Robotics II

September 12, 2016

Exercise 1

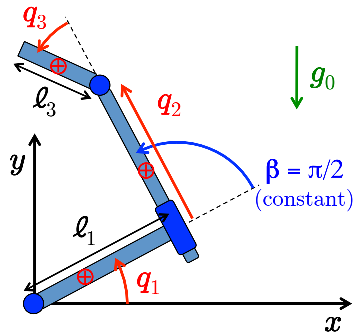

Figure 1: A RPR planar robot moving in a vertical plane.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model of the planar RPR robot in Fig. 1, using the Lagrangian coordinates $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ defined therein. Determine all equilibrium configurations $\boldsymbol{q}_0$ under no external or dissipative forces/torques nor actuation inputs.

Exercise 2

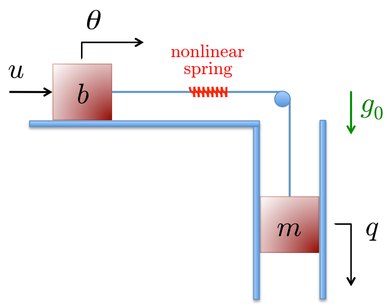

Figure 2: A two-mass system connected by a nonlinear spring, under the action of gravity.

In the mechanical system shown in Fig. 2, the first body (of mass $b > 0$ and position θ) is actuated by a force u and is connected to the second body (of mass $m > 0$ and position q) through a nonlinear spring having potential energy

$$U_e = \frac{1}{2}k(q-\theta)^2 + \frac{1}{4}k_n(q-\theta)^4, \qquad k, k_n > 0.$$

- Derive the dynamic model of this system by following a Lagrangian approach and using as generalized coordinates (θ, q).
- Determine the (unique!) equilibrium position $\bar{\theta}$ for the first mass and the required constant input force $\bar{u}$ to be applied in order to keep the second mass at a desired position (height) $\bar{q}$.
- Verify your result by computing the values of $\bar{\theta}$ and $\bar{u}$ for the following numerical data:

$$\bar{q} = 0.1\,[\mathrm{m}], \quad b = m = 3\,[\mathrm{kg}], \quad k = 1000\,[\mathrm{N/m}], \quad k_n = 10000\,[\mathrm{N/m^3}], \quad g_0 = 9.81\,[\mathrm{m/s^2}].$$

[150 minutes; open books]

Solution

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Exercise 1

For link i, $i = 1, 2, 3$, let m_i be its mass and I_i its inertia around an axis normal to the plane of motion and passing through the center of mass. Moreover, for $i = 1$ and $i = 3$, d_i is the distance of the center of mass of link i from the axis of joint i, while for the second link, d_2 will denote the (constant) distance of its center of mass from the axis of joint 3 —see Fig. 3.

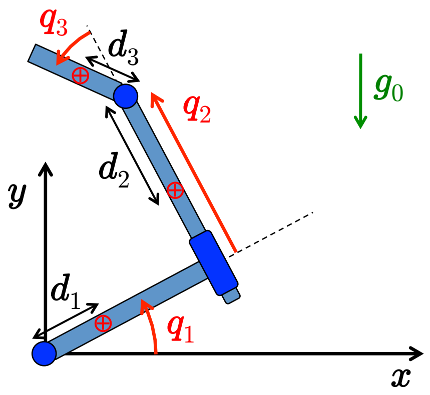

Figure 3: Parameters d_i defining the location of the center mass of the links of the RPR robot.

