Robotics 2

July 12, 2021

Exercise #1

Consider the planar 3R robot in Fig. 1. The three links have all equal length L. The robot is controlled by a joint acceleration command $\boldsymbol{u} = \ddot{\boldsymbol{q}} \in \mathbb{R}^3$. The input commands are bounded componentwise as $|u_i| \leq U_{max,i}$, for $i = 1, 2, 3$. Moreover, let $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ be the end-effector position. At a given instant $t = t_0$, the robot is in a generic state $(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) \in \mathbb{R}^6$. Assume in the following that it is always $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$. For this robot, provide (if it exists) a *feasible* solution $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$ to each of the following problems. If there are multiple feasible solutions, provide the one having *minimum norm*.

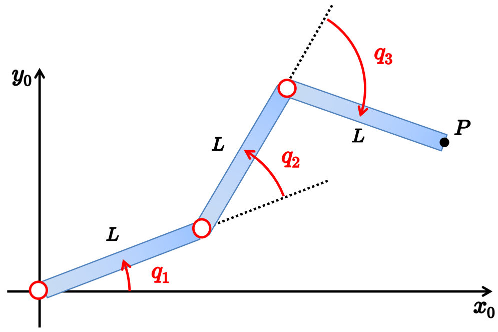

Figure 1: A planar 3R robot.

Is it possible to define $\boldsymbol{u}_0$ so that the end-effector acceleration instantaneously vanishes, i.e., $\ddot{\boldsymbol{p}}_0 = \ddot{\boldsymbol{p}}(t_0) = \boldsymbol{0}$? If so, under which conditions on the current robot state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$? Provide a specific example (or counterexample) illustrating the situation, giving the numerical values of $\boldsymbol{q}_0$, $\dot{\boldsymbol{q}}_0$, and $\boldsymbol{u}_0$ (and of the resulting $\ddot{\boldsymbol{p}}_0$, if different from zero).

Exercise #2

The dynamics of a robot arm with n joints that may be subject to actuator faults can be written in a standard form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{\tau}_f, \tag{1}$$

where $\boldsymbol{\tau}_f \in \mathbb{R}^n$ is an additional torque that models a generic actuator fault when present. Consider now as fault an *incipient block* of the joint/motor i, with $i \in \{1, \ldots, n\}$. This situation is represented by an acceleration of the faulted joint that behaves as $\ddot{q}_i = -\lambda_i \dot{q}_i$, with $\lambda_i \gg 1$, until the joint eventually stops. Show that this fault can be described as in (1), by providing the model-based expression of the fault $\tau_{f,i}$ (with $\tau_{f,j} \equiv 0$, for $j \neq i$). Find also the expression of the accelerations of the other joints, $j \neq i$, at the instant when this type of fault occurs.

Exercise #3

The 3-dof RPR robot in Fig. 2 moves on a horizontal plane. The robot should asymptotically track a smooth joint trajectory $\boldsymbol{q}_d(t)$ in the presence of a partly unknown dynamic model. In fact, only the dynamic parameters m_3, d_{c3}, and I_3 are assumed to be known. Derive first the dynamic model of this robot, neglecting any dissipative frictional effect.

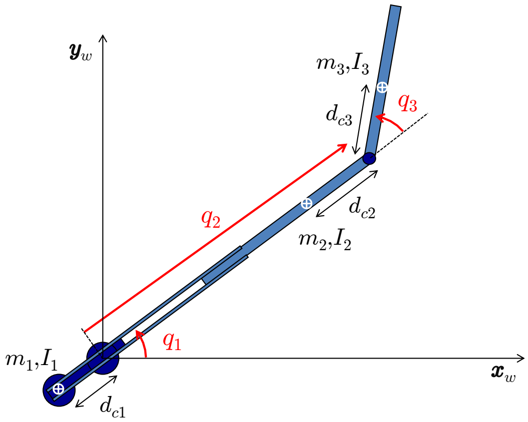

Figure 2: A planar RPR robot with its generalized coordinates and dynamic parameters.

Consider next the following adaptive trajectory tracking control law that takes advantage of the partly known dynamics

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\hat{\boldsymbol{a}}_U + \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_K + \boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D\,(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) \\ \dot{\hat{\boldsymbol{a}}}_U &= \boldsymbol{\Gamma}\,\boldsymbol{Y}_U^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}),\end{aligned} \tag{2}$$

where $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\,(\boldsymbol{q}_d - \boldsymbol{q}) = \dot{\boldsymbol{q}}_d + \boldsymbol{K}_D^{-1}\boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q})$, $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$, and $\boldsymbol{\Gamma} > 0$ are diagonal gain matrices, $\boldsymbol{a}_U \in \mathbb{R}^{p_u}$ and $\boldsymbol{a}_K \in \mathbb{R}^{p_k}$ are vectors containing, respectively, the *unknown* and the *known* dynamic coefficients of the robot model. The $3 \times p_u$ matrix $\boldsymbol{Y}_U$ and $3 \times p_k$ matrix $\boldsymbol{Y}_K$ are the associated regressors in the linear parametrization of the dynamic model. Provide the explicit expressions of all terms in the adaptive control law (2). If time remains, sketch a proof of the asymptotic stability of the trajectory tracking error with this modified adaptive law.

