$\begin{aligned} m_1\ddot{q}_1 + K(q_1 - q_2) &= F_1 \\ m_2\ddot{q}_2 + K(q_2 - q_1) &= F_2. \end{aligned} \tag{15}$$

With the control (1), the closed-loop system becomes

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{D1} & \\ & K_{D2} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix}. \tag{16}$$

At an equilibrium ($\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$), it is then

$$\bar{\boldsymbol{K}}\boldsymbol{q} = \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix} = \boldsymbol{K}_P\boldsymbol{q}_d = \bar{\boldsymbol{q}}_d, \tag{17}$$

where $\boldsymbol{K}_P = \text{diag}\{K_{P1}, K_{P2}\}$ Since the stiffness/control matrix $\bar{\boldsymbol{K}}$ is nonsingular ($\det \bar{\boldsymbol{K}} = K(K_{P1} + K_{P2}) + K_{P1}K_{P2} > 0$), equation (17) can be solved for the unique equilibrium configuration

$$\bar{\boldsymbol{q}} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{q}}_d = \frac{1}{K(K_{P1} + K_{P2}) + K_{P1}K_{P2}}\begin{pmatrix} K(K_{P1}q_{1d} + K_{P2}q_{2d}) + K_{P1}K_{P2}q_{1d} \\ K(K_{P1}q_{1d} + K_{P2}q_{2d}) + K_{P1}K_{P2}q_{2d} \end{pmatrix}. \tag{18}$$

The equilibrium position $\bar{q}_i$ of each mass is in general different from its target value q_{id}, for $i = 1, 2$. From eq. (18), it follows also that only when $q_{1d} = q_{2d} = q_d$ (a case excluded here), it is then $\bar{q}_1 = \bar{q}_2 = q_d$.

To show that the unique equilibrium state $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ is globally asymptotically stable (in fact, exponentially stable since the system is linear) one can follow in principle two ways.

The first is to leverage the linearity of the closed-loop system dynamics (16). It can be recognized that the three matrices $\boldsymbol{M} = \text{diag}\,\{m_1, m_2\}$, $\boldsymbol{D} = \text{diag}\,\{K_{D1}, K_{D2}\}$ and $\bar{\boldsymbol{K}}$ are all positive definite: this is a sufficient condition for concluding on asymptotic stability of mechanical systems in this form. Along the same lines, one can apply tools from linear systems theory to draw the same conclusion: e.g., by computing the four eigenvalues of system (16), once put into a state-space format, and verifying that their real parts are on the left-hand side of the complex plane.

The second way is to follow, as requested, a Lyapunov/LaSalle analysis. Consider the following function as natural Lyapunov candidate:

$$V = \frac{1}{2}\, m_1 \dot{q}_1^2 + \frac{1}{2}\, m_2 \dot{q}_2^2 + \frac{1}{2}\, K\,(q_1 - q_2)^2 + \frac{1}{2}\, K_{P1}\,(q_{1d} - q_1)^2 + \frac{1}{2}\, K_{P2}\,(q_{2d} - q_2)^2 \geq 0. \qquad (19)$$

This function is composed by the total energy of the system (kinetic and elastic potential in the first three terms) and by the equivalent elastic potential energy introduced by the control laws (1). Indeed, V is always non-negative and is zero only at the equilibrium $\boldsymbol{q} = \bar{\boldsymbol{q}}$, with $\dot{\boldsymbol{q}} = \boldsymbol{0}$. In fact, setting to zero the gradient of (19) with respect to $\boldsymbol{q}$, as a necessary condition for a minimum, one has

$$\nabla_{\boldsymbol{q}} V = \left(\frac{\partial V}{\partial \boldsymbol{q}}\right)^T = \bar{\boldsymbol{K}}\boldsymbol{q} - \bar{\boldsymbol{q}}_d = \boldsymbol{0},$$

which is exactly (17) and thus uniquely solved by $\bar{\boldsymbol{q}}$. Moreover, since $\partial^2 V / \partial \boldsymbol{q}^2 = \bar{\boldsymbol{K}} > 0$, this will be a minimum of V. By taking the time derivative of V and evaluating it along the trajectories of the closed-loop system (16), we obtain after few simplifications:

$$\dot{V} = \ldots = -\bar{\boldsymbol{D}}\dot{\boldsymbol{q}} \leq 0 \qquad \Rightarrow \qquad \dot{V} = 0 \iff \dot{\boldsymbol{q}} = \boldsymbol{0}.$$

When $\dot{\boldsymbol{q}} = \boldsymbol{0}$, the closed-loop system (16) simplifies to

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{q}} + \bar{\boldsymbol{K}}\boldsymbol{q} = \bar{\boldsymbol{q}}_d \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \bar{\boldsymbol{M}}^{-1}\left(\bar{\boldsymbol{q}}_d - \bar{\boldsymbol{K}}\boldsymbol{q}\right) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{0} \iff \boldsymbol{q} = \bar{\boldsymbol{q}},$$

thanks to (17). Therefore, by LaSalle theorem, the system trajectories will globally converge to the unique equilibrium state $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ (i.e., the single element in the largest invariant set contained in the set of states corresponding to $\dot{V} = 0$), which is then asymptotically stable. Again, being the considered system linear, asymptotic stability is equivalent here to exponential stability.

