Robotics II

July 11, 2019

Exercise 1

The 3R planar robot in Fig. 1 is commanded at the joint velocity level. The robot has to perform two tasks simultaneously, if possible. The first task is to keep the second link vertical and upwards at any time. The second task is to follow a desired cyclic Cartesian trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, $t \in [0, T]$, for the end-effector position. Provide the actual expressions of all terms in a task priority control law, with the given order of tasks. Determine the robot configurations for which both tasks can be perfectly executed together, and define accordingly the region of the plane where this can happen. Which would be the control law in this case? With link lengths $L_1 = L_2 = L_3 = 0.5$ [m], compute the numerical value of $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ using the task priority law at $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad] for $\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^T$ [m/s]. Finally, when errors are present during the execution of these tasks, how should the control law be modified in order to reduce them?

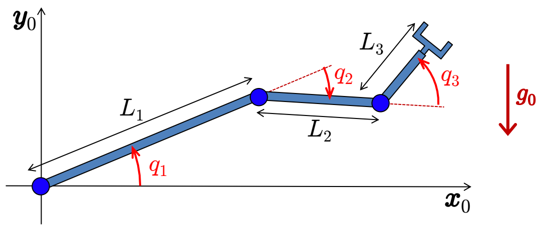

Figure 1: A 3R planar robot with its joint variables and generic link lengths.

Exercise 2

The RP planar robot shown in Fig. 2 lives in a vertical plane and may collide with some (human) obstacle when in motion. Its controller is therefore equipped with a momentum-based collision detection algorithm that generates a residual vector $\boldsymbol{r} \in \mathbb{R}^2$ as monitoring signal. Provide the explicit symbolic expressions of the two scalar components of $\boldsymbol{r}$ (introduce the needed kinematic and/or dynamic quantities). Suppose that, at time $t = t_c$, a collision occurs on the robot tip with an impact force $\boldsymbol{F}_c$ that is purely normal to the second link. What will be the instantaneous value of the time derivative of the residual vector $\dot{\boldsymbol{r}}(t_c)$?

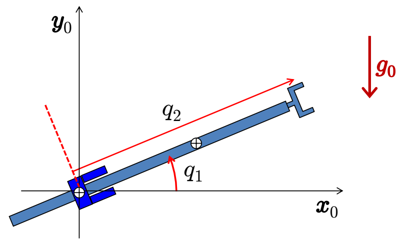

Figure 2: A RP robot moving in a vertical plane.

Exercise 3

An actuated pendulum under gravity should perform a rest-to-rest swing-up maneuver from the downward position $\theta(0) = 0$ to the upward position $\theta(T) = \pi$ in a total time T, using a bang-coast-bang acceleration profile with symmetric acceleration and deceleration phases, each of duration $T_s = T/4$. The link of the pendulum is a thin rod of length $l = 2$ [m], with uniformly distributed mass $m = 10$ [kg] and baricentral inertia $I_c = ml^2/12$ [kg·m^2]. The motor at the link base can deliver a maximum absolute torque $\tau_{max} = 200$ [Nm]. Determine the minimum time T_{min} in the chosen class of trajectories such that the motion is feasible. Sketch the resulting angular position, velocity, acceleration, and torque profiles.

[open books, 210 minutes]

Solution

July 11, 2019

Exercise 1

We will use throughout the DH coordinates indicated in Fig. 2. The first task, i.e., keeping the second link vertical and upwards, is one-dimensional ($m_1 = 1$) and is specified by

$$f_1(\boldsymbol{q}) = q_1 + q_2 = r_{d1} = \frac{\pi}{2} \qquad \Rightarrow \qquad \boldsymbol{J}_1 = \frac{\partial f_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T, \quad \dot{r}_{d1} = 0.$$

The second task, i.e., following a desired cyclic Cartesian trajectory $\boldsymbol{p}_d(t)$ with the robot tip, is two-dimensional ($m_2 = 2$) and is specified by

$$\boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} L_1c_1 + L_2c_{12} + L_3c_{123} \\ L_1s_1 + L_2s_{12} + L_3s_{123} \end{pmatrix} = \boldsymbol{r}_{d2} = \boldsymbol{p}_d(t)$$

$$\Rightarrow \quad \boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(L_1s_1 + L_2s_{12} + L_3s_{123}) & -(L_2s_{12} + L_3s_{123}) & -L_3s_{123} \\ L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123} \end{pmatrix}, \quad \dot{\boldsymbol{r}}_{d2} = \dot{\boldsymbol{p}}_d.$$

with the usual compact notation for the trigonometric functions (e.g., $c_{12} = \cos(q_1 + q_2)$).