[180 minutes (3 hours); open books]

Solution

July 12, 2021

Exercise #1

The second-order differential kinematics of a robot with n joints performing a m-dimensional task (with $m \le n$) is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{3}$$

with vector $\boldsymbol{h} \in \mathbb{R}^m$ being quadratic in $\dot{\boldsymbol{q}}$. The joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is taken here as input command $\boldsymbol{u}$. For the planar 3R robot of Fig. 1, we have $n = 3$, $m = 2$, and the terms in (3) are the 2×3 task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

its time derivative

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = -L \cdot$$
$$\begin{pmatrix} c_1\dot{q}_1 + c_{12}(\dot{q}_1+\dot{q}_2) + c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & c_{12}(\dot{q}_1+\dot{q}_2) + c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & c_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) \\ s_1\dot{q}_1 + s_{12}(\dot{q}_1+\dot{q}_2) + s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & s_{12}(\dot{q}_1+\dot{q}_2) + s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) & s_{123}(\dot{q}_1+\dot{q}_2+\dot{q}_3) \end{pmatrix},$$

and the product of matrix $\dot{\boldsymbol{J}}$ by the joint velocity $\dot{\boldsymbol{q}}$

$$\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -L \cdot$$
$$\begin{pmatrix} (c_1 + c_{12} + c_{123})\,\dot{q}_1^2 + 2\,(c_{12} + c_{123})\,\dot{q}_1\dot{q}_2 + 2\,c_{123}\,\dot{q}_1\dot{q}_3 + (c_{12} + c_{123})\,\dot{q}_2^2 + 2\,c_{123}\,\dot{q}_2\dot{q}_3 + c_{123}\,\dot{q}_3^2 \\ (s_1 + s_{12} + s_{123})\,\dot{q}_1^2 + 2\,(s_{12} + s_{123})\,\dot{q}_1\dot{q}_2 + 2\,s_{123}\,\dot{q}_1\dot{q}_3 + (s_{12} + s_{123})\,\dot{q}_2^2 + 2\,s_{123}\,\dot{q}_2\dot{q}_3 + s_{123}\,\dot{q}_3^2 \end{pmatrix},$$

having used the shorthand notation for trigonometric functions (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$).

The minimum norm solution of (3) for $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ at a generic state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ is

$$\boldsymbol{u}_0 = -\boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{h}(\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0). \tag{4}$$

In the absence of bounds on the command $\boldsymbol{u}_0$, this acceleration would return $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$ if and only if $\boldsymbol{h}(\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0) \in \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_0)\}$. In particular, when $\operatorname{rank}\{\boldsymbol{J}(\boldsymbol{q}_0)\} = 2$ (regular case), this condition is always satisfied and we only need to check whether $\boldsymbol{u}_0$ is feasible, i.e., if $|\boldsymbol{u}_{0,i}| \le U_{max,i}$, for all $i = 1, 2, 3$. If so, then we stop. The same would happen in the singular case ($\operatorname{rank}\{\boldsymbol{J}(\boldsymbol{q}_0)\} < 2$).

If instead the acceleration command (4) is unfeasible, we should attempt the use of the general solution to (3) for $\ddot{\boldsymbol{p}} = \boldsymbol{0}$, namely

$$\boldsymbol{u}_0 = -\boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{h}(\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{J}(\boldsymbol{q}_0)\right)\ddot{\boldsymbol{q}}_0, \tag{5}$$

with an extra joint acceleration $\ddot{\boldsymbol{q}}_0$ projected in the null space of $\boldsymbol{J}$. This term may possibly help in recovering feasibility when the preferred minimum norm solution (4) is unfeasible. The definition of the actual command (5) is obtained by a simple variant of the SNS (Saturation in the Null Space) method for redundant robots.

Consider for simplicity only the regular case for the Jacobian, $\operatorname{rank}\{\boldsymbol{J}(\boldsymbol{q}_0)\} = 2$. Taking into account that there is only $n-m = 1$ degree of redundancy in the problem, if two (or all) of the scalar components of (4) are out of bounds, then no feasible solution will exist in any case. Otherwise, Algorithm 1 (written in pseudo-code) will provide a feasible solution, if one exists. The bounds on the commands are organized in vector form as $\boldsymbol{U}_{max} = \begin{pmatrix} U_{max,1} & U_{max,2} & U_{max,3} \end{pmatrix}^T \in \mathbb{R}^3$. We set also $\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0)$, $\dot{\boldsymbol{J}}_0 = \dot{\boldsymbol{J}}(\boldsymbol{q}_0)$, and $\boldsymbol{h}_0 = \boldsymbol{h}(\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0) = \dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0$.

