Robotics 2

June 11, 2021

Exercise #1

Suppose that a routine is available that computes numerically the pseudoinverse of a matrix $\boldsymbol{A}$, e.g., the `pinv` function in MATLAB, $\boldsymbol{A}^{\#} = \texttt{pinv}(\boldsymbol{A})$. Given a $m \times n$, **full rank** matrix $\boldsymbol{J}$, with $m < n$, and a $n \times n$, positive definite, symmetric weighting matrix $\boldsymbol{W}$, prove formally that the weighted pseudoinverse $\boldsymbol{J}^{\#}_{\boldsymbol{W}}$ can be computed as $\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2})$. As a verification, provide a simple numerical example with $m = 2$, $n = 3$.

Exercise #2

A single link mounted on a passive elastic support is moved on a horizontal plane by a torque τ applied by a motor to the revolute joint at its base, as sketched in Fig. 1. The generalized coordinates q_1 and q_2 are defined therein, together with the relevant dynamic parameters: mass m, distance $d > 0$ of the CoM from the joint, and barycentric inertia I_L of the link; stiffness $k > 0$ of the linear spring in the support. The spring is undeformed when $q_1 = 0$. Derive first the Lagrangian dynamic model of this simple robotic system. Next, address the following two dynamic problems.

a) An input torque $\tau_0 > 0$ is applied at $t = 0$, with the system at $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (zero initial state). Determine if the spring will initially be compressed or extended and if the link will start moving clockwise or counterclockwise. Provide the expression of the initial accelerations $\ddot{q}_1(0)$ and $\ddot{q}_2(0)$ of the two coordinates .

b) Starting at $t = 0$ in a generic non-zero initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$, define a control law $\tau = \tau(\boldsymbol{q}, \dot{\boldsymbol{q}})$ such that $\boldsymbol{q}_2(t)$ will exponentially converge to zero. At steady state, determine the residual dynamics of the other coordinate $q_1(t)$ and provide a physical interpretation of this result.

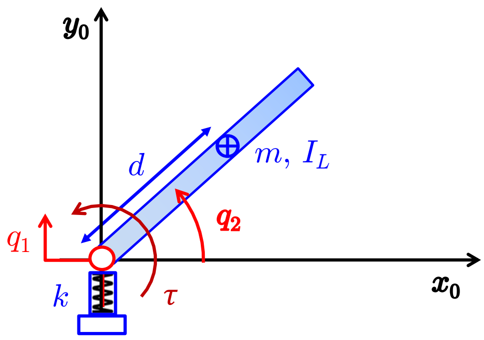

Figure 1: A single-link robotic system mounted on a flexible base.

Exercise #3

Draw a 3-dof robot whose dynamic model is given by

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \qquad \boldsymbol{M} = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ -m_3 g_0 \end{pmatrix},$$

with masses $m_i > 0$, $i = 1, 2, 3$, and $g_0 = 9.81$ [m/s^2].

Exercise #4

Consider the planar 3R robot with equal link lengths in Fig. 2 and absolute joint coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The robot is equipped with three motors producing torques $\boldsymbol{u} = (u_1, u_2, u_3)$ that perform work on the Denavit-Hartenberg angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, respectively. Provide the expression of the torque vector $\boldsymbol{u_q} = (u_{q1}, u_{q2}, u_{q3})$ that should appear on the right-hand side of the robot dynamic model written in terms of the absolute coordinates $\boldsymbol{q}$ (i.e., in $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u_q}$), as function of the components of $\boldsymbol{u}$.

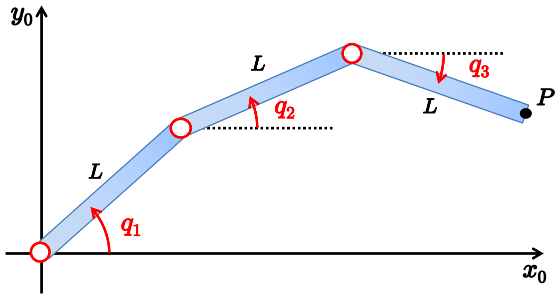

Figure 2: A planar 3R robot, with absolute joint coordinates $\boldsymbol{q}$ and equal links of length L.