The basic Task Priority (TP) method for two ordered tasks provides

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}\dot{r}_{d1} + \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1\right)\boldsymbol{v}_1, \qquad \text{with} \quad \boldsymbol{v}_1 = \left(\boldsymbol{J}_2(\boldsymbol{q})(\boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1)\right)^{\#}\left(\dot{\boldsymbol{r}}_{d2} - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}\dot{r}_{d1}\right), \tag{1}$$

where $\boldsymbol{P}_1 = \boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1$ is the (here, constant) projection matrix in the null space of the first task and no extra term has been used in the null space of the second task ($\boldsymbol{v}_2 = \boldsymbol{0}$). Since $\dot{r}_{d1} = 0$ in this case, and being $\boldsymbol{P}(\boldsymbol{J}\boldsymbol{P})^{\#} = (\boldsymbol{J}\boldsymbol{P})^{\#}$ for any projection matrix $\boldsymbol{P}$, equation (1) simplifies to

$$\dot{\boldsymbol{q}} = (\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1)^{\#}\,\dot{\boldsymbol{p}}_d. \tag{2}$$

From

$$\boldsymbol{J}_1^{\#} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}, \qquad \boldsymbol{P}_1 = \begin{pmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1 = \begin{pmatrix} -0.5L_1s_1 & 0.5L_1s_1 & -L_3s_{123} \\ 0.5L_1c_1 & -0.5L_1c_1 & L_3c_{123} \end{pmatrix}. \tag{3}$$

While the first two columns of the matrix in (3) are always dependent, it is easy to see that its rank is full unless $\sin(q_2 + q_3) = 0$. With the joint velocity command (2), the first task will always be satisfied if the constraint $f_1(\boldsymbol{q}) = \pi/2$ holds at the start, whereas the second task will be satisfied either exactly or in a least squares sense, depending on the current robot configuration and on the direction of the desired velocity $\dot{\boldsymbol{p}}_d$.

In order to verify when both tasks can be achieved simultaneously, we impose $q_1(t) + q_2(t) \equiv \pi/2$ at all times. From the direct kinematics of the robot tip $\boldsymbol{p} = \boldsymbol{f}_2(\boldsymbol{q})$, one obtains then the reduced form

$$\boldsymbol{p}_{\text{red}} = \boldsymbol{f}_2(\boldsymbol{q})|_{q_1+q_2=\pi/2} = \begin{pmatrix} L_1c_1 - L_3s_3 \\ L_1s_1 + L_2 + L_3c_3 \end{pmatrix} = \boldsymbol{p}_{\text{red}}(q_1, q_3).$$

In order to keep the constraint on the first task satisfied, we need to have $\dot{q}_2 = -\dot{q}_1$ for the second joint command. The two remaining joints $\boldsymbol{q}_{\text{red}} = \begin{pmatrix} q_1 & q_3 \end{pmatrix}^T$ will produce a tip velocity

$$\dot{\boldsymbol{p}}_{\text{red}} = \boldsymbol{J}_{\text{red}}(\boldsymbol{q}_{\text{red}})\dot{\boldsymbol{q}}_{\text{red}}, \qquad \text{with} \quad \boldsymbol{J}_{\text{red}}(\boldsymbol{q}_{\text{red}}) = \frac{\partial \boldsymbol{p}_{\text{red}}(\boldsymbol{q}_{\text{red}})}{\partial \boldsymbol{q}_{\text{red}}} = \begin{pmatrix} -L_1s_1 & -L_3c_3 \\ L_1c_1 & -L_3s_3 \end{pmatrix}, \quad \dot{\boldsymbol{q}}_{\text{red}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_3 \end{pmatrix}.$$

