Robotics 2

September 10, 2021

Exercise #1

Consider the planar 3R robot in Fig. 1. The three links have all equal length L. The robot is controlled by a joint acceleration command $\boldsymbol{u} = \ddot{\boldsymbol{q}} \in \mathbb{R}^3$. The input commands are bounded componentwise as $|u_i| \leq U_{max,i}$, for $i = 1, 2, 3$. Moreover, let $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ be the end-effector position. At a given instant $t = t_0$, the robot is in a generic state $(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) \in \mathbb{R}^6$, with $\dot{\boldsymbol{q}}_0 \notin \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_0)\}$. Which feasible command $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$ would you apply to *stop as fast as possible* the Cartesian motion of the end-effector, while *keeping its velocity aligned* with the direction of $\dot{\boldsymbol{p}}_0 = \dot{\boldsymbol{p}}(t_0) \neq \boldsymbol{0}$? If there are multiple feasible solutions, provide the one having minimum norm. Illustrate your findings with a numerical example, providing the values of $\boldsymbol{q}_0$, $\dot{\boldsymbol{q}}_0$, $\boldsymbol{u}_0$, and of the resulting acceleration $\ddot{\boldsymbol{p}}_0 = \ddot{\boldsymbol{p}}(t_0)$.

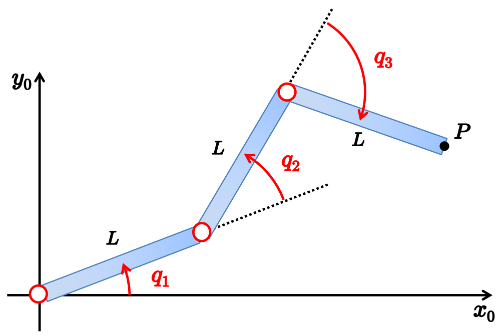

Figure 1: A planar 3R robot.

Exercise #2

For the same robot in Fig. 1, assume that the three links of equal length $L = 0.5$ [m] are all modeled as thin rods with a uniformly distributed mass of $m = 5$ [kg]. Provide the eigenvalues of the 2×2 Cartesian inertia matrix $\boldsymbol{M}_p$, when the robot is in the regular configuration $\boldsymbol{q}^* = (\pi/2, \pi/2, 0)$.

Hint: Use an equivalent expression for the Cartesian inertia matrix $\boldsymbol{M}_p = \boldsymbol{J}^{-T}\boldsymbol{M}\boldsymbol{J}^{-1}$ *that applies both to square and non-square Jacobians under the same full rank assumption.*

Exercise #3

The end-effector of a 2-dof Cartesian robot with different link masses m_1 and m_2 moves in a vertical plane $(\boldsymbol{x}, \boldsymbol{y})$ making contact with an environment. There is no force/torque sensor mounted on the robot. Design an impedance control law that shapes the response between interacting forces and tracking errors by assigning the same two real, negative and coincident eigenvalues (i.e., in $-\lambda < 0$) to the closed-loop linear dynamics along the two decoupled directions $\boldsymbol{x}$ and $\boldsymbol{y}$.

[180 minutes (3 hours); open books]

Solution

September 10, 2021

Exercise #1

The second-order differential kinematics of a robot with n joints performing a m-dimensional task (with $m \le n$) is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{1}$$

with vector $\boldsymbol{h} \in \mathbb{R}^m$ being quadratic in $\dot{\boldsymbol{q}}$. The joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is taken as the input command $\boldsymbol{u}$.

At time $t = t_0$, the task velocity is $\dot{\boldsymbol{p}}_0 = \dot{\boldsymbol{p}}(t_0) = \boldsymbol{J}\left(\boldsymbol{q}(t_0)\right)\dot{\boldsymbol{q}}(t_0) = \boldsymbol{J}_0\dot{\boldsymbol{q}}_0$, which is necessarily different from zero since one should choose a $\dot{\boldsymbol{q}}_0 \notin \mathcal{N}\left\{\boldsymbol{J}_0\right\}$. We impose to the end-effector an acceleration $\ddot{\boldsymbol{p}}_0$ (actually, a deceleration) that is aligned with $\dot{\boldsymbol{p}}_0$ and whose components are opposite in sign to the associated velocity components. Therefore, we set