To derive the robot dynamic model terms, we follow a Lagrangian approach. For obtaining the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$, we compute the kinetic energy $T = T_1 + T_2 + T_3$ of the three robot links. For the first link, it is

$$T_1 = \frac{1}{2} \left(I_1 + m_1 d_1^2\right) \dot{q}_1^2.$$

For the second link, the position of the center of mass is[1]

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 - (q_2 - d_2) \sin q_1 \\ \ell_1 \sin q_1 + (q_2 - d_2) \cos q_1 \end{pmatrix}, \tag{1}$$

where ℓ_1 is the length of link 1. Thus, the velocity of this center of mass is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + (q_2 - d_2) \cos q_1\right) \dot{q}_1 - \sin q_1 \, \dot{q}_2 \\ \left(\ell_1 \cos q_1 - (q_2 - d_2) \sin q_1\right) \dot{q}_1 + \cos q_1 \, \dot{q}_2 \end{pmatrix}$$

and its squared norm is

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \left(\ell_1^2 + (q_2 - d_2)^2\right) \dot{q}_1^2 + \dot{q}_2^2 + 2\ell_1 \, \dot{q}_1 \dot{q}_2.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_1$. As a result,

$$T_2 = \frac{1}{2} \left[\left(I_2 + m_2 \left(\ell_1^2 + (q_2 - d_2)^2\right)\right) \dot{q}_1^2 + m_2 \, \dot{q}_2^2 + 2 m_2 \ell_1 \, \dot{q}_1 \dot{q}_2 \right].$$

[1] We take into account the following identities (for $\beta = \pi/2$): $\cos(q_1 + \pi/2) = -\sin q_1$, $\sin(q_1 + \pi/2) = \cos q_1$.

Similarly, the position of the center of mass of the third link is

$$\boldsymbol{p}_{c3} = \begin{pmatrix} \ell_1 \cos q_1 - q_2 \sin q_1 - d_3 \sin(q_1 + q_3) \\ \ell_1 \sin q_1 + q_2 \cos q_1 + d_3 \cos(q_1 + q_3) \end{pmatrix}. \tag{2}$$

Its velocity is

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -(\ell_1 \sin q_1 + q_2 \cos q_1)\,\dot{q}_1 - \sin q_1\,\dot{q}_2 - d_3 \cos(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \\ (\ell_1 \cos q_1 - q_2 \sin q_1)\,\dot{q}_1 + \cos q_1\,\dot{q}_2 - d_3 \sin(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \end{pmatrix}$$

and the squared norm becomes

$$\|\boldsymbol{v}_{c3}\|^2 = \left(\ell_1^2 + q_2^2\right)\dot{q}_1^2 + \dot{q}_2^2 + d_3^2(\dot{q}_1 + \dot{q}_3)^2 + 2\ell_1\dot{q}_1\dot{q}_2 - 2d_3 \sin q_3(\dot{q}_1 + \dot{q}_3)(\ell_1\dot{q}_1 + \dot{q}_2) + 2d_3 q_2 \cos q_3(\dot{q}_1 + \dot{q}_3)\dot{q}_1.$$

The (scalar) angular velocity of link 3 is simply $(\dot{q}_1 + \dot{q}_3)$. As a result,

$$\begin{aligned} T_3 &= \tfrac{1}{2}\left[\left(I_3 + m_3 d_3^2\right)(\dot{q}_1 + \dot{q}_3)^2 + m_3\left(\ell_1^2 + q_2^2\right)\dot{q}_1^2 + m_3\,\dot{q}_2^2\right. \\ &\qquad \left. + 2m_3\ell_1\,\dot{q}_1\dot{q}_2 + 2m_3 d_3(\dot{q}_1 + \dot{q}_3)\left(q_2 \cos q_3 \dot{q}_1 - \sin q_3(\ell_1\dot{q}_1 + \dot{q}_2)\right)\right]. \end{aligned}$$