As a result, the robot will be able to generate also any desired $\dot{\boldsymbol{p}}_{\text{red}} = \dot{\boldsymbol{p}}_d \in \mathbb{R}^2$, provided that

$$\det \boldsymbol{J}_{\text{red}} = L_1 L_3 \cos(q_3 - q_1) \neq 0 \qquad \Longleftrightarrow \qquad q_3 \neq q_1 \pm \frac{\pi}{2}. \tag{4}$$

The actual region of the plane where the two tasks can be performed simultaneously is illustrated in Fig. 3 for some specific but arbitrary values of the link lengths. The second link is always kept vertical and upwards. The circular annulus has outer radius R_{out}, inner radius R_{in}, and center C_{WS} on the axis $\boldsymbol{y}_0$, computed by simple geometric reasoning as

$$R_{\text{out}} = \frac{(L_1 + L_2 + L_3) - (-L_1 + L_2 - L_3)}{2} = L_1 + L_3, \qquad R_{\text{in}} = R_{\text{out}} - 2L_3 = |L_1 - L_3|,$$

and

$$C_{\text{WS}} = \frac{(L_1 + L_2 + L_3) + (-L_1 + L_2 - L_3)}{2} = L_2.$$

For $L_1 = L_2 = L_3 = L$, this is a full circle ($R_{\text{in}} = 0$) of radius $R_{\text{out}} = 2L$, centered at $C_{\text{WS}} = L$ on axis $\boldsymbol{y}_0$.

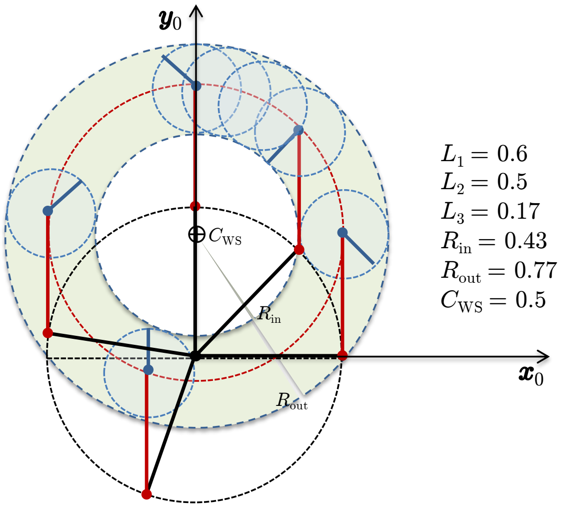

Figure 3: The Cartesian region of compatibility for both tasks (drawn for a specific set of link lengths).

As mentioned, when $q_1 + q_2 = \pi/2$ and (4) hold, then the TP method (2) will generate the exact (and unique) solution for both tasks. In these conditions, the same solution is obtained with $\dot{\boldsymbol{q}}_{\text{red}} = \boldsymbol{J}_{\text{red}}^{-1}(\boldsymbol{q}_{\text{red}})\dot{\boldsymbol{p}}_d$ and $\dot{q}_2 = -\dot{q}_{red,1}$. Equivalently, because of the assumed consistency of the two tasks, the problem can be solved also by the Extended Jacobian method (since $n = m_1 + m_2 = 3$):

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{p}_1 \\ \dot{\boldsymbol{p}}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q})\dot{\boldsymbol{r}}_d = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} 0 \\ \dot{\boldsymbol{p}}_d \end{pmatrix}. \tag{5}$$

We have in fact $\det \boldsymbol{J}_E(\boldsymbol{q}) = -L_1 L_3 \sin(q_2 + q_3)$. So, when the second link is kept vertical and upwards ($q_1 + q_2 = \pi/2$), the two singularities of the Extended Jacobian matrix ($q_2 + q_3 = \{0, \pi\}$) correspond exactly to having $q_3 = q_1 \pm \pi/2$, i.e., the violation of condition (4).