Exercise #5

With reference to Fig. 3, the end-point P of a planar 2R robot should execute a desired trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, specified by a circular path in the Cartesian plane with a time-varying desired tangential velocity. Motion occurs on a horizontal plane and the circle should be traced counterclockwise. An initial position and/or velocity error between the robot end-point and the desired trajectory is present at $t = 0$. Later on, external disturbances may also affect occasionally the execution of the desired motion task. Design a torque control law for the robot such that the trajectory tracking error dynamics is exponentially stable, linear, and decoupled along the normal and tangential directions to the path. The error behaviors in these two directions (represented, respectively, by the $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ axes of a time-varying frame that moves with the desired task) should be critically damped, with a dominant reaction time to position errors in the normal direction which is about five times faster than in the tangential one. Assume that no kinematic singularities are encountered (as suggested also by the placement of the circle in Fig. 3).

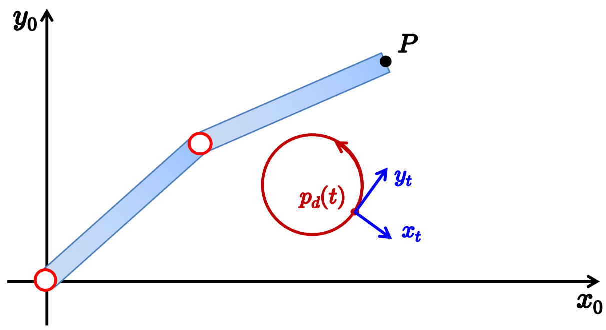

Figure 3: Tracking a circular Cartesian trajectory with a planar 2R robot.

[240 minutes (4 hours); open books]

Solution

June 11, 2021

Exercise #1

The weighted pseudoinverse $\boldsymbol{J}^{\#}_{\boldsymbol{W}}$ of a $m \times n$ matrix $\boldsymbol{J}$, with $\text{rank}\,\boldsymbol{J} = m < n$ and a symmetric matrix $\boldsymbol{W} > 0$, is given by

$$\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right)^{-1},$$

with the three relations holding

$$\boldsymbol{J}\boldsymbol{J}^{\#}_{\boldsymbol{W}}\boldsymbol{J} = \boldsymbol{J}, \qquad \boldsymbol{J}^{\#}_{\boldsymbol{W}}\boldsymbol{J}\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}, \qquad \left(\boldsymbol{J}\boldsymbol{J}^{\#}_{\boldsymbol{W}}\right)^T = \boldsymbol{J}\boldsymbol{J}^{\#}_{\boldsymbol{W}}, \tag{1}$$

but not the fourth one (i.e., in general $(\boldsymbol{J}^{\#}_{\boldsymbol{W}}\boldsymbol{J})^T \neq \boldsymbol{J}^{\#}_{\boldsymbol{W}}\boldsymbol{J}$). On the other hand, for a full row rank matrix $\boldsymbol{A}$ the `pinv` routine provides as output

$$\boldsymbol{A}^{\#} = \texttt{pinv}(\boldsymbol{A}) = \boldsymbol{A}^T\left(\boldsymbol{A}\boldsymbol{A}^T\right)^{-1}.$$

Therefore,

$$\begin{aligned}\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) &= \boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^T\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^T\right)^{-1}\\ &= \boldsymbol{W}^{-1/2}\,\boldsymbol{W}^{-1/2}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\,\boldsymbol{W}^{-1/2}\boldsymbol{J}^T\right)^{-1}\\ &= \boldsymbol{W}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right)^{-1} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}.\end{aligned}$$

As a numerical example with $m = 2$, $n = 3$, for

$$\boldsymbol{J} = \begin{pmatrix} 1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}, \qquad \boldsymbol{W} = \begin{pmatrix} 7 & & \\ & 8 & \\ & & 9\end{pmatrix},$$

we obtain

$$\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) = \begin{pmatrix} -0.9792 & 0.4583\\ -0.0417 & 0.0833\\ 0.6875 & -0.2083\end{pmatrix} = \boldsymbol{J}^{\#}_{\boldsymbol{W}},$$

which satisfies indeed the defining identities (1).