$$\ddot{\boldsymbol{p}}_0 = -\lambda\,\dot{\boldsymbol{p}}_0 = -\lambda\,\boldsymbol{J}_0\dot{\boldsymbol{q}}_0, \qquad \text{with } \lambda \ge 0,$$

and choose the largest possible (non-negative) value for the scalar λ such that the minimum norm joint acceleration solution $\boldsymbol{u}_0$ to (1) is feasible. It is then

$$\boldsymbol{u}_0 = \boldsymbol{J}_0^{\#}\left(\ddot{\boldsymbol{p}}_0 - \boldsymbol{h}_0\right) = -\lambda\,\boldsymbol{J}_0^{\#}\boldsymbol{J}_0\,\dot{\boldsymbol{q}}_0 - \boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0. \tag{2}$$

Define now the two n-dimensional vectors[1]

$$\boldsymbol{a} = -\boldsymbol{J}_0^{\#}\boldsymbol{J}_0\,\dot{\boldsymbol{q}}_0, \qquad \boldsymbol{b} = -\boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0, \tag{3}$$

and organize the bounds on the commands in vector form as

$$\boldsymbol{U}_{max} = \begin{pmatrix} U_{max,1} \\ U_{max,2} \\ \vdots \\ U_{max,n} \end{pmatrix}.$$

The problem is formulated as a simple linear program (LP) as follows:

$$\max\ \lambda \qquad \text{s.t.} \quad -\boldsymbol{U}_{max} \le \boldsymbol{a}\,\lambda + \boldsymbol{b} \le \boldsymbol{U}_{max}, \quad \lambda \ge 0, \tag{4}$$

where vector inequalities are to be considered component-wise. Note first that $\boldsymbol{a} \neq \boldsymbol{0}$ (although some of its components may possibly vanish). In fact, $\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$ is a realizable velocity, as generated by $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$; thus, the pseudoinverse of such task velocity cannot produce a zero joint velocity. The feasible set may be empty, in which case no instantaneous acceleration solution exists. Moreover, if the optimal value of problem (4) is $\lambda = 0$, the end-effector will not be able to instantaneously decelerate; the problem has again no actual solution at $t = t_0$. Nonetheless, it is convenient to keep the value $\lambda = 0$ in the feasible set, so as to guarantee the existence of a solution to problem (4) whenever its (closed) feasible set is non-empty.

[1] One can also define the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with a positive sign in front. Being the bounds on the command $\boldsymbol{u}$ symmetric, the linear inequalities in (4) would remain the same.

The optimal solution λ^* to (4) is easily found. For $i = 1, \ldots, n$, let

$$\lambda_i = \begin{cases} -\infty & \text{if } b_i < -U_{max,i} \text{ and } a_i \leq 0, \\ \dfrac{U_{max,i} - b_i}{a_i} & \text{if } b_i < -U_{max,i} \text{ and } a_i > 0, \\ \max\left\{-\dfrac{U_{max,i} + b_i}{a_i}, \dfrac{U_{max,i} - b_i}{a_i}\right\} & \text{if } U_{max,i} \leq b_i \leq U_{max,i} \text{ and } a_i \neq 0, \\ +\infty & \text{if } U_{max,i} \leq b_i \leq U_{max,i} \text{ and } a_i = 0, \\ -\dfrac{U_{max,i} + b_i}{a_i} & \text{if } b_i > U_{max,i} \text{ and } a_i < 0, \\ -\infty & \text{if } b_i > U_{max,i} \text{ and } a_i \geq 0. \end{cases} \tag{5}$$

We compute then

$$\lambda^* = \min_{i=1,\ldots,n} \lambda_i, \tag{6}$$

with the following conclusions:

$\lambda^* > 0 \quad \Rightarrow \quad \lambda^*$ is the optimal solution, with a feasible acceleration $\boldsymbol{u}_0^* = \boldsymbol{a}\lambda^* + \boldsymbol{b}$;

$\lambda^* = 0 \quad \Rightarrow \quad$ the resulting joint acceleration is $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#} \dot{\boldsymbol{J}}_0 \, \dot{\boldsymbol{q}}_0$, yielding $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$;

$\lambda^* = -\infty \quad \Rightarrow \quad$ there is no solution to the problem (the feasible set is empty).