With the link lengths $L_1 = L_2 = L_3 = 0.5$ [m] and for the given desired tip velocity $\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^T$ [m/s], when the robot is, e.g., in the configuration $\boldsymbol{q}_b = \begin{pmatrix} 0 & \pi/2 & \pi/3 \end{pmatrix}^T$ (condition (4) holds), then

$$\boldsymbol{J}_E(\boldsymbol{q}_b) = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2(\boldsymbol{q}_b) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -0.75 & -0.75 & -0.25 \\ 0.067 & -0.433 & -0.433 \end{pmatrix}, \quad (\boldsymbol{J}_2(\boldsymbol{q}_b)\boldsymbol{P}_1)^{\#} = \begin{pmatrix} -3.4641 & 2 \\ 3.4641 & -2 \\ -4 & 0 \end{pmatrix},$$

and the joint velocity provided by (2) or by (5) is

$$\dot{\boldsymbol{q}}_b = \left(\boldsymbol{J}_2(\boldsymbol{q}_b)\boldsymbol{P}_1\right)^{\#}\dot{\boldsymbol{p}}_d = \boldsymbol{J}_E^{-1}(\boldsymbol{q}_b)\begin{pmatrix} 0 \\ \dot{\boldsymbol{p}}_d \end{pmatrix} = \begin{pmatrix} -1.3464 \\ 1.3464 \\ -0.4 \end{pmatrix} \text{[rad/s]} \quad \Rightarrow \quad \begin{cases} \boldsymbol{J}_1\dot{\boldsymbol{q}}_b = 0 = \dot{r}_{d1} \\ \boldsymbol{J}_2(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_b = \begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix} = \dot{\boldsymbol{r}}_{d2}. \end{cases}$$

On the other hand, when the robot is in the requested configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad], the two tasks are inconsistent (condition (4) is violated). In this situation, the robot end effector is on the outer boundary of the Cartesian region of compatibility, and the desired tip velocity points outside. The task priority law (2) provides in this case

$$\dot{\boldsymbol{q}}_0 = \left(\boldsymbol{J}_2(\boldsymbol{q}_0)\boldsymbol{P}_1\right)^{\#}\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 & 2/3 \\ 0 & -2/3 \\ 0 & 4/3 \end{pmatrix}\begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \text{[rad/s]} \;\Rightarrow\; \begin{cases} \boldsymbol{J}_1\dot{\boldsymbol{q}}_0 = 0 = \dot{p}_{d1} \\ \boldsymbol{J}_2(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} = \dot{\boldsymbol{p}}_0 \neq \dot{\boldsymbol{p}}_d. \end{cases}$$

Note that the computed solution $\dot{\boldsymbol{q}}_0$ will realize only part of the desired tip velocity $\dot{\boldsymbol{p}}_d$ requested as secondary task, namely the component of $\dot{\boldsymbol{p}}_d \in \mathcal{R}\{\boldsymbol{J}_2(\boldsymbol{q}_0)\}$ (see Fig. 4).

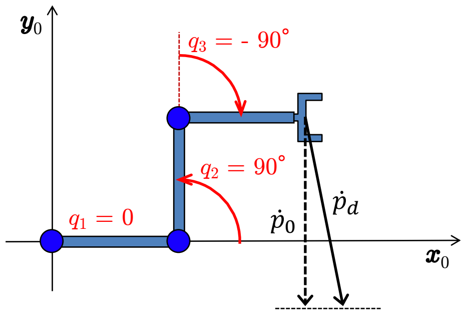

Figure 4: The specified secondary task velocity $\dot{\boldsymbol{p}}_d$ for the actual 3R planar robot in the configuration $\boldsymbol{q}_0$ and the realized one $\dot{\boldsymbol{p}}_0$.

Finally, suppose that errors $e_1 = r_{d1} - f_1(\boldsymbol{q}) = \pi/2 - (q_1 + q_2) \neq 0$ and/or $\boldsymbol{e}_2 = \boldsymbol{r}_{d2} - \boldsymbol{f}_2(\boldsymbol{q}) = \boldsymbol{p}_d - \boldsymbol{f}_2(\boldsymbol{q}) \neq \boldsymbol{0}$ are present during the simultaneous execution of the tasks. The task priority scheme (1) will be modified by introducing an error feedback term in both tasks as