Finally, a modification of the control laws (1) is needed in order to enforce $\bar{\boldsymbol{q}} = \boldsymbol{q}_d$, i.e., to eliminate the constant final position errors at steady state. A straightforward solution would be to cancel the effect of elasticity on both masses, namely defining the new control laws as

$$F_1 = K_{P1}\,(q_{1d} - q_1) - K_{D1}\dot{q}_1 + K\,(q_1 - q_2)\,, \qquad F_2 = K_{P2}\,(q_{2d} - q_2) - K_{D2}\dot{q}_2 + K\,(q_2 - q_1)\,. \quad (20)$$

This would fully decouple the behavior of the two controlled masses, being the closed-loop system

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{D1} & \\ & K_{D2} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{P1} & \\ & K_{P2} \end{pmatrix} \begin{pmatrix} q_1 - q_{1d} \\ q_2 - q_{2d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (21)$$

It is easy to verify that mass m_i would independently reach its target position q_{id} at the equilibrium, for $i = 1, 2$.

However, the control modification (20) destroys the decentralized structure of (1), being the input force to each mass also a function of the position of the other mass. If the original decentralized structure has to be kept, one can include integral terms in the two controllers, i.e.,

$$F_i = K_{Pi}\left(q_{id} - \dot{q}_i\right) - K_{Di}\dot{q}_i + K_{Ii}\int\left(q_{id} - \dot{q}_i\right)dt, \qquad i = 1, 2,$$

and then study the conditions for the control gains K_{Pi}, K_{Di} and K_{Ii} ($i = 1, 2$) that guarantee asymptotic stability of the closed-loop system. This choice has the advantage of requiring no information at all about the parameters of the dynamic system (except for some bounds). A simpler solution is to include feedforward terms in the control laws (1), i.e.,

$$F_i = K_{Pi}\left(q_{id} - \dot{q}_i\right) - K_{Di}\dot{q}_i + F_{i,ffw}, \qquad i = 1, 2, \tag{22}$$

with

$$\begin{aligned} \boldsymbol{F}_{ffw} = \begin{pmatrix} F_{1,ffw} \\ F_{2,ffw} \end{pmatrix} &= \bar{\boldsymbol{K}}\boldsymbol{q}_d - \bar{\boldsymbol{q}}_d \\ &= \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix}\begin{pmatrix} q_{1d} \\ q_{2d} \end{pmatrix} - \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix} \\ &= \begin{pmatrix} K & -K \\ -K & K \end{pmatrix}\begin{pmatrix} q_{1d} \\ q_{2d} \end{pmatrix} = \begin{pmatrix} K\left(q_{1d} - q_{2d}\right) \\ K\left(q_{2d} - q_{1d}\right) \end{pmatrix}. \end{aligned} \tag{23}$$

The new equilibrium conditions are obtained by modifying accordingly (17) as

$$\bar{\boldsymbol{K}}\boldsymbol{q} = \bar{\boldsymbol{q}}_d + \boldsymbol{F}_{ffw} = \bar{\boldsymbol{q}}_d + \bar{\boldsymbol{K}}\boldsymbol{q}_d - \bar{\boldsymbol{q}}_d = \bar{\boldsymbol{K}}\boldsymbol{q}_d, \qquad \text{with } \bar{\boldsymbol{K}} > 0,$$

which has the unique solution $\boldsymbol{q} = \boldsymbol{q}_d$ as desired. The Lyapunov/LaSalle analysis for the controller (22),(23) follows then in a similar way by using

$$V' = V + \left(\boldsymbol{a}_d - \boldsymbol{q}\right)^T \boldsymbol{F}_{ffw}, \qquad \text{with } \boldsymbol{a}_d = \begin{pmatrix} \dfrac{3q_{1d} + q_{2d}}{4} \\ \dfrac{q_{1d} + 3q_{2d}}{4} \end{pmatrix},$$

which can be shown to be a suitable Lyapunov candidate, i.e., $V' \geq 0$ and $V' = 0$ if and only if $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, obtaining eventually global asymptotic (exponential) stability of the unique closed-loop equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$. The actual verification is left as an exercise to the reader.

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