Robotics II

January 11, 2018

Exercise 1

The RP planar robot in Fig. 1, with coordinates $\boldsymbol{q} = (q_1, q_2)$ and parameters m_2, d_{c2}, I_1 and I_2 defined therein, should execute a task defined by a time-varying trajectory $y_d(t) \in \mathbb{R}$ for the height of its end-effector.

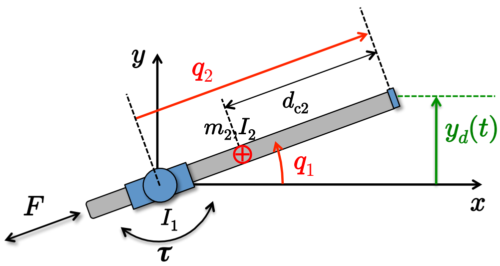

Figure 1: A RP planar robot with the relevant parameters and variables.

Assuming as input command the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$, determine the explicit expressions of the kinematic control laws that execute the task in nominal conditions, recover exponentially from any task error, and

- minimize $\frac{1}{2}\|\dot{\boldsymbol{q}}\|^2$: which is the theoretical pitfall of this solution?
- minimize the weighted norm $\frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{W}\dot{\boldsymbol{q}}$, with constant $\boldsymbol{W} = \mathrm{diag}\{w_1, w_2\} > 0$; what happens for very large ratios w_1/w_2 (in the limit $\to \infty$); and for $w_2/w_1 \to \infty$?
- minimize the kinetic energy $T = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, being $\boldsymbol{M}(\boldsymbol{q}) > 0$ the robot inertia matrix.

Exercise 2

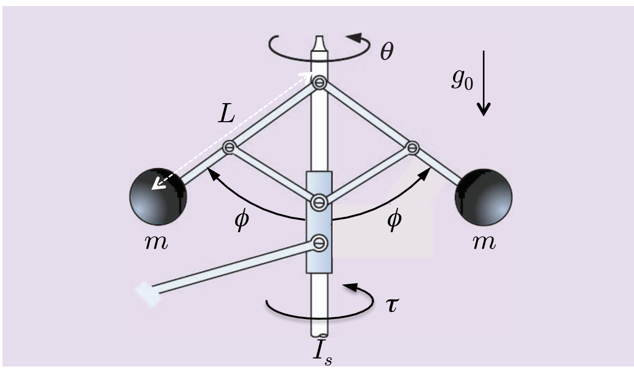

Figure 2: The Boulton-Watt governor and a scheme with definition of parameters and variables.

Figure 2 shows a picture and a simplified scheme of the famous Boulton-Watt centrifugal governor, a system invented to regulate the rotational speed of a steam engine by a mechanical leverage (feedback) opening a valve that provides steam under pressure to the engine. We consider here only the so-called *open-loop* dynamic behavior of the system, under the action of an external torque $\tau \in \mathbb{R}$ applied to the main rotating shaft.

Assume that:

- the main shaft has an inertia I_s around its rotation axis
- the two balls have identical mass m that is concentrated at the end of a link of length L
- the links and all other linkages have negligible masses
- a viscous friction torque with coefficient $f_v > 0$ is acting on the main shaft
- all other frictional effects are negligible.

Derive the complete dynamic model of this system using a Lagrangian formalism. Assuming knowledge of the geometric parameter L, provide a linear parametrization of the dynamics in terms of its dynamic coefficients. Find the value of the constant torque τ_Ω to be applied for sustaining a steady-state rotation at a given angular speed $\Omega > 0$. Finally, design a nonlinear feedback for τ so as to achieve partial feedback linearization of the system, i.e., exact linearization by feedback of only part of the closed-loop dynamics, in this case of one of the two coordinates.

Exercise 3

Consider the design of impedance control laws and force control laws for the 1-dof example, shown in Fig. 3, namely a single mass m that moves on a frictionless horizontal plane under the action of a commanded force $f \in \mathbb{R}$ and of a contact force $f_c \in \mathbb{R}$.

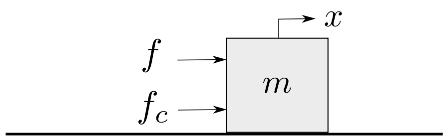

Figure 3: A mass m subject to a commanded force f and a contact force f_c.