In the optimal solution $\boldsymbol{u}_0^*$, at least one joint acceleration will saturate one of its bounds. When $\lambda^* = 0$, the end-effector will keep instantaneously the same velocity $\dot{\boldsymbol{p}}_0$, with no deceleration. When there is no solution to the problem, the end-effector will no longer be able to move exactly along the direction of $\dot{\boldsymbol{p}}_0$ (in either way). Some of the various possible situations for a generic single component λ_i are illustrated in Fig. 2. Figure 3 shows geometrically some resulting cases for $\boldsymbol{u}_0^*$ with $n = 2$ components (thus, when $m = 1$).

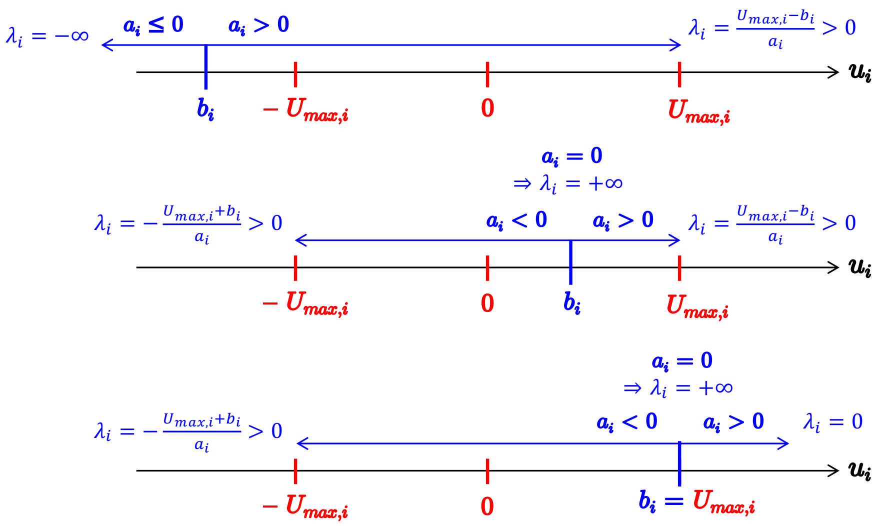

Figure 2: Examples of evaluation of λ_i for a generic single component.

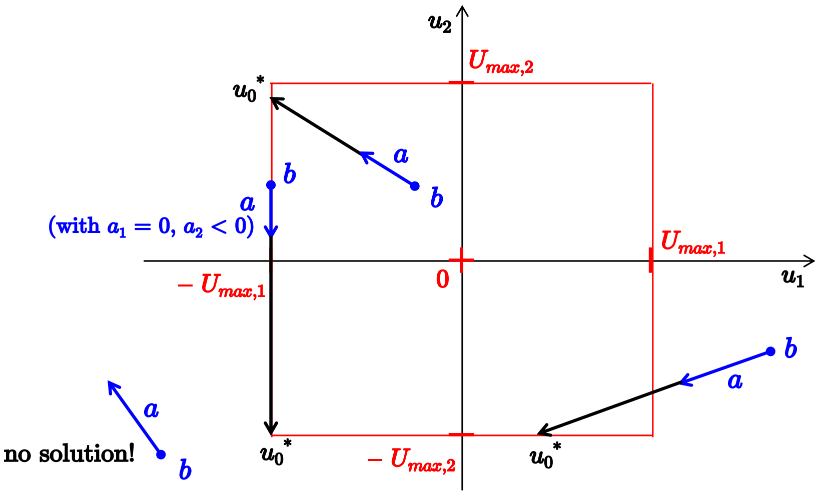

Figure 3: Examples of existence or not of a solution $\boldsymbol{u}_0^*$ and its geometrical evaluation when $n = 2$.