Algorithm 1 SNS method for finding a feasible solution, if it exists (case $n = 3$, $m = 2$)

if the command $\boldsymbol{u}_0$ in (4) satisfies $|\boldsymbol{u}_0| \le \boldsymbol{U}_{max}$ (componentwise) **then**
  STOP % *the minimum norm solution (4) is feasible and returns already* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
else
  $j^* = \arg\max_{i=1,2,3} \left| \left(\boldsymbol{J}_0^{\#} \boldsymbol{h}_0 \right)_i \right|$ % *it is then* $|u_{0,j^*}| > U_{max,j^*}$*; by assumption,*
  % u_{0,j^*} *will be the only command out of bounds*
  set $u_{0,j^*} = \text{sign}\left(-\boldsymbol{J}_0^{\#} \boldsymbol{h}_0\right)_{j^*} U_{max,j^*}$ % *the overdriven command is now saturated*
  define the 2×2 matrix $\bar{\boldsymbol{J}}_{\{-j^*\}}$ by deleting the j^*-th column $\boldsymbol{J}_{j^*}$ from $\boldsymbol{J}_0$,
   $\boldsymbol{u}_{\{-j^*\}} \in \mathbb{R}^2$ by deleting u_{j^*} from $\boldsymbol{u}$, and $\boldsymbol{U}_{max,\{-j^*\}} \in \mathbb{R}^2$ deleting U_{max,j^*} from $\boldsymbol{U}_{max}$
  set $\boldsymbol{a} = \boldsymbol{J}_{j^*} u_{0,j^*} + \boldsymbol{h}_0$ % *... one needs to solve* $\bar{\boldsymbol{J}}_{\{-j^*\}} \boldsymbol{u}_{\{-j^*\}} + \boldsymbol{a} = \boldsymbol{0}$
  compute $\boldsymbol{u}_{0,\{-j^*\}} = -\bar{\boldsymbol{J}}_{\{-j^*\}}^{\#} \boldsymbol{a}$ % *... a unique solution if rank* $\left\{ \bar{\boldsymbol{J}}_{\{-j^*\}} \right\} = 2$!!
  if $\left| \boldsymbol{u}_{0,\{-j^*\}} \right| \le \boldsymbol{U}_{max,\{-j^*\}}$ (componentwise) **then**
   recompose $\boldsymbol{u}_{0,\text{SNS}}$ from u_{0,j^*} and $\boldsymbol{u}_{0,\{-j^*\}}$
   STOP % *a new feasible solution* $\boldsymbol{u}_{0,\text{SNS}}$ *has been found, returning* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
  else
   EXIT % *there is no feasible solution providing* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
  end if
end if

We provide next several numerical examples illustrating different situations[1]. In all cases, we have set $L = 1$ [m] for the link lengths and chosen

$$\boldsymbol{U}_{max} = \begin{pmatrix} 15\,\pi \\ 10\,\pi \\ 10\,\pi \end{pmatrix} = \begin{pmatrix} 47.1239 \\ 31.4159 \\ 31.4159 \end{pmatrix} \; [\text{rad/s}^2]$$

as bounds for the acceleration commands. The same data will be used also for Problem #1b of this Exercise. Computations are performed in MATLAB.

1. *Regular configuration* $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], *joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi, \pi, 0)$ [rad/s]

   We evaluate the terms in (3)

$$\boldsymbol{J}_0 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} \pi & 2\pi & 2\pi \\ -2\pi & -2\pi & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 3\,\pi^2 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 29.6088 \\ -39.4784 \end{pmatrix},$$

   and compute the pseudoinverse as

$$\boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T \left(\boldsymbol{J}_0\, \boldsymbol{J}_0^T \right)^{-1} = \begin{pmatrix} -2/3 & 1/3 \\ -1/3 & -1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

   Note that the end-effector velocity $\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} -2\pi & -\pi \end{pmatrix}^T$ [m/s] is different from zero. Applying the minimum norm solution (4), we obtain the feasible command

$$\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\, \boldsymbol{h}_0 = \begin{pmatrix} 10\,\pi^2/3 \\ -\pi^2/3 \\ -11\,\pi^2/3 \end{pmatrix} = \begin{pmatrix} 32.8987 \\ -3.2899 \\ -36.1885 \end{pmatrix} \; [\text{rad/s}^2],$$

   which returns $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$.

[1] Indeed, only one example was requested in the text.

2. *Same regular configuration* $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], *new velocity* $\dot{\boldsymbol{q}}_0 = (\pi, \pi, -\pi/4)$ [rad/s]

   The Jacobian $\boldsymbol{J}_0$ and its pseudoinverse $\boldsymbol{J}_0^\#$ are the same as before. The different terms to be computed are

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} 3\pi/4 & 7\pi/4 & 7\pi/4 \\ -2\pi & -2\pi & 0 \end{pmatrix} = \begin{pmatrix} 2.3562 & 5.4978 & 5.4978 \\ -6.2832 & -6.2832 & 0 \end{pmatrix}$$

   and

$$\boldsymbol{h}_0 = \begin{pmatrix} 33\,\pi^2/16 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 20.3561 \\ -39.4784 \end{pmatrix}.$$

   The end-effector velocity is $\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -2\pi & -3\pi/4 \end{pmatrix}^T \neq \boldsymbol{0}$. The minimum norm solution (4) is now

$$\boldsymbol{u}_0 = -\boldsymbol{J}_0^\#\,\boldsymbol{h}_0 = \begin{pmatrix} 65\,\pi^2/24 \\ -31\,\pi^2/48 \\ -161\,\pi^2/48 \end{pmatrix} = \begin{pmatrix} 26.7302 \\ -6.3741 \\ -33.1043 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

   which is unfeasible, being the third joint acceleration $u_{0,3} = -33.1043$ larger (in module) than its bound $U_{max,3} = 31.4159$. The other two acceleration commands remain instead within their bounds, so that we can apply Algorithm 1 to check if a feasible solution can be found by the SNS method. Being $j^* = 3$, we set $u_{0,3} = -U_{max,3} = -10\,\pi = -31.4159$ (saturation to the closest limit, the negative one) and compute

$$\bar{\boldsymbol{J}}_{\{-3\}} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{a} = -\boldsymbol{J}_3\,U_{max,3} + \boldsymbol{h}_0 = -\begin{pmatrix} 0 \\ -1 \end{pmatrix} 10\,\pi + \begin{pmatrix} 33\,\pi^2/16 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 20.3561 \\ -8.0625 \end{pmatrix}.$$