Note that the same formula holds also in the singular case ($\text{rank}\,\boldsymbol{J} < m$), being then

$$\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\texttt{pinv}\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right) = \boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}).$$

For instance, with

$$\boldsymbol{J} = \begin{pmatrix} 1 & 2 & 3\\ 4 & 8 & 12\end{pmatrix}, \qquad \text{rank}\,\boldsymbol{J} = 1,$$

and the same previous diagonal $\boldsymbol{W} > 0$, we obtain

$$\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) = \begin{pmatrix} 0.0051 & 0.0205\\ 0.0090 & 0.0358\\ 0.0119 & 0.0477\end{pmatrix} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}.$$

Exercise #2

We compute the kinetic energy T of the link and the potential energy U_e due to the elastic spring. The position and velocity of the center of mass of the link are

$$\boldsymbol{p}_c = \begin{pmatrix} d\cos q_2 \\ q_1 + d\sin q_2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_c = \dot{\boldsymbol{p}}_c = \begin{pmatrix} -d\sin q_2\, \dot{q}_2 \\ \dot{q}_1 + d\cos q_2\, \dot{q}_2 \end{pmatrix},$$

while the angular velocity of the link has only the z-component $\omega_z = \dot{q}_2$. Therefore, the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{q}_1^2 + d^2\dot{q}_2^2 + 2d\cos q_2\, \dot{q}_1\dot{q}_2\right) + \frac{1}{2} I_L\, \dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the inertia matrix given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m & md\cos q_2 \\ md\cos q_2 & I_L + md^2 \end{pmatrix}.$$

The velocity terms in the dynamic model are computed through the standard Christoffel's symbols. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix}^T - \boldsymbol{0} \right\} = \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix},$$

yielding

$$c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} = -md\sin q_2\, \dot{q}_2^2.$$

On the other hand, it is easy to see that

$$\boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.$$

The elastic potential U_e and its gradient are

$$U_e = \frac{1}{2} k\, q_1^2 \qquad \Rightarrow \qquad \nabla_{\boldsymbol{q}} U_e = \begin{pmatrix} k\, q_1 \\ 0 \end{pmatrix}$$

As a result, the dynamic model of the robotic system is

$$\begin{pmatrix} m & md\cos q_2 \\ md\cos q_2 & I_L + md^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -md\sin q_2\, \dot{q}_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} k\, q_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tau \end{pmatrix}. \tag{2}$$

For case a), at the initial state $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ and with an input $\tau = \tau_0 > 0$, we solve from (2) for the acceleration $\ddot{\boldsymbol{q}}(0)$ as

$$\begin{pmatrix} \ddot{q}_1(0) \\ \ddot{q}_2(0) \end{pmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{q}(0)) \begin{pmatrix} 0 \\ \tau_0 \end{pmatrix} = \frac{1}{\det \boldsymbol{M}(\boldsymbol{q}(0))} \begin{pmatrix} -md\cos q_2(0) \\ m \end{pmatrix} \tau_0 = \frac{\tau_0}{I_L} \begin{pmatrix} -d \\ 1 \end{pmatrix}. \tag{3}$$

As a consequence, the spring will be initially compressed ($\ddot{q}_1(0) < 0$) and the link will start moving counterclockwise ($\ddot{q}_2(0) > 0$).

For case b), we isolate $\ddot{q}_1$ from the first equation in (2),

$$\ddot{q}_1 = \frac{1}{m}\left(md\left(\sin q_2\, \dot{q}_2^2 - \cos q_2\, \ddot{q}_2\right) - k\, q_1\right),$$

and replace it in the second, obtaining thus

$$\left(I_L + md^2(1-\cos^2 q_2)\right)\ddot{q}_2 + md^2 \sin q_2 \cos q_2\, \dot{q}_2^2 - dk\, q_1 \cos q_2 = \tau. \tag{4}$$

Consider now the nonlinear feedback control law

$$\tau = \left(I_L + md^2(1-\cos^2 q_2)\right)\left(-k_d\, \dot{q}_2 + k_p\left(q_{2d} - q_2\right)\right) + md^2 \sin q_2 \cos q_2\, \dot{q}_2^2 - dk\, q_1 \cos q_2, \tag{5}$$

with $k_p > 0$ and $k_d > 0$, and any constant value for q_{2d}. Note that the (inertial) factor multiplying $\ddot{q}_2$ in this control law is always positive. Then, plugging the torque (5) into (4) will exactly linearize and stabilize the dynamics of the coordinate q_2 in a global sense, yielding

$$\ddot{q}_2 + k_d \dot{q}_2 + k_p q_2 = k_p q_{2,d} \qquad \Rightarrow \qquad q_2(t) \to q_{2d} \quad \text{exponentially.}$$