For the planar 3R robot of Fig. 1, we have $n = 3$, $m = 2$, and the terms in (1) are the 2×3 task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}, \tag{7}$$

its time derivative

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = -L \cdot \begin{pmatrix} c_1\dot{q}_1 + c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ s_1\dot{q}_1 + s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and the product of matrix $\dot{\boldsymbol{J}}$ by the joint velocity $\dot{\boldsymbol{q}}$

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -L \begin{pmatrix} (c_1 + c_{12} + c_{123})\,\dot{q}_1^2 + 2\,(c_{12} + c_{123})\,\dot{q}_1\dot{q}_2 + 2\,c_{123}\,\dot{q}_1\dot{q}_3 + (c_{12} + c_{123})\,\dot{q}_2^2 + 2\,c_{123}\,\dot{q}_2\dot{q}_3 + c_{123}\,\dot{q}_3^2 \\ (s_1 + s_{12} + s_{123})\,\dot{q}_1^2 + 2\,(s_{12} + s_{123})\,\dot{q}_1\dot{q}_2 + 2\,s_{123}\,\dot{q}_1\dot{q}_3 + (s_{12} + s_{123})\,\dot{q}_2^2 + 2\,s_{123}\,\dot{q}_2\dot{q}_3 + s_{123}\,\dot{q}_3^2 \end{pmatrix},$$

having used the shorthand notation for trigonometric functions (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$).

As a first numerical example, set $L = 1$ [m] for the link lengths and choose

$$\boldsymbol{U}_{max} = \begin{pmatrix} 15\,\pi \\ 10\,\pi \\ 10\,\pi \end{pmatrix} = \begin{pmatrix} 47.1239 \\ 31.4159 \\ 31.4159 \end{pmatrix} \;\; [\text{rad/s}^2]$$

as values for the (symmetric) bounds for the acceleration commands[2]. At time $t = t_0$, consider the robot state

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 \\ \pi/2 \\ \pi/2 \end{pmatrix} \;\; [\text{rad}], \qquad \dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \end{pmatrix} \;\; [\text{rad/s}].$$

[2]These bounds are the same used in Exercise 1 of the exam of July 12, 2021.

We compute then from the previous formulas

$$\boldsymbol{J}_0 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} -\pi \\ -\pi/2 \end{pmatrix} \text{ [m/s]}$$

and

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} \pi/2 & \pi & \pi \\ -\pi & -\pi & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \dot{\boldsymbol{J}}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} 3\,\pi^2/4 \\ -\pi^2 \end{pmatrix} = \begin{pmatrix} 7.4022 \\ -9.8696 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

The pseudoinverse of the task Jacobian is

$$\boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T \left(\boldsymbol{J}_0 \boldsymbol{J}_0^T \right)^{-1} = \begin{pmatrix} -2/3 & 1/3 \\ -1/3 & -1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Therefore, from (3) we obtain

$$\boldsymbol{a} = -\boldsymbol{J}_0^{\#} \dot{\boldsymbol{p}}_0 = -\begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = -\boldsymbol{J}_0^{\#} \boldsymbol{h}_0 = \begin{pmatrix} 5\,\pi^2/6 \\ -\pi^2/12 \\ -11\,\pi^2/12 \end{pmatrix} = \begin{pmatrix} 8.2247 \\ -0.8225 \\ -9.0471 \end{pmatrix}.$$

None of the components of vector $\boldsymbol{b}$ (related to the Cartesian drift acceleration $\boldsymbol{h}_0$) is outside the acceleration bounds specified by $\boldsymbol{U}_{max}$. As a result, according to the law (5–6), an optimal solution certainly exists and is given by

$$\lambda^* = 19.4764 \text{ [s}^{-1}\text{]} \quad \Rightarrow \quad \boldsymbol{u}_0^* = \begin{pmatrix} -22.3688 \\ -31.4159 \\ -9.0471 \end{pmatrix} \text{ [rad/s}^2\text{]} \quad \Rightarrow \quad \ddot{\boldsymbol{p}}_0 = \begin{pmatrix} 61.1869 \\ 30.5935 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

As expected, there is at least a component of $\boldsymbol{u}_0^*$ that is saturated (only the second one, at its negative lower bound). The obtained task acceleration is $\ddot{\boldsymbol{p}}_0 = -\lambda^* \dot{\boldsymbol{p}}_0$, as expected.