In particular:

- The impedance controllers should work with a generic time-varying, smooth position reference $x_d(t)$, either with or without the use of a load cell that can measure the contact force f_c. Illustrate the properties of the obtained closed-loop systems.
- What happens when $x_d(t)$ degenerates to a constant? What happens during free motion, when $f_c = 0$?
- For $m = 5$ [kg], design the control parameters of the impedance law so that the dynamics of the position error $e = x_d - x$ in the closed-loop system is characterized by a pair of asymptotically stable complex poles with natural frequency $\omega_n = 10$ [rad/s] and critical damping ratio $\zeta = 0.7071$.
- On the other hand, the force controllers should be able to regulate the (measured) contact force f_c to a constant value f_d, using any combination of desired force feedforward and force error feedback. Illustrate the properties of the obtained closed-loop systems.
- What happens during free motion, when $f_c = 0$ and a constant contact force f_d is desired?

[150 minutes; open books]

Solution

January 11, 2018

Exercise 1

The problem deals with kinematic redundancy since the RP robot has $n = 2$ joints and the required task is scalar $m = 1$. The task output function and its Jacobian are

$$y(\boldsymbol{q}) = q_2 \sin q_1, \quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial y(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix}. \tag{1}$$

The 1×2 task Jacobian loses rank (vanishes) iff $q_1 = \{0, \pi\}$ and $q_2 = 0$ simultaneously.

The minimization of the squared norm of $\dot{\boldsymbol{q}}$ is achieved by the use of the pseudoinverse of the task Jacobian. Out of singularities, $\boldsymbol{J}^{\#} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ and the kinematic control law takes the expression

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\dot{y}_d + k(y_d - y(\boldsymbol{q}))\right) = \frac{1}{s_1^2 + q_2^2 c_1^2} \begin{pmatrix} q_2 c_1 \\ s_1 \end{pmatrix} \left(\dot{y}_d + k(y_d - q_2 \sin q_1)\right), \tag{2}$$

where $k > 0$ is a control gain that guarantees exponential recovery from transient errors, i.e., $\dot{e}(t) = -k e(t)$, with $e = y_d - q_2 \sin q_1 \neq 0$, during task execution. The pitfall of (2) is that the norm $\|\dot{\boldsymbol{q}}\|$ involves mixed angular (the revolute joint velocity $\dot{q}_1$) and linear (the prismatic joint velocity $\dot{q}_2$) quantities, so its straight minimization is ill-defined conceptually. In fact, the denominator in (2) contains the sum of an non-dimensional term (s_1^2) and of a term with (squared) length units. Stated differently, changing the representing units (e.g., from 1 m to 100 cm) will change the 'optimal' solution.

The minimization of the weighted norm $\frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}}$, leading to weighted pseudoinversion of the task Jacobian, may solve this theoretical issue. In particular, the units of the (positive) elements in the diagonal of $\boldsymbol{W}$ can be used to make terms non-dimensional (e.g., by choosing w_1 in (squared) length units). Out of singularities, $\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}$ and the kinematic control law takes the expression

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}(\boldsymbol{q})\left(\dot{y}_d + k(y_d - y(\boldsymbol{q}))\right) = \frac{1}{\dfrac{q_2^2 c_1^2}{w_1} + \dfrac{s_1^2}{w_2}} \begin{pmatrix} \dfrac{q_2 c_1}{w_1} \\ \dfrac{s_1}{w_2} \end{pmatrix} \left(\dot{y}_d + k(y_d - q_2 \sin q_1)\right), \tag{3}$$

with $k > 0$ as before. Indeed, different values of the weights w_1 and w_2 will lead to different joint velocity solutions. It is easy to verify that is the relative ratio between w_1 and w_2 that really matters. For very large ratios w_1/w_2, the cost of moving the (revolute) joint 1 will be dominant and therefore the solution (3) will tend to minimize its motion while performing the task. In the limit, when $w_1 \to \infty$, it follows from (3) that $\dot{q}_1 \to 0$, while $\dot{q}_2 \propto 1/s_1$: therefore, executing the task will become more and more problematic as the second link gets closer to the horizontal. Similarly, for $w_2/w_1 \to \infty$ the second (prismatic) joint will be very expensive to move, while $\dot{q}_1 \propto 1/q_2 c_1$: the control effort will increase dramatically when the second link is close to being vertical ($c_1 \simeq 0$) and/or fully retracted ($q_2 \simeq 0$).