   Thus

$$\boldsymbol{u}_{0,\{-3\}} = -\bar{\boldsymbol{J}}_{\{-3\}}^\#\,\boldsymbol{a} = -\bar{\boldsymbol{J}}_{\{-3\}}^{-1}\,\boldsymbol{a} = \begin{pmatrix} 28.4186 \\ -8.0625 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{u}_{0,\text{SNS}} = \begin{pmatrix} 28.4186 \\ -8.0625 \\ -31.4159 \end{pmatrix}.$$

   The SNS result $\boldsymbol{u}_{0,\text{SNS}}$ is now a feasible solution and returns again $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$.

3. *Singular configuration* $\boldsymbol{q}_0 = (0, 0, \pi)$ [rad], *joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi/2, -\pi, \pi/2)$ [rad/s]

   The robot has the second link stretched and the third folded. This is a singular configuration since

$$\boldsymbol{J}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\,\{\boldsymbol{J}_0\} = 1, \qquad \mathcal{R}\{\boldsymbol{J}_0\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

   The end-effector is placed at $\boldsymbol{p}_0 = \boldsymbol{f}(\boldsymbol{q}_0) = (1, 0)$. Note also that the end-effector velocity is now

$$\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi/2 \\ -\pi \\ \pi/2 \end{pmatrix} \in \mathcal{N}\{\boldsymbol{J}_0\}.$$

   We evaluate the other terms in (3):

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} 0 & \pi/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \begin{pmatrix} -\pi^2/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4.9348 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{h}_0 \notin \mathcal{R}\{\boldsymbol{J}_0\}.$$

Computing the pseudoinverse of the singular task Jacobian,

$$\boldsymbol{J}_0^{\#} = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \\ 0 & -0.5 \end{pmatrix},$$

we obtain that the minimum norm solution (4) is simply $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\,\boldsymbol{h}_0 = \boldsymbol{0}$. Therefore, the end-effector acceleration cannot be modified in any case, remaining equal to

$$\ddot{\boldsymbol{p}}_0 = \boldsymbol{h}_0 = \begin{pmatrix} -4.9348 \\ 0 \end{pmatrix} \neq \boldsymbol{0}.$$

4. *Another singular case* $\boldsymbol{q}_0 = (0, \pi, -\pi)$ [rad], *same joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi/2, -\pi, \pi/2)$ [rad/s]

   In this last example, the robot has the second and third links both folded. We are again in a singularity, being

$$\boldsymbol{J}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank}\{\boldsymbol{J}_0\} = 1, \qquad \mathcal{R}\{\boldsymbol{J}_0\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

   The end-effector position $\boldsymbol{p}_0$ is the same as in the previous example. On the other hand, the end-effector velocity is now

$$\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ \pi \end{pmatrix} \neq \boldsymbol{0} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{p}}_0 \in \mathcal{R}\{\boldsymbol{J}_0\}.$$

   Moreover, being

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} -\pi & -\pi/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \dot{\boldsymbol{J}}_0 \dot{\boldsymbol{q}}_0 = \boldsymbol{0} \quad (!!),$$

   the minimum norm solution (4) will be $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\,\boldsymbol{h}_0 = \boldsymbol{0}$ as in the previous example. However, the (feasible) solution $\boldsymbol{u}_0$ will produce now $\ddot{\boldsymbol{p}}_0 = \boldsymbol{h}_0 = \boldsymbol{0}$, keeping thus the end-effector at the same instantaneous velocity $\dot{\boldsymbol{p}}_0$.

Exercise #2

Denote the inverse of the (symmetric) robot inertia matrix as

$$\boldsymbol{H}(\boldsymbol{q}) = \boldsymbol{M}^{-1}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{h}_1^T(\boldsymbol{q}) \\ \boldsymbol{h}_2^T(\boldsymbol{q}) \\ \vdots \\ \boldsymbol{h}_n^T(\boldsymbol{q}) \end{pmatrix},$$

with $\boldsymbol{h}_i(\boldsymbol{q})$ being the i-th column of $\boldsymbol{H}(\boldsymbol{q})$, for $i = 1, 2, \ldots, n$. Also, denote by $h_{ii}(\boldsymbol{q})$ the i-th element on the diagonal of $\boldsymbol{H}(\boldsymbol{q})$, for $i = 1, 2, \ldots, n$.

With reference to the dynamic model (1), the incipient blocking fault of the motor at the joint i is modeled by a vector $\boldsymbol{\tau}_f \in \mathbb{R}^n$ with the single nonzero i-th component having the expression

$$\tau_{f,i} = \frac{1}{h_{ii}(\boldsymbol{q})} \left(\boldsymbol{h}_i^T(\boldsymbol{q}) \left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) \right) + \lambda_i \dot{q}_i \right), \qquad \lambda_i \gg 1, \tag{6}$$

whereas $\tau_{f,j} = 0$, for all $j \neq i$.