To verify further the method, consider now the following joint velocity at time $t = t_0$,

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 2\pi \\ \pi/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6.2832 \\ 1.5708 \\ 0 \end{pmatrix} \text{ [rad/s]},$$

with the first component four times higher than before, all the rest being the same. The changed terms are

$$\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -7.8540 \\ -1.5708 \end{pmatrix} \text{ [m/s]}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} 1.5708 & 7.8540 & 7.8540 \\ -7.8540 & -7.8540 & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 22.2066 \\ -61.6850 \end{pmatrix} \text{ [m/s}^2\text{]},$$

and thus

$$\boldsymbol{a} = \begin{pmatrix} -4.7124 \\ -3.1416 \\ 1.5708 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 35.3661 \\ -13.1595 \\ -48.5256 \end{pmatrix}.$$

The component b_3 of vector $\boldsymbol{b}$ exceeds now the lover bound $-U_{max,3} = -31.4159$ [rad/s^2]. However, $a_3 > 0$ and thus an optimal solution exists. According to the law (5–6), we obtain

$$\lambda^* = 5.8112 \text{ [s}^{-1}\text{]} \quad \Rightarrow \quad \boldsymbol{u}_0^* = \begin{pmatrix} 7.9814 \\ -31.4159 \\ -39.3973 \end{pmatrix} \text{ [rad/s}^2\text{]} \quad \Rightarrow \quad \ddot{\boldsymbol{p}}_0 = \begin{pmatrix} 45.6411 \\ 9.1282 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

As before, the second component of $\boldsymbol{u}_0^*$ is saturated at its negative lower bound. The rate of decrease of the Cartesian velocity is now slower[3], because the optimal λ^* is also smaller and the rate of decrease of $\ddot{\boldsymbol{p}}_0$ depends on λ^* only. In fact, it is immediate to see that

$$\ddot{\boldsymbol{p}} = -\lambda\,\dot{\boldsymbol{p}} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}}(t) = e^{-\lambda(t-t_0)}\,\dot{\boldsymbol{p}}(t_0) \simeq (1-\lambda\,dt)\,\dot{\boldsymbol{p}}_0 \qquad \Rightarrow \qquad \frac{\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_0}{dt} \simeq -\lambda\,\dot{\boldsymbol{p}}_0,$$

for a sufficiently small $dt = t - t_0 > 0$.

As a last example, we double the joint velocity at time $t = t_0$ with respect to the first case,

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi \\ \pi \\ 0 \end{pmatrix} = \begin{pmatrix} 3.1416 \\ 3.1416 \\ 0 \end{pmatrix} \text{ [rad/s]},$$

all the rest being again the same[4]. The changed terms are now

$$\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -6.2832 \\ -3.1416 \end{pmatrix} \text{ [m/s]}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} 3.1416 & 6.2832 & 6.2832 \\ -6.2832 & -6.2832 & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 29.6088 \\ -39.4784 \end{pmatrix} \text{ [m/s}^2\text{]},$$

and

$$\boldsymbol{a} = \begin{pmatrix} -3.1416 \\ -3.1416 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 32.8987 \\ -3.2899 \\ -36.1885 \end{pmatrix}.$$

Again, the third component of $\boldsymbol{b}$ exceeds its lower bound $-U_{max,3} = -31.4159$ [rad/s^2]. However, since $a_3 = 0$, no solution exists in this case. In fact, according to the law (5–6), it is

$$\lambda_1 = 25.4720, \quad \lambda_2 = 8.9528, \quad \text{but} \quad \lambda_3 = -\infty \qquad \Rightarrow \qquad \lambda^* = -\infty.$$

Exercise #2

Note first that the $m \times m$ Cartesian inertia matrix of a robot with a $m \times n$ Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that has full rank m can always be written as[5]

$$\boldsymbol{M}_p(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{8}$$

where $\boldsymbol{M}(\boldsymbol{q}) > 0$ is the $n \times n$ inertia matrix in the configuration space. The derivation of (8) for the case $m < n$ (redundant robot) with a full rank Jacobian is simple. Let the robot dynamics in joint space be

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{9}$$

and the second-order differential kinematics to the Cartesian space be

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{10}$$

Extracting $\ddot{\boldsymbol{q}}$ from (9), using the transformation of generalized forces $\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}$, and substituting in (10) yields

$$\begin{aligned}\ddot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &= \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)\boldsymbol{F} - \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

