Robotics II

July 11, 2016

Exercise 1

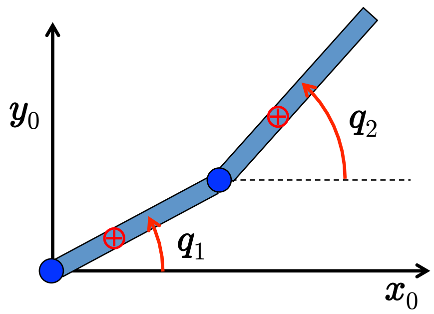

Figure 1: A 2R planar robot moving on a horizontal plane.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ of a planar 2R robot using the *absolute* coordinates $\boldsymbol{q} = (q_1, q_2)$ defined in Fig. 1. Will Coriolis and/or centrifugal terms be present in the quadratic velocity term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the dynamic model? Why? In the absence of dissipative effects, what is the relation between the input torques $\boldsymbol{u}$ acting in this case on the right-hand side of the Euler-Lagrange equations and the torques $\boldsymbol{u}_\theta = (u_{\theta 1}, u_{\theta 2})$ produced by two motors, one for each joint, directly connected to the respective joint axes?

Exercise 2

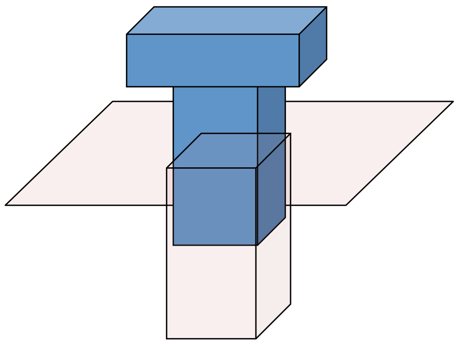

Figure 2: A peg with a square section entering a hole with little or no clearance.

For the task of inserting a peg with a square section into a square hole, as depicted in Fig. 2, define *i)* a suitable *task frame*, *ii)* the *natural constraints* imposed by the geometry of the rigid and frictionless environment on the generalized (i.e., linear and angular) motion/force quantities expressed in this task frame, and *iii)* the *virtual constraints* that can be taken as reference values by a hybrid force-velocity control law for the smooth execution of the task.

Exercise 3

For the peg-in-hole task considered in Exercise 2, consider the problem of regulating the contact force in one direction against a side of the hole. The hole has now a compliant behavior, as modeled by a spring of stiffness $k_e > 0$. With reference to Fig. 3, which displays a horizontal section of the peg in contact with one side of the hole, let $m > 0$ be the mass of the peg and $d > 0$ the position of the undeformed contact along the ${}^t\boldsymbol{x}$ direction.

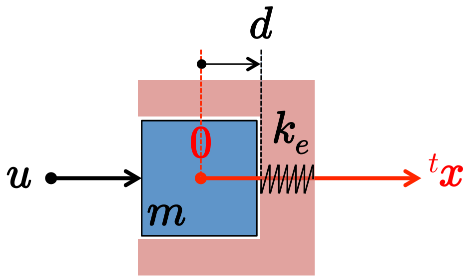

Figure 3: Interaction with a compliant environment at one side of the peg-in-hole of Fig 2.

Design two controllers for the scalar command u such that the contact force F_c is regulated to a desired value $F_d > 0$, with an asymptotically stable transient behavior. In particular:

- suppose first that no force sensing is available and design a *compliance control* law;
- introduce an ideal force sensor (e.g., on the peg surface) and design a *force control* law;
- discuss robustness of the two designs w.r.t. uncertainties in the knowledge of k_e, d and m.