For the third objective, we need first to derive the inertia matrix of the RP robot. From the expression of the kinetic energy $T = T_1 + T_2$, with

$$T_1 = \frac{1}{2} I_1 \dot{q}_1^2, \quad T_2 = \frac{1}{2} m_2 \left\| \frac{d}{dt} \begin{pmatrix} (q_2 - d_{c2}) \cos q_1 \\ (q_2 - d_{c2}) \sin q_1 \end{pmatrix} \right\|^2 + \frac{1}{2} I_2 \dot{q}_1^2 = \frac{1}{2}\left(I_2 + m_2 (q_2 - d_{c2})^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2,$$

we obtain a diagonal inertia matrix as

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d_{c2})^2 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m_{11}(q_2) & 0 \\ 0 & m_{22} \end{pmatrix}. \tag{4}
$$

The minimization of the kinetic energy T is then a special case of a weighted pseudoinversion of the task Jacobian, with one weight being configuration dependent. Thus, out of singularities, the inertia-weighted kinematic control law takes the expression

$$
\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\left(\dot{y}_d + k(y_d - y(\boldsymbol{q}))\right) = \frac{1}{\dfrac{q_2^2 c_1^2}{m_{11}(q_2)} + \dfrac{s_1^2}{m_{22}}} \begin{pmatrix} \dfrac{q_2 c_1}{m_{11}(q_2)} \\ \dfrac{s_1}{m_{22}} \end{pmatrix} \left(\dot{y}_d + k(y_d - q_2 \sin q_1)\right). \tag{5}
$$

Note that the two addends in the first denominator have both consistent units of [kg^{-1}].

Exercise 2

Let $\boldsymbol{q} = (\theta, \phi)$. Following a Lagrangian approach, under the given assumptions, we compute the kinetic energy $T = T_s + 2T_m$ for the main shaft and the two equal balls. We have

$$
T_s = \frac{1}{2}\, I_s \dot{\theta}^2, \qquad T_m = \frac{1}{2}\, mL^2 \left(\dot{\phi}^2 + \dot{\theta}^2 \sin^2\phi\right),
$$

and thus the diagonal inertia matrix

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_s + 2mL^2 \sin^2\phi & 0 \\ 0 & 2mL^2 \end{pmatrix}. \tag{6}
$$

Using the Christoffel symbols, the Coriolis and centrifugal terms are easily computed from (6) as

$$
\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 4mL^2 \sin\phi \cos\phi\, \dot{\theta}\, \dot{\phi} \\ -2mL^2 \sin\phi \cos\phi\, \dot{\theta}^2 \end{pmatrix} = mL^2 \sin(2\phi) \begin{pmatrix} 2\,\dot{\theta}\,\dot{\phi} \\ -\,\dot{\theta}^2 \end{pmatrix} \tag{7}
$$

For the potential energy due to gravity, $U = U_s + 2U_m$, we have (up to a constant)

$$
U_s = 0, \qquad U_m = -mg_0 L \cos\phi,
$$

and thus

$$
\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ 2mg_0 L \sin\phi \end{pmatrix}. \tag{8}
$$

Including also viscous friction on the main shaft, the dynamic equations are

$$
\begin{aligned}
\left(I_s + 2mL^2 \sin^2\phi\right)\ddot{\theta} + 4mL^2 \sin\phi \cos\phi\, \dot{\theta}\,\dot{\phi} + f_v \dot{\theta} &= \tau \\
2mL^2\, \ddot{\phi} - 2mL^2 \sin\phi \cos\phi\, \dot{\theta}^2 + 2mg_0 L \sin\phi &= 0.
\end{aligned} \tag{9}
$$

Assuming knowledge of the geometric parameter L, equation (9) can be expressed in the linearly parametrized form

$$
\begin{pmatrix} \ddot{\theta} & 2L^2 \sin^2\phi\, \ddot{\theta} + 2L^2 \sin(2\phi)\, \dot{\theta}\, \dot{\phi} & \dot{\theta} \\ 0 & 2L^2\, \ddot{\phi} - L^2 \sin(2\phi)\, \dot{\theta}^2 + 2g_0 L \sin\phi & 0 \end{pmatrix} \begin{pmatrix} I_s \\ m \\ f_v \end{pmatrix} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{\pi} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}, \tag{10}
$$

with the vector $\boldsymbol{\pi} \in \mathbb{R}^3$ of dynamic coefficients.