[3] This happens independently from the value of $\|\ddot{\boldsymbol{p}}_0\|$, which is smaller here than in the first case.
[4] This case coincides with the first one considered in Exercise 1 of the exam of July 12, 2021.
[5] The expression (8) appears also in the lecture slides on robot redundancy (block 2, part 2, p. 10) and on collision detection and reaction (block 19, p. 40).

or

$$\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{p}} + \boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{F}, \tag{11}$$

with $\boldsymbol{M}_p(\boldsymbol{q})$ as in (8) and

$$\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{M}_p(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\, \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right), \qquad \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{M}_p(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\, \boldsymbol{g}(\boldsymbol{q}).$$

Indeed, the dynamic description (11) is incomplete when $m < n$ and should be complemented by additional $n - m$ dynamic equations (e.g., judiciously extracted from the original complete dynamics (9) in the joint space). On the other hand, when the Jacobian is square ($m = n$) and nonsingular, the terms in the (now complete) Cartesian dynamic model (11) simplify to

$$\boldsymbol{M}_p(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})$$

and

$$\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\, \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{M}_p(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\, \boldsymbol{g}(\boldsymbol{q}).$$

The Jacobian of the planar 3R robot in Fig. 1 is given by (7). The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is extracted from the kinetic energy of the three links of the robot. Under the assumption of equal uniform thin rods, the center of mass of each link is at $d_{ci} = L/2$ on its kinematic axis and the barycentral inertia (around the axis normal to the plane) equals $I_i = (1/12)\, mL^2$. For the first link, we have then

$$T_1 = \frac{1}{2}\left(m d_{c1}^2 + I_1\right)\dot{q}_1^2 = \frac{1}{2}\left(m\left(\frac{L}{2}\right)^2 + \frac{1}{12}\, mL^2\right)\dot{q}_1^2 = \frac{1}{2}\,\frac{mL^2}{3}\,\dot{q}_1^2.$$

For the second link, we have

$$\boldsymbol{p}_{c2} = L \begin{pmatrix} c_1 + \frac{1}{2}\, c_{12} \\ s_1 + \frac{1}{2}\, s_{12} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = L \begin{pmatrix} -s_1\dot{q}_1 - \frac{1}{2}\, s_{12}\,(\dot{q}_1 + \dot{q}_2) \\ c_1\dot{q}_1 + \frac{1}{2}\, c_{12}\,(\dot{q}_1 + \dot{q}_2) \end{pmatrix},$$

and thus

$$\begin{aligned} T_2 &= \frac{1}{2}\left(m\,\|\boldsymbol{v}_{c2}\|^2 + I_2\,(\dot{q}_1 + \dot{q}_2)^2\right) \\ &= \frac{1}{2}\left(mL^2\left(\dot{q}_1^2 + \frac{1}{4}\,(\dot{q}_1 + \dot{q}_2)^2 + c_2\,\dot{q}_1\,(\dot{q}_1 + \dot{q}_2)\right) + \frac{1}{12}\, mL^2\,(\dot{q}_1 + \dot{q}_2)^2\right) \\ &= \frac{1}{2}\, mL^2\left(\left(\frac{4}{3} + c_2\right)\dot{q}_1^2 + \left(\frac{2}{3} + c_2\right)\dot{q}_1\dot{q}_2 + \frac{1}{3}\,\dot{q}_2^2\right). \end{aligned}$$

Finally, for the third link

$$\boldsymbol{p}_{c3} = L \begin{pmatrix} c_1 + c_{12} + \frac{1}{2}\, c_{123} \\ s_1 + s_{12} + \frac{1}{2}\, s_{123} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = L \begin{pmatrix} -\left(s_1\dot{q}_1 + s_{12}\,(\dot{q}_1 + \dot{q}_2) + \frac{1}{2}\, s_{123}\,(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\right) \\ c_1\dot{q}_1 + c_{12}\,(\dot{q}_1 + \dot{q}_2) + \frac{1}{2}\, c_{123}\,(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and so