$$\begin{aligned} \dot{\boldsymbol{q}}_c &= \boldsymbol{J}_1^{\#}\left(\dot{r}_{d1} + k_1 e_1\right) + \boldsymbol{P}_1\left(\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1\right)^{\#}\left(\dot{\boldsymbol{r}}_{d2} + \boldsymbol{K}_2\boldsymbol{e}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}\left(\dot{r}_{d1} + k_1 e_1\right)\right) \\ &= \boldsymbol{J}_1^{\#} k_1 e_1 + \left(\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1\right)^{\#}\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_2\boldsymbol{e}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#} k_1 e_1\right), \end{aligned} \tag{6}$$

with a scalar gain $k_1 > 0$ and a (typically, diagonal) matrix gain $\boldsymbol{K}_2 > 0$. Since $\boldsymbol{J}_1\dot{\boldsymbol{q}}_c = k_1 e_1$, we always have $\dot{e}_1 = -k_1 e_1$ and the error on the first task will exponentially converge to zero. On the other hand, the control law (6) will generate the largest possible reduction (or, in the worst case, the smallest increase) of the error on the second task, without ever affecting the first task.

Exercise 2

Based on the dynamic model of the RP planar robot

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \qquad \text{with } \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}, \tag{7}$$

we need to derive the dynamic elements that appear in the expression of the residual vector

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left[\boldsymbol{p}(t) - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r} \right) ds - \boldsymbol{p}(0) \right], \tag{8}$$

where $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is the generalized momentum and matrix $\boldsymbol{K}_I > 0$ is diagonal. Without loss of generality, we can assume that the robot is at rest at the beginning of the experiment, i.e, $\boldsymbol{p}(0) = \boldsymbol{0}$.

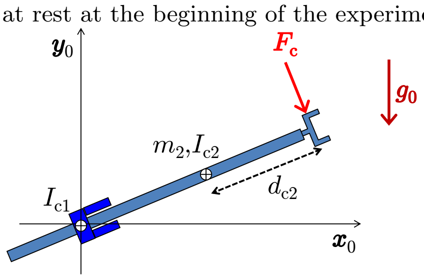

Figure 5: Definition of the relevant dynamic parameters for the RP robot of Fig. 2. Also shown is a collision force $\boldsymbol{F}_c$ acting at the tip along the normal direction to the second link.

With reference to the dynamic parameters defined in Fig. 5, for the kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

since $\boldsymbol{p}_{c2} = (q_2 - d_{c2}) \begin{pmatrix} \cos q_1 & \sin q_1 \end{pmatrix}^T$ and $\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2}$, we have

$$T_1 = \frac{1}{2} I_{c1}\, \dot{q}_1^2, \qquad T_2 = \frac{1}{2} I_{c2}\, \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}\left(I_{c2} + m_2 (q_2 - d_{c2})^2\right) \dot{q}_1^2 + \frac{1}{2} m_2\, \dot{q}_2^2.$$

The robot inertia matrix is then

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1} + I_{c2} + m_2 (q_2 - d_{c2})^2 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{9}$$

From this, we compute the Coriolis/centrifugal terms using the matrices of Christoffel symbols

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right) \right], \qquad i = 1, 2.$$

We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2 (q_2 - d_{c2}) \\ m_2 (q_2 - d_{c2}) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2 (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix},$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \dot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} 2 m_2 \,(q_2 - d_{c2})\, \dot{q}_1 \dot{q}_2 \\ -m_2 \,(q_2 - d_{c2})\, \dot{q}_1^2 \end{pmatrix}. \tag{10}$$

A factorization $\boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}$ in (7) such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric is given by

$$\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_2 \,(q_2 - d_{c2})\, \dot{q}_2 & m_2 \,(q_2 - d_{c2})\, \dot{q}_1 \\ -m_2 \,(q_2 - d_{c2})\, \dot{q}_1 & 0 \end{pmatrix}. \tag{11}$$

For the potential energy

$$U = U_1 + U_2 = U(\boldsymbol{q}),$$

we have

$$U_1 = \text{constant}, \qquad U_2 = m_2 g_0 \,(q_2 - d_{c2}) \sin q_1,$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} m_2 g_0 \,(q_2 - d_{c2}) \cos q_1 \\ m_2 g_0 \sin q_1 \end{pmatrix}. \tag{12}$$