In fact, since the acceleration vector $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is given by

$$\ddot{\boldsymbol{q}} = \boldsymbol{H}(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}_f\right),$$

using (6), the scalar component of the acceleration at joint i will be

$$\begin{aligned}
\ddot{q}_i &= \boldsymbol{h}_i^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tau_{f,i}\right) \\
&= \boldsymbol{h}_i^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) - h_{ii}(\boldsymbol{q}) \cdot \frac{1}{h_{ii}(\boldsymbol{q})}\left(\boldsymbol{h}_i^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \lambda_i \dot{q}_i\right) \\
&= -\lambda_i \dot{q}_i,
\end{aligned}$$

as requested. Moreover, the acceleration at any other joint $j \neq i$ is

$$\ddot{q}_j = \boldsymbol{h}_j^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) - h_{ji}(\boldsymbol{q})\,\tau_{f,i},$$

showing that the fault of motor i will affect also the acceleration of the other joints, because of the inertial couplings of the inverse of the robot inertia matrix (the off-diagonal terms $h_{ji}(\boldsymbol{q})$, $j \neq i$).

Exercise #3

The robot in Fig. 2 has $n = 3$ joints and moves on the horizontal plane. Neglecting friction effects, its dynamic model is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}.$$

In computing the kinetic energy T of the robot, we take into account that the motion is planar; thus, linear velocities will be 2D vectors in the plane $(\boldsymbol{x}_w, \boldsymbol{y}_w)$, while angular velocities will be just scalars (in the $\boldsymbol{z}_w$-direction). For the first link, it is

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2.$$

For the second link, since

$$\boldsymbol{p}_{c2} = (q_2 - d_{c2})\begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} &= \dot{q}_2 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} + (q_2 - d_{c2})\,\dot{q}_1 \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix}\begin{pmatrix} \dot{q}_2 \\ (q_2 - d_{c2})\,\dot{q}_1 \end{pmatrix} = \boldsymbol{R}(q_1)\begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix}, \end{aligned}$$

it follows

$$T_2 = \frac{1}{2} I_2\,\omega_2^2 + \frac{1}{2} m_2\,\boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \frac{1}{2}\left(I_2\,\dot{q}_1^2 + m_2\left((q_2 - d_{c2})^2\,\dot{q}_1^2 + \dot{q}_2^2\right)\right).$$

For the third link, from

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_2 \cos q_1 + d_{c3}\cos(q_1 + q_3) \\ q_2 \sin q_1 + d_{c3}\sin(q_1 + q_3) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \cos q_1 \dot{q}_2 - q_2 \sin q_1 \dot{q}_1 - d_{c3}\sin(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \\ \sin q_1 \dot{q}_2 + q_2 \cos q_1 \dot{q}_1 + d_{c3}\cos(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \end{pmatrix},$$

we obtain

$$
\begin{aligned}
T_3 &= \frac{1}{2} I_3\, \omega_3^2 + \frac{1}{2} m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} \\
&= \frac{1}{2} I_3 \left(\dot{q}_1 + \dot{q}_3\right)^2 + \frac{1}{2}\, m_3 \left(q_2^2\, \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \left(\dot{q}_1 + \dot{q}_3\right)^2 + 2\, d_{c3} \left(q_2 \cos q_3\, \dot{q}_1 - \sin q_3\, \dot{q}_2\right) \left(\dot{q}_1 + \dot{q}_3\right) \right).
\end{aligned}
$$

Thus, being

$$
T = \sum_{i=1}^{3} T_i = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix},
$$

we can write the single elements of the symmetric inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ as follows:

$$
\begin{aligned}
m_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 \\
&\quad - 2\, m_2 d_{c2}\, q_2 + (m_2 + m_3)\, q_2^2 + 2\, m_3 d_{c3}\, q_2 \cos q_3 \\
m_{12} &= -m_3 d_{c3} \sin q_3 \\
m_{13} &= I_3 + m_3 d_{c3}^2 + m_3 d_{c3}\, q_2 \cos q_3 \\
m_{22} &= m_2 + m_3 \\
m_{23} &= -m_3 d_{c3} \sin q_3 \\
m_{33} &= I_3 + m_3 d_{c3}^2.
\end{aligned}
\tag{7}
$$

The inertial term in the dynamic model can be rewritten in terms of $p = 5$ coefficients that collect the dynamic parameters of the robot,

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{pmatrix} a_1 + 2a_2 q_2 + a_3 q_2^2 + 2a_4 q_2 \cos q_3 & -a_4 \sin q_3 & a_5 + a_4 q_2 \cos q_3 \\ -a_4 \sin q_3 & a_3 & -a_4 \sin q_3 \\ a_5 + a_4 q_2 \cos q_3 & -a_4 \sin q_3 & a_5 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a},
$$

with the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^5$ defined by

$$
\begin{aligned}
a_1 &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 \\
a_2 &= -m_2 d_{c2} \\
a_3 &= m_2 + m_3 \\
a_4 &= m_3 d_{c3} \\
a_5 &= I_3 + m_3 d_{c3}^2.
\end{aligned}
\tag{8}
$$

Similarly, the entire dynamic model (because of the absence of gravity) is linearly parametrized in terms of the same dynamic coefficients $\boldsymbol{a}$, with a suitable $n \times p = 3 \times 5$ regressor matrix $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$. However, the parametrization (8) does not separate the unknown from the known dynamic coefficients in the inertia matrix (and thus in the dynamic model). On the other hand, the proposed adaptive law (2) does not need updating coefficients that involve only known dynamic parameters, i.e., m_3, d_{c3}, and I_3.