[210 minutes; open books]

Solution

July 11, 2016

Exercise 1

To obtain the robot dynamic model terms, we follow a Lagrangian approach. Since the robot motion occurs at constant potential energy (on a horizontal plane), we need only to compute the kinetic energy $T = T_1 + T_2$. For link i, $i = 1, 2$, let m_i be its mass, d_i the distance of its center of mass from the axis of joint i, and I_i the baricentral inertia of the link around an axis normal to the plane of motion. For the first link, it is

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_1^2\right)\dot{q}_1^2.$$

For the second link, the position of the center of mass when using the absolute coordinates is

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 + d_2 \cos q_2 \\ \ell_1 \sin q_1 + d_2 \sin q_2 \end{pmatrix},$$

where ℓ_1 is the length of link 1. Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\ell_1 \sin q_1\, \dot{q}_1 - d_2 \sin q_2\, \dot{q}_2 \\ \ell_1 \cos q_1\, \dot{q}_1 + d_2 \cos q_2\, \dot{q}_2 \end{pmatrix},$$

and its squared norm becomes

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \ell_1^2 \dot{q}_1^2 + d_2^2 \dot{q}_2^2 + 2\ell_1 d_2 \cos(q_2 - q_1)\, \dot{q}_1 \dot{q}_2.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_2$. As a result,

$$T_2 = \frac{1}{2}\left(m_2 \ell_1^2 \dot{q}_1^2 + \left(I_2 + m_2 d_2^2\right)\dot{q}_2^2 + 2 m_2 \ell_1 d_2 \cos(q_2 - q_1)\, \dot{q}_1 \dot{q}_2\right).$$

Therefore,

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + m_1 d_1^2 + m_2 \ell_1^2 & m_2 \ell_1 d_2 \cos(q_2 - q_1) \\ m_2 \ell_1 d_2 \cos(q_2 - q_1) & I_2 + m_2 d_2^2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}}.$$

Defining for compactness the following dynamic coefficients

$$a_1 = I_1 + m_1 d_1^2 + m_2 \ell_1^2, \qquad a_2 = I_2 + m_2 d_2^2, \qquad a_3 = m_2 \ell_1 d_2,$$

we can rewrite the inertia matrix as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_3 \cos(q_2 - q_1) \\ a_3 \cos(q_2 - q_1) & a_2 \end{pmatrix}.$$

For the (quadratic) velocity terms in the dynamic model, we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix}, \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \quad i = 1, 2,$$

where

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right\}, \qquad i = 1, 2.$$

Computing

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{\begin{pmatrix} 0 & 0 \\ a_3\sin(q_2-q_1) & -a_3\sin(q_2-q_1)\end{pmatrix} + \begin{pmatrix} 0 & a_3\sin(q_2-q_1) \\ 0 & -a_3\sin(q_2-q_1)\end{pmatrix}\right. \\
&\qquad \left. - \begin{pmatrix} 0 & a_3\sin(q_2-q_1) \\ a_3\sin(q_2-q_1) & 0\end{pmatrix}\right\} = \begin{pmatrix} 0 & 0 \\ 0 & -a_3\sin(q_2-q_1)\end{pmatrix}, \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \cdots = \begin{pmatrix} a_3\sin(q_2-q_1) & 0 \\ 0 & 0\end{pmatrix},
\end{aligned}$$

we finally obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_3\sin(q_2-q_1)\,\dot{q}_2^2 \\ a_3\sin(q_2-q_1)\,\dot{q}_1^2\end{pmatrix}.$$

There are no Coriolis, but only centrifugal terms in the dynamic equations. This is due to the choice of absolute coordinates, so that a motion of q_1 only, with q_2 kept constant, will not change the orientation of link 2.