From the expressions (9) and (11–12), we finally obtain

$$r_1(t) = k_{I1}\left[\left(I_{c1} + I_{c2} + m_2 (q_2 - d_{c2})^2\right)\dot{q}_1 - \int_0^t \left(\tau_1 - m_2 g_0 \,(q_2 - d_{c2}) \cos q_1 + r_1\right) ds\right],$$

and

$$r_2(t) = k_{I2}\left[m_2 \dot{q}_2 - \int_0^t \left(\tau_2 + m_2\,(q_2 - d_{c2})\,\dot{q}_1^2 - m_2 g_0 \sin q_1 + r_2\right) ds\right].$$

Suppose now that, at time $t = t_c$, a collision force $\boldsymbol{F}_c$ acts at the robot tip in the orthogonal direction to the second link and with an intensity $F \neq 0$ (see again Fig. 5). The Jacobian $\boldsymbol{J}_c(\boldsymbol{q})$ associated to the contact point and the contact force are then

$$\boldsymbol{J}_c(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix}, \qquad \boldsymbol{F}_c = F \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix},$$

while the resulting torque at the joint is computed as

$$\boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \begin{pmatrix} F\, q_2 \\ 0 \end{pmatrix}.$$

From the nominal behavior of the residual vector $\boldsymbol{r}$, being $\boldsymbol{r}(t) = \boldsymbol{0}$ for all $t \in [0, t_c]$, it follows that

$$\dot{\boldsymbol{r}}(t_c) = \boldsymbol{K}_I\left(\boldsymbol{\tau}_c(t_c) - \boldsymbol{r}(t_c)\right) \qquad \Rightarrow \qquad \begin{cases} \dot{r}_1(t_c) = k_{I1} F(t_c)\, q_2(t_c) \\ \dot{r}_2(t_c) = 0. \end{cases}$$

Although the collision occurs on the second link, the second component of the residual will not be affected immediately; in fact, $\boldsymbol{F}_c$ is not producing work on q_2, due to the specific direction assumed for the impact force.

Exercise 3

The acceleration profile for the rest-to-rest motion trajectory $\theta(t)$ is *assigned* to be of the bang-coast-bang type, having symmetric initial and final acceleration/deceleration phases, each of duration $T_s = T/4$ and with $\ddot{\theta} = \pm A$, and a cruising phase that lasts for half of the motion time, i.e., $T/2$, with constant velocity $\dot{\theta} = V$. From this motion structure, it is easy to compute the following quantities:

$$V = \dot{\theta}\left(\frac{T}{4}\right) = A\,\frac{T}{4}, \qquad \Delta\theta_s = \theta\left(\frac{T}{4}\right) = \frac{1}{2} A \left(\frac{T}{4}\right)^2 = \frac{AT^2}{32}, \qquad \Delta\theta = \theta(T) = 2\Delta\theta_s + V\frac{T}{2} = \frac{3AT^2}{16}.$$

Thus, for a desired total displacement $\Delta\theta > 0$ and a given motion time T, we have for the acceleration A and cruise velocity V

$$A = \frac{16\Delta\theta}{3T^2} > 0 \qquad \Rightarrow \qquad V = \frac{4\Delta\theta}{3T} > 0. \tag{13}$$

The swing-up maneuver from $\theta(0) = 0$ to $\Delta\theta = \theta(T) = \pi$ in time T needs then an acceleration/deceleration $A = \pm 16\pi/(3T^2)$ in the first and third motion phases. Note that, when the acceleration phase ends at time $t = T_s = T/4$, the performed motion will be $\Delta\theta_s = \Delta\theta/6 = \pi/6$. By symmetry, when the deceleration phase begins at time $t = T - T_s = 3T/4$, the performed motion completed so far will be $\Delta\theta - \Delta\theta_s = 5\Delta\theta/6 = 5\pi/6$.