Therefore,

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} b_{11}(q_2, q_3) & b_{12}(q_3) & b_{13}(q_2, q_3) \\ & b_{22} & b_{23}(q_3) \\ \mathit{symm} & & b_{33} \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

from which the elements b_{ij} of the 3×3, symmetric, positive definite inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ can be extracted:

$$\begin{aligned} b_{11}(q_2, q_3) &= I_1 + m_1 d_1^2 + I_2 + m_2 d_2^2 + I_3 + m_3 d_3^2 + (m_2 + m_3)\ell_1^2 \\ &\quad - 2m_2 d_2 q_2 + (m_2 + m_3) q_2^2 + 2m_3 d_3\left(q_2 \cos q_3 - \ell_1 \sin q_3\right) \\ &= \pi_1 + \pi_2\,q_2 + \pi_3\,q_2^2 + 2\pi_4\left(q_2 \cos q_3 - \ell_1 \sin q_3\right) \\ b_{12}(q_3) &= (m_2 + m_3)\ell_1 - m_3 d_3 \sin q_3 = \pi_3\,\ell_1 - \pi_4 \sin q_3 \\ b_{13}(q_2, q_3) &= I_3 + m_3 d_3^2 + m_3 d_3\left(q_2 \cos q_3 - \ell_1 \sin q_3\right) = \pi_5 + \pi_4\left(q_2 \cos q_3 - \ell_1 \sin q_3\right) \\ b_{22} &= m_2 + m_3 = \pi_3 \\ b_{23}(q_3) &= -m_3 d_3 \sin q_3 = -\pi_4 \sin q_3 \\ b_{33} &= I_3 + m_3 d_3^2 = \pi_5. \end{aligned} \tag{3}$$

In the expressions (3) of the elements of $\boldsymbol{B}(\boldsymbol{q})$, we have assumed that the kinematic parameter ℓ_1 is known and found thus a linear parameterization in terms of five dynamic coefficients π_i, $i = 1, \ldots, 5$.

For obtaining the vector $\boldsymbol{g}(\boldsymbol{q})$, we compute the potential energy $U = U_1 + U_2 + U_3$ due to gravity for the three robot links. Using the expressions of the y-component of the vectors $\boldsymbol{p}_{ci}$, $i = 1, 2, 3$ (see also eqs. (1) and (2)), we have

$$\begin{aligned} U_1(q_1) &= m_1 g_0\,p_{c1,y} = m_1 g_0\,d_1 \sin q_1 \\ U_2(q_1, q_2) &= m_2 g_0\,p_{c2,y} = m_2 g_0\left(\ell_1 \sin q_1 + (q_2 - d_2)\cos q_1\right) \\ U_3(q_1, q_2, q_3) &= m_3 g_0\,p_{c3,y} = m_3 g_0\left(\ell_1 \sin q_1 + q_2 \cos q_1 + d_3 \cos(q_1 + q_3)\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\boldsymbol{g}(\boldsymbol{q}) &= \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T \\
&= \begin{pmatrix} (m_1 d_1 + (m_2+m_3)\ell_1)\, g_0 \cos q_1 - (m_2(q_2-d_2) + m_3 q_2)\, g_0 \sin q_1 - m_3 d_3 g_0 \sin(q_1+q_3) \\ (m_2+m_3) g_0 \cos q_1 \\ -m_3 d_3 g_0 \sin(q_1+q_3) \end{pmatrix}.
\end{aligned}
\tag{4}
$$

Solving for $\boldsymbol{g}(\boldsymbol{q}_0) = \mathbf{0}$ provides all equilibrium configurations $\boldsymbol{q}_0$. From the second component in (4), it follows that $q_{0,1} = \pm\pi/2$, which confirms also the intuition that the prismatic joint axis should be horizontal at the equilibrium. Plugging this into the third equation leads again to $q_{0,3} = \pm\pi/2$. Imposing these two conditions in the first equation, $g_1(\boldsymbol{q}_0) = 0$ provided the additional condition $q_2 = m_2 d_2/(m_2+m_3)$. Thus, all unforced equilibrium configurations for this RPR robot are of the form

$$
\boldsymbol{q}_0 = \begin{pmatrix} \pm\dfrac{\pi}{2} & \dfrac{m_2\, d_2}{m_2+m_3} & \pm\dfrac{\pi}{2} \end{pmatrix}^T.
$$