In a steady-state equilibrium with constant angular velocity $\dot{\theta} = \Omega > 0$, we have $\ddot{\theta} = 0$ and $\ddot{\phi} = \dot{\phi} = 0$. This yields from (9)

$$\tau_\Omega = f_v \Omega, \qquad L \sin\phi \cos\phi \; \Omega^2 + g_0 \sin\phi = 0 \qquad \Rightarrow \qquad \cos\phi_e = \frac{g_0}{L\,\Omega^2}. \tag{11}$$

The input torque τ_Ω has to compensate just for the energy loss due to friction, in order to keep a uniform motion via constant angular velocity. Moreover, the equilibrium angle ϕ_e results from the balance of the gravity force and the centrifugal force. Its value increases (in the range $(0, \pi/2)$) together with Ω.

Finally, by applying the nonlinear feedback law

$$\tau = \left(I_s + 2mL^2 \sin^2\phi\right) a + 4mL^2 \sin\phi \cos\phi \; \dot{\theta}\, \dot{\phi} + f_v \dot{\theta} \tag{12}$$

where $a \in \mathbb{R}$ is the new control input (an acceleration), system (9) is transformed into

$$\begin{aligned} \ddot{\theta} &= a \\ \ddot{\phi} - \sin\phi \cos\phi \, \dot{\theta}^2 + \frac{g_0}{L} \sin\phi &= 0. \end{aligned} \tag{13}$$

The dynamics of θ is now exactly linear (a double integrator), while partial control of the motion of ϕ can be achieved only through the centrifugal term in the second equation, being $\dot{\theta}^2 = \left(\int a \, dt\right)^2$.

Exercise 3

The dynamic equation of the system in Fig. 3 is

$$m\ddot{x} = f + f_c. \tag{14}$$

Impedance control. The so-called inverse dynamics control law becomes in this simple case

$$f = ma - f_c, \tag{15}$$

and transforms system (14) into the double integrator

$$\ddot{x} = a. \tag{16}$$

The auxiliary input a has to be designed so that the controlled mass m, under the action of the contact force f_c, matches the behavior of an impedance model characterized by a desired (apparent) mass $m_d > 0$, desired damping $k_d > 0$, and desired stiffness $k_p > 0$, all acting with respect to a smooth motion reference $x_d(t)$, or

$$m_d \left(\ddot{x} - \ddot{x}_d\right) + k_d \left(\dot{x} - \dot{x}_d\right) + k_p \left(x - x_d\right) = f_c. \tag{17}$$

Equating $\ddot{x}$ in (16) and in the reference behavior (17), solving for a and substituting in (15) yields the control force

$$f = \frac{m}{m_d} \left(\ddot{x}_d + k_d \left(\dot{x}_d - \dot{x}\right) + k_p \left(x_d - x\right)\right) + \left(\frac{m}{m_d} - 1\right) f_c. \tag{18}$$

The feedback law (18) requires in general a measure of the contact force f_c.

In the reference model (17), the position error $e = x_d - x$ does not converge to zero if there is a contact force f_c. Otherwise, e will asymptotically go to zero —indeed exponentially, in view of the linearity of the system dynamics. In particular, for $k_d^2 < 4k_p m_d$, the obtained second-order linear system (17) is characterized by a pair of asymptotically stable complex poles with natural frequency and damping ratio given by

$$\omega_n = \sqrt{\frac{k_p}{m_d}}, \qquad \zeta = \frac{k_d}{2\sqrt{k_p m_d}}. \tag{19}$$

Reducing the desired mass m_d, for given values of stiffness and damping, will increase both the natural frequency ω_n and the damping ratio ζ, and thus improve transients. On the other hand, for a given mass m_d, an increase of the stiffness k_p should be accompanied by an increase of the damping k_d in order to prevent more oscillatory transients. If the desired mass equals the natural (original) mass, i.e., $m_d = m$, a measure of the contact force f_c is no longer needed in the impedance controller (18).

Wishing to achieve $\omega_n = 10$ and $\zeta = 0.7071 = 1/\sqrt{2}$, equations (19) provide

$$k_p = 100\, m_d, \qquad k_d = 10\sqrt{2}\, m_d, \qquad \text{for any } m_d > 0. \tag{20}$$

Being $m = 5$ [kg], if we take in particular $m_d = m = 5$, we obtain as gains

$$k_p = 500, \qquad k_d = 50\sqrt{2} = 70.71, \tag{21}$$

and a measure of f_c will not be needed.