At steady state, one has $\dot{q}_2 = \ddot{q}_2 = 0$. Thus, the first dynamic equation in (2) provides

$$m\, \ddot{q}_1 + k\, q_1 = 0.$$

This is the dynamics of an undamped mass m suspended on a spring of stiffness k. If initially excited at some $t = \bar{t} > 0$, i.e., for $q_1(\bar{t}) \neq 0$ and/or $\dot{q}_1(\bar{t}) \neq 0$, the mass will oscillate forever as

$$q_1(t) = q_1(\bar{t}) \cos\omega(t-\bar{t}) + \frac{\dot{q}_1(\bar{t})}{\omega} \sin\omega(t-\bar{t}), \qquad \omega = \sqrt{\frac{k}{m}} > 0, \quad \forall t \geq \bar{t}.$$

Accordingly, the control law (5) boils down at steady state to the oscillatory command

$$\tau_{ss}(t) = -\left(dk \cos q_{2d}\right) q_1(t),$$

which will prevent the link from rotating.

Exercise #3

It is easy to recognize that the dynamic model refers to a Cartesian 3P robot with orthogonal axes, see Fig. 4. Only the third (vertical) prismatic joint is subject to gravity. The structure is also called a portal robot. It supports and moves heavy payloads and is usually equipped with an additional 3R spherical wrist mounted on the end effector.

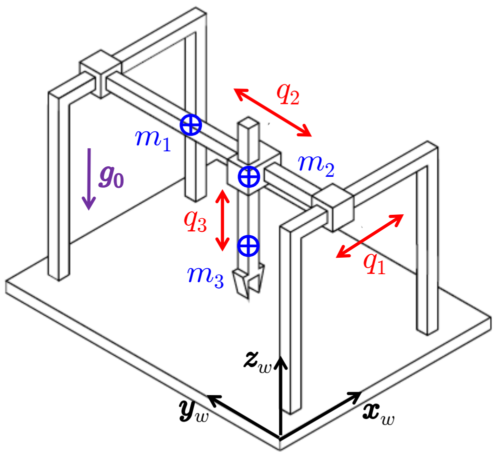

Figure 4: A Cartesian 3P robot with a portal structure.

Exercise #4

This is a straightforward application of the principle of virtual work. The (absolute) coordinates q_i, $i = 1,2,3$, in Fig. 2 are related to the (relative) joint variables θ_i, $i = 1,2,3$, of the Denavit-Hartenberg (DH) notation by the linear transformation

$$\begin{aligned} q_1 &= \theta_1 \\ q_2 &= \theta_1 + \theta_2 \\ q_3 &= \theta_1 + \theta_2 + \theta_3 \end{aligned} \qquad \Rightarrow \qquad \boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} = \boldsymbol{T}\boldsymbol{\theta}.$$

The joint torques $\boldsymbol{u}$ produced by the three motors and performing work on the DH angles $\boldsymbol{\theta}$ and the torques $\boldsymbol{u}_{\boldsymbol{q}}$ performing work on the $\boldsymbol{q}$ coordinates that appear in the dynamic model

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u}_{\boldsymbol{q}},$$

are related by the identity

$$\boldsymbol{u}_q^T \dot{\boldsymbol{q}} = \boldsymbol{u}^T \dot{\boldsymbol{\theta}} \qquad \Rightarrow \qquad \boldsymbol{u}_q^T \dot{\boldsymbol{q}} = \boldsymbol{u}_{\boldsymbol{q}}^T \boldsymbol{T} \dot{\boldsymbol{\theta}} = \boldsymbol{u}^T \dot{\boldsymbol{\theta}}, \quad \forall \dot{\boldsymbol{\theta}}$$

or

$$\boldsymbol{u} = \boldsymbol{T}^T \boldsymbol{u}_{\boldsymbol{q}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{u}_{\boldsymbol{q}} \qquad \Rightarrow \qquad \begin{aligned} u_1 &= u_{q1} + u_{q2} + u_{q3} \\ u_2 &= u_{q2} + u_{q3} \\ u_3 &= u_{q3} \end{aligned}$$

and its inverse

$$\boldsymbol{u}_{\boldsymbol{q}} = \boldsymbol{T}^{-T} \boldsymbol{u} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{u} \qquad \Rightarrow \qquad \begin{aligned} u_{q1} &= u_1 - u_2 \\ u_{q2} &= u_2 - u_3 \\ u_{q3} &= u_3. \end{aligned}$$