Exercise 2

The kinetic and potential energies of the mechanical system in Fig. 2 are given by

$$
T = \frac{1}{2} b\, \dot{\theta}^2 + \frac{1}{2} m\, \dot{q}^2, \qquad U = U_e + U_g = \frac{1}{2} k (q-\theta)^2 + \frac{1}{4} k_n (q-\theta)^4 - m g_0 q.
$$

Thus, applying the Euler-Lagrange equations to $L = T - U$ yields the two second-order differential equations

$$
b\ddot{\theta} + k(\theta - q) + k_n(\theta - q)^3 = u \tag{5}
$$

$$
m\ddot{q} - m g_0 + k(q-\theta) + k_n(q-\theta)^3 = 0, \tag{6}
$$

which are nonlinear due to the cubic terms in the deformation $\delta = q - \theta$ of the spring.

For the equilibrium conditions, we set $\ddot{\theta} = \ddot{q} = 0$ in eqs. (5)–(6). From the second one, we obtain the following algebraic equation:

$$
k_n (q-\theta)^3 + k(q-\theta) - m g_0 = 0.
$$

This is a cubic equation of the form

$$
\delta^3 + p\delta + r = 0, \qquad \text{with} \quad p = \frac{k}{k_n}, \quad r = -\frac{m g_0}{k_n}, \tag{7}
$$

which is known to have the single real solution[2]

$$
\delta = \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{p^3}{27}}}. \tag{8}
$$

[2]The (depressed) cubic equation in (7) was studied already in the XVI century. The formula (8) is attributed to the mathematician Gerolamo Cardano, but in fact is due to Scipione del Ferro and Niccolò Fontana (also known as Tartaglia).

Therefore, for a given desired position $q = \bar{q}$ of the mass m, we compute from (8) the spring deformation $\bar{\delta}$ at steady state and set

$$\bar{\theta} = \bar{q} - \bar{\delta}. \tag{9}$$

Moreover, from eq. (5) at steady state we obtain the required input force

$$\bar{u} = k(\bar{\theta} - \bar{q}) + k_n(\bar{\theta} - \bar{q})^3 = k\bar{\delta} + k_n\bar{\delta}^3 = -mg_0. \tag{10}$$

The input force balances the weight of the mass m, as reflected through the elastic force τ_e of the deformed spring. Note that that the mass b plays no role in this analysis.

Using now the given numerical data, and in particular $\bar{q} = 0.1$, $k = 1000$, $k_n = 10000$ and $m = 3$, we compute

$$\delta = 0.0292, \qquad \bar{\theta} = 0.0708, \qquad \bar{u} = -29.4300. \tag{11}$$

As a result, the static deformation of the nonlinear spring at the equilibrium is equal to slightly less than 3 cm. The deformation-force characteristics $\tau_e = k\delta + k_n\delta^3$ of the chosen spring is shown in Fig. 4, where we have indicated also the actual deformation $\bar{\delta}$ at equilibrium. In correspondence to this value, we can easily check that the elastic force is $\tau_e = -\bar{u}$.

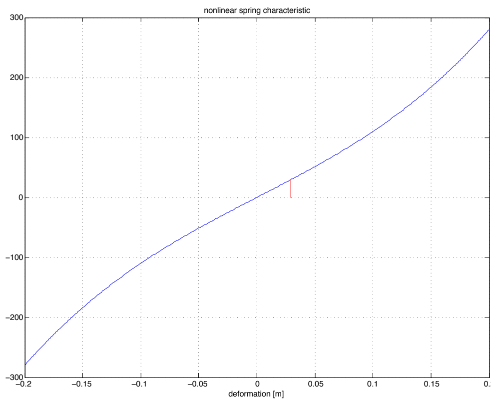

Figure 4: The nonlinear deformation-force characteristics of the spring for $k = 10^3$ [N/m] and $k_n = 10^4$ [N/m^3]. The equilibrium condition for the given problem data is reported in red.

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