In regulation tasks (with $x_d(t) = x_d =$ constant), by choosing again $m_d = m$, the control law 18) collapses to just a PD action on the position error e,

$$f = k_p\,(x_d - x) - k_d\dot{x}. \tag{22}$$

This scheme is also called *compliance control*, since the main design parameter left is the desired stiffness k_p. Also in this case, the system will converge to $x = x_d$ if (and only if) there is no contact force. With $f_c \neq 0$ but constant, the position $x_e \neq x_d$ that satisfies

$$k_p(x_d - x_e) + f_c = 0 \quad \Rightarrow \quad x_e = x_d + \frac{f_c}{k_p} \tag{23}$$

will be an asyptotically (exponentially) stable closed-loop equilibrium, as can be possibly checked with the Lyapunov candidate $V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_p(x - x_e)^2 \geq 0$ (using in this case LaSalle theorem for the analysis).

Force control. If we desire to regulate explicitly the contact force to a desired constant value f_d, it is necessary to build a force error $e_f = f_d - f_c$ into the control law. After using (15), define the auxiliary input a as

$$a = \frac{1}{m_d}\left(k_f\,(f_d - f_c) - k_d\dot{x}\right), \tag{24}$$

with force error gain $k_f > 0$ and velocity damping coefficient $k_d > 0$. The associated control force is then

$$f = \frac{m}{m_d}\left(k_f\,(f_d - f_c) - k_d\dot{x}\right) - f_c. \tag{25}$$

A contact force measure is needed in this case, even if we choose $m_d = m$. The closed-loop system becomes

$$m_d\ddot{x} + k_d\dot{x} = k_f(f_d - f_c). \tag{26}$$

During free motion, i.e., as long as $f_c = 0$, the mass will eventually move at the constant speed $\dot{x}_e = k_f f_d / k_d$. Therefore, the gain k_d can be tuned so as to keep this speed low (say, during an approaching phase before contacting a hard environment).

An analysis of the general behavior of system (26) for $f_c \neq 0$ is impossible without assigning a model that describes the source of the contact force f_c. Even if we can measure it, as assumed when designing (25), we do not know the evolution of this disturbance nor can impose a desired behavior to it. Should the force error e_f converge to zero at steady state, it follows from eq. (26) that also the mass velocity $\dot{x}$ would go to zero. However, the position x_e reached at the equilibrium would depend on the actual history of the external contact force (see an example in Appendix).

Assume then that contact forces are generated by a compliant environment with stiffness $k_c > 0$, placed beyond the (undeformed) position $x = x_c > 0$. Then, the model for the reaction force of the environment is

$$f_e = \begin{cases} -k_c(x - x_c), & \text{for } x \geq x_c, \\ 0, & \text{else.} \end{cases} \tag{27}$$

During contact, the force applied to the mass is $f_c = -f_e$. Thus, from (26) and (27) it follows

$$m_d\ddot{x} + k_d\dot{x} = k_f\left(f_d - k_c(x - x_c)\right) \qquad \Rightarrow \qquad m_d\ddot{x} + k_d\dot{x} + k_f k_c x = k_f\left(f_d + k_c x_c\right). \tag{28}$$

The steady-state position reached by the second-order asymptotically stable system (28) in response to the (positive) step input $k_f\left(f_d + k_c x_c\right)$ and the associated steady-state contact force will be

$$x_e = x_c + \frac{f_d}{k_c} \qquad \Rightarrow \qquad f_c = \Big(-f_e = k_c(x_e - x_c)\Big) = f_d. \tag{29}$$

A slight variant of the force control law (25) is obtained by replacing the cancelation of the actual contact force in (15) by a compensation/feedforward of the desired contact force, i.e., $f = ma - f_d$. Using again (24), we obtain

$$f = \frac{m}{m_d}\left(k_f\left(f_d - f_c\right) - k_d\dot{x}\right) - f_d, \tag{30}$$

and, as a result, the closed-loop system

$$m_d\ddot{x} + k_d\dot{x} = \left(k_f - \frac{m_d}{m}\right)(f_d - f_c). \tag{31}$$

Using the contact force model (27) leads finally to

$$m_d\ddot{x} + k_d\dot{x} + \left(k_f - \frac{m_d}{m}\right)k_c x = \left(k_f - \frac{m_d}{m}\right)\left(f_d + k_c x_c\right). \tag{32}$$

It is immediate to see that the analysis of (32) can be completed as for (28), provided that the slightly more restrictive design condition $k_f > m_d/m > 0$ is satisfied. Under this hypothesis, the steady-state conditions for the asymptotically stable system (32) are the same given in (29).