Exercise #5

The solution is provided by a feedback linearization control law designed to stabilize in a decoupled way the normal and tangential trajectory errors in the time-varying reference frame $(\boldsymbol{x}_t, \boldsymbol{y}_t)$ associated to the task. For this, having defined the position error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$, the error vector of interest in the rotated task frame is

$${}^t\boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\,\boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\,(\boldsymbol{p}_d - \boldsymbol{p}),$$

where

$${}^0\boldsymbol{R}_t(t) = \begin{pmatrix} \cos\alpha_d(t) & -\sin\alpha_d(t) \\ \sin\alpha_d(t) & \cos\alpha_d(t) \end{pmatrix}$$

is the 2×2 (planar) rotation matrix characterizing the current orientation (by the angle $\alpha_d(t)$) of the task frame $(\boldsymbol{x}_t, \boldsymbol{y}_t)$ moving with $\boldsymbol{p}_d(t)$ w.r.t. the absolute frame $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. Accordingly, the time derivative of ${}^t\boldsymbol{e}$ is[1]

$${}^t\dot{\boldsymbol{e}} = {}^0\boldsymbol{R}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\dot{\boldsymbol{R}}_t^T(t)\,\boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\,\left(\dot{\boldsymbol{e}} + \boldsymbol{S}^T(\dot{\alpha}_d(t))\,\boldsymbol{e}\right), \tag{6}$$

[1] In the following, the two formats of the expression of ${}^t\dot{\boldsymbol{e}}$ in (6), and later of ${}^t\ddot{\boldsymbol{e}}$ in (7), can be used equivalently: either the one containing time derivatives of the rotation matrix, or the one where these are substituted by their explicit computation.

with

$$\boldsymbol{S}(\dot{\alpha}_d(t)) = \begin{pmatrix} 0 & -\dot{\alpha}_d(t) \\ \dot{\alpha}_d(t) & 0 \end{pmatrix}.$$

Similarly, its second time derivative is

$$\begin{aligned} {}^t\ddot{\boldsymbol{e}} &= {}^0\boldsymbol{R}_t^T(t)\,\ddot{\boldsymbol{e}} + 2\,{}^0\dot{\boldsymbol{R}}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\ddot{\boldsymbol{R}}_t^T(t)\,\boldsymbol{e} \\ &= {}^0\boldsymbol{R}_t^T(t)\,\ddot{\boldsymbol{e}} + 2\,{}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + {}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\ddot{\alpha}_d(t))\,\boldsymbol{e} - {}^0\boldsymbol{R}_t^T(t)\boldsymbol{D}(\dot{\alpha}_d(t))\,\boldsymbol{e} \\ &= {}^0\boldsymbol{R}_t^T(t)\,\left(\ddot{\boldsymbol{e}} + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}\right), \end{aligned} \tag{7}$$

being

$$\boldsymbol{D}(\dot{\alpha}_d^2(t)) = -\left(\boldsymbol{S}^T(\dot{\alpha}_d(t))\right)^2 = \begin{pmatrix} \dot{\alpha}_d^2(t) & 0 \\ 0 & \dot{\alpha}_d^2(t) \end{pmatrix}.$$

Therefore, to satisfy the problem specifications, we should impose to the controlled robot the following linear and decoupled error dynamics

$${}^t\ddot{\boldsymbol{e}} + {}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} = \boldsymbol{0}, \tag{8}$$

with diagonal, positive definite gain matrices (task gains)

$${}^t\boldsymbol{K}_P = \begin{pmatrix} k_{P,norm} & 0 \\ 0 & k_{P,tang} \end{pmatrix} > 0, \qquad {}^t\boldsymbol{K}_D = \begin{pmatrix} k_{D,norm} & 0 \\ 0 & k_{D,tang} \end{pmatrix} > 0,$$

where the subscripts *norm* and *tang* are used to denote, respectively, the normal direction $\boldsymbol{x}_t$ and the tangential direction $\boldsymbol{y}_t$ of the current task frame.