Therefore, looking at the expressions of the elements m_{ij} in (7) and at the definitions of the coefficients a_i in (8), we can organize and separate terms in the inertia matrix as

$$\begin{aligned}
m_{11} &= \left(I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2\right) + I_3 + m_3 d_{c3}^2 - 2\, m_2 d_{c2}\, q_2 + (m_2 + m_3)\, q_2^2 + 2\, m_3 d_{c3}\, q_2 \cos q_3 \\
&= a_{U1} + a_{K1} + 2\, a_{U2}\, q_2 + (a_{U3} + a_{K2})\, q_2^2 + 2\, a_{K3}\, q_2 \cos q_3 \\
m_{12} &= -m_3 d_{c3} \sin q_3 = -a_{K3} \sin q_3 \\
m_{13} &= I_3 + m_3 d_{c3}^2 + m_3 d_{c3}\, q_2 \cos q_3 = a_{K1} + a_{K3}\, q_2 \cos q_3 \\
m_{22} &= m_2 + m_3 = a_{U3} + a_{K2} \\
m_{23} &= -m_3 d_{c3} \sin q_3 = -a_{K3} \sin q_3 \\
m_{33} &= I_3 + m_3 d_{c3}^2 = a_{K1},
\end{aligned}$$

with $p_u = 3$ *unknown* dynamic coefficients

$$\begin{aligned}
a_{U1} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 \\
a_{U2} &= -m_2 d_{c2} \\
a_{U3} &= m_2,
\end{aligned} \tag{9}$$

organized as components of a vector $\boldsymbol{a}_U \in \mathbb{R}^3$, and, respectively, $p_k = 3$ *known* dynamic coefficients

$$\begin{aligned}
a_{K1} &= I_3 + m_3 d_{c3}^2 \\
a_{K2} &= m_3 \\
a_{K3} &= m_3 d_{c3},
\end{aligned} \tag{10}$$

organized as components of a vector $\boldsymbol{a}_K \in \mathbb{R}^3$. Despite the total number of dynamic coefficients is now higher than before ($p_u + p_k = 6 > 5 = p$), the number of those to be updated in the adaptive law is actually lower ($p_u = 3$).

To complete the dynamic modeling, we have to derive also the Coriolis and centrifugal terms. This will be done by referring directly to the double parametrization by $\boldsymbol{a}_U$ and $\boldsymbol{a}_K$, in view of their final use in the adaptive control law (2). Rewrite the robot inertia matrix as

$$\begin{aligned}
&\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) & \boldsymbol{m}_3(\boldsymbol{q}) \end{pmatrix} = \\
&\begin{pmatrix} a_{U1} + a_{K1} + 2\, a_{U2}\, q_2 + (a_{U3} + a_{K2})\, q_2^2 + 2\, a_{K3}\, q_2 \cos q_3 & -a_{K3} \sin q_3 & a_{K1} + a_{K3}\, q_2 \cos q_3 \\ -a_{K3} \sin q_3 & a_{U3} + a_{K2} & -a_{K3} \sin q_3 \\ a_{K1} + a_{K3}\, q_2 \cos q_3 & -a_{K3} \sin q_3 & a_{K1} \end{pmatrix}.
\end{aligned} \tag{11}$$

Using the Christoffel's symbols for computing the components $c_i(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$,

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad i = 1, 2, 3,$$

we obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 2\,a_{U2}+2\,(a_{U3}+a_{K2})\,q_2+2\,a_{K3}\cos q_3 & -2\,a_{K3}\,q_2\sin q_3 \\ 0 & 0 & -a_{K3}\cos q_3 \\ 0 & a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \end{pmatrix} + \begin{pmatrix} & \dots & \end{pmatrix}^T - \boldsymbol{O}\right)$$

$$= \begin{pmatrix} 0 & a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3 & 0 & 0 \\ -a_{K3}\,q_2\sin q_3 & 0 & -a_{K3}\,q_2\sin q_3 \end{pmatrix}$$

$$\Rightarrow \qquad c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = 2\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1\dot{q}_2 - a_{K3}\,q_2\sin q_3\,(2\,\dot{q}_1+\dot{q}_3)\,\dot{q}_3,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ 0 & 0 & -a_{K3}\cos q_3 \end{pmatrix} + \begin{pmatrix} \dots \end{pmatrix}^T - \begin{pmatrix} 2\,a_{U2}+2\,(a_{U3}+a_{K2})\,q_2+2\,a_{K3}\cos q_3 & 0 & a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ a_{K3}\cos q_3 & 0 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3) & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ -a_{K3}\cos q_3 & 0 & -a_{K3}\cos q_3 \end{pmatrix}$$

$$\Rightarrow \qquad c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1^2 - a_{K3}\cos q_3\,(2\,\dot{q}_1+\dot{q}_3)\,\dot{q}_3,$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ 0 & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} & \dots & \end{pmatrix}^T \right.$$