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Appendix (extra material to Exercise 3)

Consider a scheme for the contact force generation modeled by

$$\dot{f}_c = \alpha(f_d - f_c), \qquad \text{with } \alpha > 0, \tag{33}$$

and assume, e.g., $f_c(0) = f_{c0} > f_d$ (the initial contact force is larger than the one desired). Then

$$f_c(t) = f_d - (f_d - f_{c0}) \exp^{-\alpha t} \quad \text{and} \quad e_f(t) = f_d - f_c(t) = (f_d - f_{c0}) \exp^{-\alpha t} = e_{f0} \exp^{-\alpha t}. \tag{34}$$

Assuming $x(0) = \dot{x}(0) = 0$ and discarding the special case $\alpha = k_d/m_d$, the solution of (26) can be found by Laplace techniques and is given by the following position trajectory

$$x(t) = \frac{k_f e_{f0}}{k_d\,\alpha} + \frac{k_f e_{f0}}{k_d - \alpha\, m_d} \left(\frac{m_d}{k_d} \exp^{-\frac{k_d}{m_d} t} - \frac{1}{\alpha} \exp^{-\alpha t} \right), \tag{35}$$

and associated velocity

$$\dot{x}(t) = \frac{k_f e_{f0}}{k_d - \alpha\, m_d} \left(\exp^{-\alpha t} - \exp^{-\frac{k_d}{m_d} t} \right). \tag{36}$$

It follows from (35) that, at steady state,

$$x_e = \lim_{t \to \infty} x(t) = \frac{k_f e_{f0}}{k_d\,\alpha}, \tag{37}$$

which shows an explicit dependence on the parameter α of the contact force model (33). Figure 4 shows two possible evolutions of the applied force error term $k_f(f_d - f_c)$ (in blue) and of the resulting mass position x (in green), for $\alpha = 2$ and $\alpha = 3$, with the other parameters being $f_d = 3$ [N], $f_{c0} = 2$ [N] (and thus, $e_f = f_d - f_{c0} = 1$ [N]), $k_f = 1.4$, $m_d = 1$ [kg], and $k_d = 1$ [kg/s].

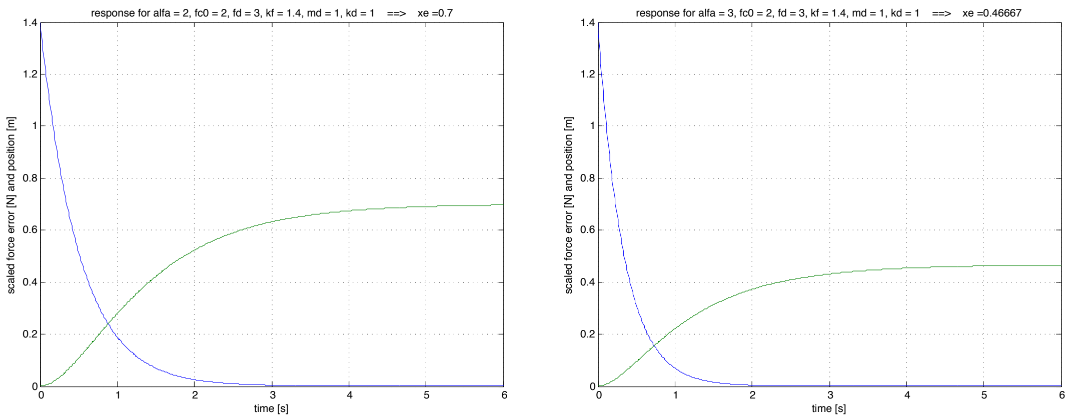

Figure 4: Simulation results of (26) of a controlled mass m_d subject to the contact force f_c in (34), for $\alpha = 2$ [left] and $\alpha = 3$ [right]. The plots are the position x (shown in green) and the force error term $k_f(f_d - f_c) = k_f e_f$ (in blue). The reached position x_e is the one computed in (37).

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