$$
\begin{aligned}
T_3 &= \frac{1}{2}\left(m\left\|\boldsymbol{v}_{c3}\right\|^2 + I_3\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2\right)\\
&= \frac{1}{2}\left(mL^2\left(\dot{q}_1^2+\left(\dot{q}_1+\dot{q}_2\right)^2+\frac{1}{4}\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2+2\,c_2\,\dot{q}_1\left(\dot{q}_1+\dot{q}_2\right)\right.\right.\\
&\qquad\left.\left.+\;c_{23}\,\dot{q}_1\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)+c_3\left(\dot{q}_1+\dot{q}_2\right)\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)\right)+\frac{1}{12}\,mL^2\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2\right)\;.\\
&= \frac{1}{2}\,mL^2\left(\left(\frac{7}{3}+2\,c_2+c_{23}+c_3\right)\dot{q}_1^2+\left(\frac{8}{3}+2\,c_2+c_{23}+2c_3\right)\dot{q}_1\dot{q}_2+\left(\frac{2}{3}+c_{23}+c_3\right)\dot{q}_1\dot{q}_3\right.\\
&\qquad\left.+\left(\frac{4}{3}+c_3\right)\dot{q}_2^2+\left(\frac{2}{3}+c_3\right)\dot{q}_2\dot{q}_3+\frac{1}{3}\,\dot{q}_3^2\right).
\end{aligned}
$$

Therefore, from

$$
T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}
$$

we obtain

$$
\boldsymbol{M}(\boldsymbol{q}) = mL^2\begin{pmatrix} 4+3\,c_2+c_{23}+c_3 & \frac{5}{3}+\frac{3}{2}\,c_2+\frac{1}{2}\,c_{23}+c_3 & \frac{1}{3}+\frac{1}{2}\,c_{23}+\frac{1}{2}\,c_3\\ \vdots & \frac{5}{3}+c_3 & \frac{1}{3}+\frac{1}{2}\,c_3\\ \text{symm} & \cdots & \frac{1}{3}\end{pmatrix}. \tag{12}
$$

Since $L = 0.5$ [m] and $mL^2 = 5\cdot 0.5^2 = 1.25$ [kgm^2], evaluating eqs. (7) and (12) at the configuration $\boldsymbol{q}^* = (\pi/2, \pi/2, 0)$ gives:

$$
\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q}^*) = \begin{pmatrix} -0.5 & 0 & 0\\ -1 & -1 & -0.5\end{pmatrix},
$$

$$
\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{q}^*) = \begin{pmatrix} \frac{25}{4} & \frac{10}{3} & \frac{25}{24}\\ \frac{10}{3} & \frac{10}{3} & \frac{25}{24}\\ \frac{25}{24} & \frac{25}{24} & \frac{5}{12}\end{pmatrix} \simeq \begin{pmatrix} 6.25 & 3.3333 & 1.0417\\ 3.3333 & 3.3333 & 1.0417\\ 1.0417 & 1.0417 & 0.4167\end{pmatrix}.
$$

The Cartesian inertia matrix in (8) is

$$
\boldsymbol{M}_p = \boldsymbol{M}_p(\boldsymbol{q}^*) = \begin{pmatrix} 35/3 & 0\\ 0 & 35/24\end{pmatrix} \simeq \begin{pmatrix} 11.6667 & 0\\ 0 & 1.4583\end{pmatrix}.
$$

Its two eigenvalues are then

$$
\lambda_1 = 11.6667, \qquad \lambda_2 = 1.4583.
$$

Note that the Cartesian inertia is fully decoupled in the configuration $\boldsymbol{q}^*$. This is by no means the generic case, although the matrix $\boldsymbol{M}_p(\boldsymbol{q})$ is always symmetric, and also positive definite as long as the Jacobian is full rank (its eigenvalues are always real, and strictly positive outside singularities).

Exercise #3

With reference to Fig. 4, the dynamic model of the 2-dof Cartesian robot in contact with a generic environment is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{\tau} + \boldsymbol{F}, \tag{13}$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ m_2 g_0 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix},$$

where $\boldsymbol{\tau} \in \mathbb{R}^2$ is the control input force at the prismatic joints and $\boldsymbol{F} \in \mathbb{R}^2$ is the contact force exerted from the environment on the robot end effector.