$$\left. - \begin{pmatrix} -2a_{K3}\,q_2\sin q_3 & -a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ -a_{K3}\cos q_3 & 0 & -a_{K3}\cos q_3 \\ -a_{K3}\,q_2\sin q_3 & -a_{K3}\cos q_3 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} a_{K3}\,q_2\sin q_3 & a_{K3}\cos q_3 & 0 \\ a_{K3}\cos q_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \qquad c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) = a_{K3}\,q_2\sin q_3\,\dot{q}_1^2 + 2\,a_{K3}\cos q_3\,\dot{q}_1\dot{q}_2,$$

yielding

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 2\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1\dot{q}_2 - a_{K3}\,q_2\sin q_3\,(2\,\dot{q}_1+\dot{q}_3)\,\dot{q}_3 \\ -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1^2 - a_{K3}\cos q_3\,(2\,\dot{q}_1+\dot{q}_3)\,\dot{q}_3 \\ a_{K3}\,q_2\sin q_3\,\dot{q}_1^2 + 2\,a_{K3}\cos q_3\,\dot{q}_1\dot{q}_2 \end{pmatrix} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}},$$

with the factorizing matrix

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathrm{col}\left\{\dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\right\} = \tag{12}$$

$$\begin{pmatrix} \begin{matrix}(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_2 \\ -a_{K3}\,q_2\sin q_3\,\dot{q}_3\end{matrix} & (a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1 & -a_{K3}\,q_2\sin q_3\,(\dot{q}_1+\dot{q}_3) \\ \begin{matrix}-\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot{q}_1 \\ -a_{K3}\cos q_3\,\dot{q}_3\end{matrix} & 0 & -a_{K3}\cos q_3\,(\dot{q}_1+\dot{q}_3) \\ a_{K3}\,q_2\sin q_3\,\dot{q}_1 + a_{K3}\cos q_3\,\dot{q}_2 & a_{K3}\cos q_3\,\dot{q}_1 & 0 \end{pmatrix}$$

being such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric (check this!).

As a result, the complete dynamic model can be linearly parametrized as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_U + \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_K \tag{13}$$

with the $n \times p_u = 3 \times 3$ regressor matrix $\boldsymbol{Y}_U$ for the unknown coefficients (conveniently separating the contributions by the inertial and by the velocity terms)

$$\begin{aligned} \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) &= \boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}) + \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} \ddot{q}_1 & 2\,q_2\,\ddot{q}_1 & q_2^2\,\ddot{q}_1 \\ 0 & 0 & \ddot{q}_2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2\,\dot{q}_1\dot{q}_2 & q_2\,\dot{q}_1\dot{q}_2 \\ 0 & -\dot{q}_1^2 & -q_2\,\dot{q}_1^2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{14}$$

and, similarly, with the $n \times p_k = 3 \times 3$ regressor matrix $\boldsymbol{Y}_K$ for the known coefficients

$$\begin{aligned} \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) &= \boldsymbol{Y}_{K,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}) + \boldsymbol{Y}_{K,c}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} \ddot{q}_1 + \ddot{q}_3 & q_2^2\,\ddot{q}_1 & q_2\cos q_3\,(2\,\ddot{q}_1 + \ddot{q}_3) - \sin q_3\,\ddot{q}_2 \\ 0 & \ddot{q}_2 & -\sin q_3\,(\ddot{q}_1 + \ddot{q}_3) \\ \ddot{q}_1 + \ddot{q}_3 & 0 & q_2\cos q_3\,\ddot{q}_1 - \sin q_3\,\ddot{q}_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 2\,q_2\,\dot{q}_1\dot{q}_2 & 2\,\cos q_3\,\dot{q}_1\dot{q}_2 - q_2\sin q_3\,(2\,\dot{q}_1 + \dot{q}_3)\,\dot{q}_3 \\ 0 & -q_2\,\dot{q}_1^2 & -\cos q_3\,(\dot{q}_1 + \dot{q}_3)^2 \\ 0 & 0 & q_2\sin q_3\,\dot{q}_1^2 + 2\cos q_3\,\dot{q}_1\dot{q}_2 \end{pmatrix}. \end{aligned} \tag{15}$$

To complete the design of the (partly) adaptive control law (2), we need to evaluate the two above regressors using suitable arguments. In particular, for the regressor $\boldsymbol{Y}_U$ we have

$$\boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}_r) + \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)\,. \tag{16}$$

Inside the second (velocity-dependent) addend, we have to split the quadratic velocity terms by exploiting the factorization given by matrix $\boldsymbol{S}$ in (12). For this, let

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}_U(\boldsymbol{q}) + \boldsymbol{M}_K(\boldsymbol{q})$$

be a decomposition of the inertia matrix in elements that depends (linearly) only on $\boldsymbol{a}_U$ and, respectively, only on $\boldsymbol{a}_K$. Accordingly, one can decompose also

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_K(\boldsymbol{q},\dot{\boldsymbol{q}}).$$