Consider the dynamic model of the planar 2R robot (without gravity term, being this on a horizontal plane)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

and the direct (and first- and second-order differential) kinematics of the robot

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}), \qquad \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Assuming no singularities of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ are encountered, we apply the control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{9}$$

with

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e}\right) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}. \tag{10}$$

This gives

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e}\right) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}$$

or

$${}^0\boldsymbol{R}_t^T(t)\left(\ddot{\boldsymbol{e}} + \boldsymbol{S}^T(\dot{\alpha}_d(t))\,2\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}\right) + {}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} = \boldsymbol{0},$$

which is equivalent to the expression (8) of the desired behavior for the task error ${}^t\boldsymbol{e}(t)$.

Note that the commanded acceleration $\boldsymbol{a}$ in (10) can be equivalently written using the time derivatives of the task rotation matrix ${}^0\boldsymbol{R}_t(t)$ within the expressions of ${}^t\dot{\boldsymbol{e}}$ in (6) and ${}^t\ddot{\boldsymbol{e}}$ in (7). This would lead to

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} + 2\,{}^0\dot{\boldsymbol{R}}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\ddot{\boldsymbol{R}}_t^T(t)\,\boldsymbol{e}\right), \tag{11}$$

which produces indeed the same target result (8). Moreover, equation (10) can be rewritten only in terms of the position and velocity errors $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$ expressed in the base frame as

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(t)\,\dot{\boldsymbol{e}} + \boldsymbol{K}_P(t)\,\boldsymbol{e} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(t)\,(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) + \boldsymbol{K}_P(t)\,(\boldsymbol{p}_d - \boldsymbol{p}), \tag{12}$$

where we defined the two time-varying gain matrices associated to the task

$$\boldsymbol{K}_P(t) = {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_P\,{}^0\boldsymbol{R}_t^T(t) + {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T(t)\,\boldsymbol{S}^T(\dot{\alpha}_d(t)) + \boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))$$

and

$$\boldsymbol{K}_D(t) = {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T(t) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t)).$$

We note that the desired task acceleration $\ddot{\boldsymbol{p}}_d$ needs not to be rotated in the expression (12), and that extra terms appear in these time-varying gain matrices, related to the changing orientation of the task frame. If the same trajectory were assigned along a linear path, then $\dot{\alpha}_d = \ddot{\alpha}_d = 0$ and these gain matrices would become constant

$${}^t\boldsymbol{K}_P = {}^0\boldsymbol{R}_t\,\boldsymbol{K}_P\,{}^0\boldsymbol{R}_t^T, \qquad {}^t\boldsymbol{K}_D = {}^0\boldsymbol{R}_t\,\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T.$$

Finally, in order to assign a critical damping and the desired time scale separation between the normal and tangential error components, we use standard results on linear second-order dynamic systems. In order to impose two asymptotically stable eigenvalues/poles in

$$-a \pm i\,b = -\omega\left(\zeta \pm i\sqrt{1-\zeta^2}\right) \qquad \text{with } \omega > 0,\ \zeta > 0,\ a = \zeta\omega > 0,\ b = \sqrt{1-\zeta^2}\omega \geq 0,$$

to the characteristic equation $s^2 + k_D s + k_P = 0$ that governs the dynamics of each component of the tracking error in the task frame, one needs to choose

$$k_D = 2\zeta\omega, \qquad k_P = \omega^2.$$

The coefficient ζ affects the way oscillations at the natural frequency ω are damped. Moreover, the error response over time is enveloped by a decaying exponential $e^{-\zeta\omega t}$. Therefore, to achieve our design target for the transient errors, we choose first a common critical damping $\zeta = 0.7$ (or larger) for both the tangential and normal directions. Then, once a sufficiently high frequency $\omega_{tang} > 0$ has been selected for the tangential direction, we set

$$k_{P,tang} = \omega_{tang}^2, \qquad k_{D,tang} = 2\zeta\omega_{tang},$$

and

$$k_{P,norm} = \omega_{norm}^2 = (5\,\omega_{tang})^2 = 25\,\omega_{tang}^2, \qquad k_{D,norm} = 2\zeta\omega_{norm} = 10\zeta\omega_{tang}.$$

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