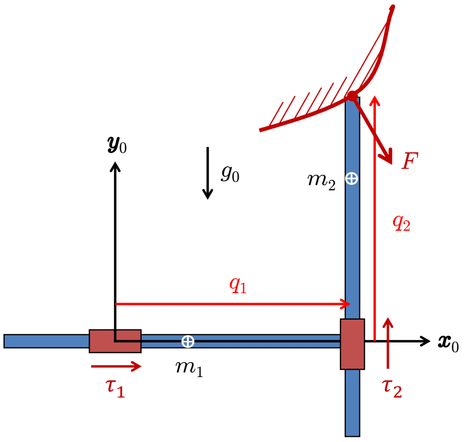

Figure 4: A 2-dof Cartesian robot making contact with an environment.

The desired linear and decoupled impedance model is

$$\boldsymbol{M}_d\,\ddot{\boldsymbol{e}} + \boldsymbol{D}_d\,\dot{\boldsymbol{e}} + \boldsymbol{K}_d\,\boldsymbol{e} = \boldsymbol{F}, \tag{14}$$

where

$$\boldsymbol{e} = \boldsymbol{p} - \boldsymbol{p}_d, \qquad \boldsymbol{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

with $\boldsymbol{p}_d \in \mathbb{R}^2$ being the desired position of the robot end effector, and

$$\boldsymbol{M}_d = \begin{pmatrix} M_{dx} & 0 \\ 0 & M_{dy} \end{pmatrix} > 0, \qquad \boldsymbol{D}_d = \begin{pmatrix} D_{dx} & 0 \\ 0 & D_{dy} \end{pmatrix} > 0, \qquad \boldsymbol{K}_d = \begin{pmatrix} K_{dx} & 0 \\ 0 & K_{dy} \end{pmatrix} > 0.$$

Since there is no force/torque sensor available, we have to choose necessarily the actual (Cartesian) inertia as desired (apparent) inertia:

$$\boldsymbol{M}_d = \boldsymbol{M} \qquad \Longleftrightarrow \qquad M_{dx} = m_1 + m_2, \quad M_{dy} = m_2. \tag{15}$$

From (13) and (14), with (15), the required control law is

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{M}\ddot{\boldsymbol{p}}_d + \boldsymbol{g} - \boldsymbol{D}_d\,\dot{\boldsymbol{e}} - \boldsymbol{K}_d\,\boldsymbol{e} \\ &= \begin{pmatrix} (m_1 + m_2)\,\ddot{x}_d \\ m_2\,\ddot{y}_d \end{pmatrix} + \begin{pmatrix} 0 \\ m_2 g_0 \end{pmatrix} + \begin{pmatrix} D_{dx}\,(\dot{x}_d - \dot{x}) + K_{dx}\,(x_d - x) \\ D_{dy}\,(\dot{y}_d - \dot{y}) + K_{dy}\,(y_d - y) \end{pmatrix}, \end{aligned} \tag{16}$$

which has the standard form of a PD action with a feedforward acceleration term (if $\ddot{\boldsymbol{p}}_d \neq \boldsymbol{0}$) and gravity cancellation. For the choice of the gains, we rewrite the impedance model (14), using again (15), in the Laplace domain,

$$\left(\boldsymbol{M}s^2 + \boldsymbol{D}_d s + \boldsymbol{K}_d\right)\boldsymbol{e}(s) = \boldsymbol{F}(s),$$

and impose the desired dynamic characteristics between $\boldsymbol{F}(s)$ and $\boldsymbol{e}(s)$ in each Cartesian direction:

$$\left(\boldsymbol{I}s^2 + \boldsymbol{M}^{-1}\boldsymbol{D}_d\ s + \boldsymbol{M}^{-1}\boldsymbol{K}_d\right) = \begin{pmatrix} (s+\lambda)^2 & 0 \\ 0 & (s+\lambda)^2 \end{pmatrix} = \begin{pmatrix} s^2 + 2\lambda s + \lambda^2 & 0 \\ 0 & s^2 + 2\lambda s + \lambda^2 \end{pmatrix}.$$

As a result,

$$D_{dx} = 2\,(m_1 + m_2)\,\lambda, \qquad K_{dx} = (m_1 + m_2)\,\lambda^2,$$

and

$$D_{dy} = 2m_2\lambda, \qquad K_{dy} = m_2\lambda^2.$$

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