It can be easily shown that the three matrices

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}, \qquad \dot{\boldsymbol{M}}_U - 2\boldsymbol{S}_U, \qquad \dot{\boldsymbol{M}}_K - 2\boldsymbol{S}_K,$$

satisfy all the skew-symmetry property (as requested by the adaptive control law —see below). In particular, for this robot we have

$$\boldsymbol{M}_U(\boldsymbol{q}) = \begin{pmatrix} a_{U1} + 2\,a_{U2}\,q_2 + a_{U3}\,q_2^2 & 0 & 0 \\ 0 & a_{U3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} (a_{U2}+a_{U3}\,q_2)\,\dot{q}_2 & (a_{U2}+a_{U3}\,q_2)\,\dot{q}_1 & 0 \\ -\,(a_{U2}+a_{U3}\,q_2)\,\dot{q}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify the skew-symmetry of $\dot{\boldsymbol{M}}_U - 2\boldsymbol{S}_U$. As a result, the second term $\boldsymbol{Y}_{U,c}$ in (16) is obtained by using the identity

$$\boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}_r = \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)\,\boldsymbol{a}_U = \begin{pmatrix} 0 & \dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r} & q_2\,(\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r}) \\ 0 & -\dot{q}_1\dot{q}_{1r} & -q_2\,\dot{q}_1\dot{q}_{1r} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{U1} \\ a_{U2} \\ a_{U3} \end{pmatrix}. \tag{17}$$

Summarizing, the regressor matrix $\boldsymbol{Y}_U$ needed in control (2), together with its transpose for the adaptation law, is computed by (16) using $\boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}_r)$ from (14) and $\boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)$ from (17):

$$\boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{1r} & 2\,q_2\,\ddot{q}_{1r}+\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r} & q_2^2\,\ddot{q}_{2r}+q_2\,(\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r}) \\ 0 & -\dot{q}_1\dot{q}_{1r} & \ddot{q}_{2r}-q_2\,\dot{q}_1\dot{q}_{1r} \\ 0 & 0 & 0 \end{pmatrix}.$$

The same considerations can be repeated for $\boldsymbol{M}_K$, $\boldsymbol{S}_K$, and $\boldsymbol{Y}_{K,c}$, leading to $\boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)$ (these computations are left to the reader).

The modified adaptive control law (2) provides (global) asymptotic stability of the trajectory tracking error. The proof follows the same argument as in the complete adaptive case (simpler in the absence of friction terms), once the above partitioned notation for unknown and known terms has been introduced. Consider in fact the same positive definite Lyapunov candidate

$$V = \frac{1}{2}\boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{e}^T\boldsymbol{K}_P\,\boldsymbol{e} + \frac{1}{2}\tilde{\boldsymbol{a}}_U^T\,\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{a}}_U \geq 0, \tag{18}$$

where $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\boldsymbol{s} = \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}} = \dot{\boldsymbol{e}} + \boldsymbol{\Lambda}\boldsymbol{e}$, $\boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P > 0$, $\tilde{\boldsymbol{a}}_U = \boldsymbol{a}_U - \hat{\boldsymbol{a}}_U$, and the gain matrices $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and $\boldsymbol{\Gamma} > 0$ have been chosen as diagonal. Note that only the unknown dynamic coefficients, those that need to be updated online, and their estimates appear in the Lyapunov candidate. The time derivative of (18) is

$$\dot{V} = \frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_U^T\,\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{a}}}_U \tag{19}$$

The closed-loop dynamics, i.e., eq. (13) with the control (2), is

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} &= \boldsymbol{Y}_U\,(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\hat{\boldsymbol{a}}_U + \boldsymbol{Y}_K\,(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_K + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ &= \hat{\boldsymbol{M}}_U(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}_U(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{M}_K(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{S}_K(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}. \end{aligned} \tag{20}$$

Subtracting both sides of eq. (20) from the identity

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r = \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_U + \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_K$$

yields

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} &= \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\boldsymbol{a}_U - \hat{\boldsymbol{a}}_U) - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ &= \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\tilde{\boldsymbol{a}}_U - (\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}) \end{aligned} \tag{21}$$

where the known term $\boldsymbol{Y}_K \boldsymbol{a}_K$ has been cancelled. Substituting $\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}}$ from (21) into (19) and using the update law $\dot{\hat{\boldsymbol{a}}}_U = \boldsymbol{\Gamma}\, \boldsymbol{Y}_U^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \boldsymbol{s}$ gives

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\, \boldsymbol{s}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\, \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) \boldsymbol{s} + \boldsymbol{s}^T \boldsymbol{Y}_U(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \tilde{\boldsymbol{a}}_U - \boldsymbol{s}^T \left(\boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}\right) \\
&\quad + 2\, \boldsymbol{e}^T \boldsymbol{K}_P \dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_U^T\, \boldsymbol{\Gamma}^{-1} \cdot \boldsymbol{\Gamma}\, \boldsymbol{Y}_U^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \boldsymbol{s} \\
&= -\left(\dot{\boldsymbol{e}} + \boldsymbol{K}_D^{-1} \boldsymbol{K}_P\, \boldsymbol{e}\right)^T \left(\boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}\right) + 2\, \boldsymbol{e}^T \boldsymbol{K}_P \dot{\boldsymbol{e}} \\
&= -\dot{\boldsymbol{e}}^T \boldsymbol{K}_D\, \dot{\boldsymbol{e}} - \boldsymbol{e}^T \boldsymbol{K}_P \boldsymbol{K}_D^{-1} \boldsymbol{K}_P\, \boldsymbol{e} \leq 0,
\end{aligned}$$

where we used the skew-symmetry of $\dot{\boldsymbol{M}} - 2\, \boldsymbol{S}$ and the diagonality of the gain matrices. The proof is completed by Barbalat lemma and LaSalle theorem.

* * * * *