With the above in mind, consider the dynamics of the actuated pendulum

$$I\ddot{\theta} + m g_0 d \sin\theta = \tau, \tag{14}$$

where $\theta = 0$ corresponds to the downward equilibrium and the dynamic parameters are given by

$$d = \frac{l}{2} = 1 \text{ [m]}, \; mg_0d = 98.1 \text{ [kg·m}^2\text{]}, \; I = I_c + md^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3} = \frac{40}{3} = 13.33 \text{ [kg·m}^2\text{]}.$$

By inverse dynamics on (14), the torque needed to perform the desired motion during the three phases is:

$$\tau(t) = \begin{cases} IA + mg_0d\sin\theta(t), & \theta \in [0, \pi/6), & \text{phase I: } t \in [0, T/4), \\ mg_0d\sin\theta(t), & \theta \in [\pi/6, 5\pi/6), & \text{phase II: } t \in [T/4, 3T/4), \\ -IA + mg_0d\sin\theta(t), & \theta \in [\pi/6, \pi], & \text{phase III: } t \in [3T/4, T]. \end{cases} \quad (15)$$

The gravity contribution to the inverse dynamics torque is maximum at the midpoint of motion, i.e., at $\theta = \pi/2$, is independent of the total motion time, and is equal to $\tau_g = mg_0d < \tau_{max}$. Note that if it were $\tau_g > \tau_{max}$, then actuation would be too weak to perform the intended task (even when moving the pendulum very slowly, with an arbitrarily long motion time T).

Further, from (13) and (15) it is easy to see that, when speeding up motion by uniformly reducing T, the inertial torque component in the first phase will increase quadratically and the maximum required torque will be attained at the end of the first phase, where the gravity contribution is the largest (and has the same sign of the acceleration). Thus, for feasibility we require that

$$IA + mg_0d\sin\Delta\theta_s = \frac{16\pi I}{3T^2} + mg_0d\sin\frac{\pi}{6} \le \tau_{max},$$

and the optimal motion time will be defined as the lower bound for all feasible motion times,

$$T \ge \sqrt{\frac{16\pi I}{3\left(\tau_{max} - mg_0d\sin(\pi/6)\right)}} = T_{min}.$$

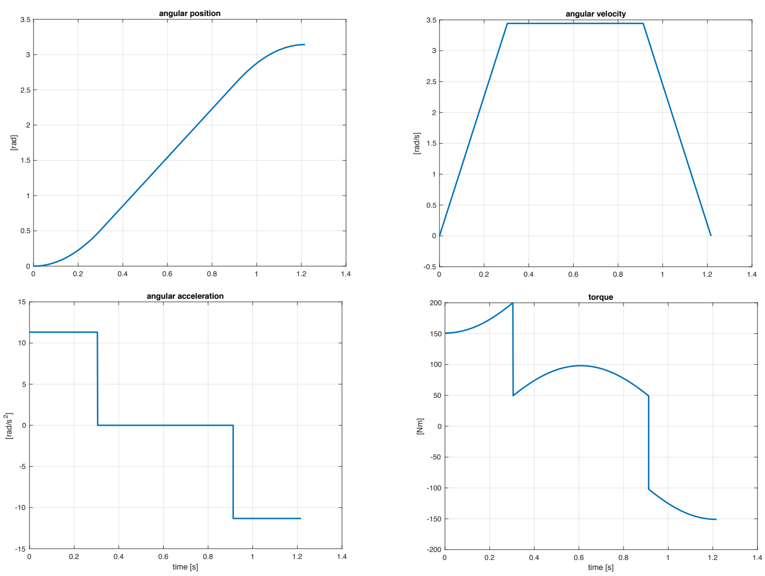

Figure 6: Kinematic (position, velocity, and acceleration) and dynamic (torque) profiles of the minimum time rest-to-rest swing-up maneuver.

Plugging in the numerical data, we find the optimal time $T_{min} = 1.2165$ [s]. The maximum torque during motion is indeed $\tau_{max} = 200$ [Nm], reached at the single instant $t = T_{min}/4 = 0.3041$ [s]. Accordingly, we obtain from (13) $A = 11.3212$ [rad/s^2] and $V = 3.4432$ [rad/s]. Figure 6 shows the resulting time profiles of the angular position, velocity and acceleration, and of the commanded torque $\tau(t)$.

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