Moreover, the transformation from *relative* joint coordinates $\boldsymbol{\theta}$ (those of a classical DH convention) to *absolute* joint coordinates $\boldsymbol{q}$ is

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_1+\theta_2 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\dot{\boldsymbol{\theta}} = \boldsymbol{T}\dot{\boldsymbol{\theta}}.$$

Therefore, due to the principle of virtual works, the mapping between the generalized forces $\boldsymbol{u}$ performing work on $\dot{\boldsymbol{q}}$ and the generalized forces $\boldsymbol{u}_\theta$ performing work on $\dot{\boldsymbol{\theta}}$ is given by

$$\boldsymbol{u}_\theta = \begin{pmatrix} u_{\theta 1} \\ u_{\theta 2}\end{pmatrix} = \boldsymbol{T}^T\boldsymbol{u} = \begin{pmatrix} u_1+u_2 \\ u_2\end{pmatrix} \qquad \Leftrightarrow \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2\end{pmatrix} = \boldsymbol{T}^{-T}\boldsymbol{u}_\theta = \begin{pmatrix} u_{\theta 1}-u_{\theta 2} \\ u_{\theta 2}\end{pmatrix}.$$

The robot dynamic model written in the $\boldsymbol{q}$ coordinates and driven by the motor torques $\boldsymbol{u}_\theta$ is

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} = \boldsymbol{T}^{-T}\boldsymbol{u}_\theta.$$

Exercise 2

With reference to the task frame shown in Fig. 4, the natural (geometric) contraints and the virtual constraints in the peg-hole frictionless interaction are respectively

$$\text{natural:} \quad \begin{cases} {}^t v_x &= 0 \\ {}^t v_y &= 0 \\ {}^t \omega_x &= 0 \\ {}^t \omega_y &= 0 \\ {}^t \omega_z &= 0 \\ {}^t F_z &= 0 \end{cases} \qquad\qquad \text{virtual:} \quad \begin{cases} {}^t F_x &= F_{xd} \\ {}^t F_y &= F_{yd} \\ {}^t M_x &= M_{xd} \\ {}^t M_y &= M_{yd} \\ {}^t M_z &= M_{zd} \\ {}^t v_z &= v_{zd} > 0. \end{cases}$$

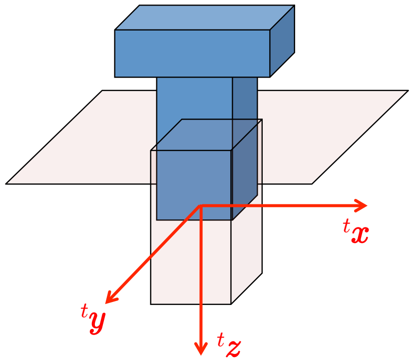

Figure 4: The task frame used in the definition of natural and virtual constraints for hybrid force/motion analysis and control.

Except for the last virtual constraint, which specifies the desired positive speed of insertion of the peg in the hole, all the desired interaction forces and moments can be set to zero, if a smooth behavior is to be realized with minimum mechanical stress on the peg. On the other hand, if a firm contact needs to be maintained with one side of the hole (e.g., in the presence of some uncertain clearance), then the choice of a positive value for F_{xd} and/or F_{yd} would be useful (this is the situation considered, e.g., in Exercise 3).

Exercise 3

Let x be the position along $^t\boldsymbol{x}$ of the peg side that undergoes contact with the hole side. We assume that the peg is already in contact ($x \geq d$). The dynamic model of the peg-environment interaction is then

$$m\ddot{x} + k_e(x - d) = u,$$

whereas the contact force applied by the peg to the hole side is

$$F_c = k_e(x - d), \qquad \text{for } x \geq d$$

Instead, when $x < d$ (no contact) then $F_c = 0$.

If we want to have $F_c = F_d$ at the closed-loop equilibrium, we need to drive the mass m to a position x_d computed as follows:

$$F_c = k_e\,(x - d)|_{x=x_d} = F_d \qquad \Rightarrow \qquad x_d = d + \frac{1}{k_e}F_d. \tag{1}$$

However, a simple *compliance control* law of the PD type

$$u = k_p(x_d - x) - k_d\dot{x}, \qquad \text{with } k_p > 0 \text{ and } k_d > 0, \tag{2}$$

will have a closed-loop equilibrium $x = x_E$ that does not provide the desired contact force F_d at steady state. In fact, the unique equilibrium $x_E(> d)$ will be the solution of

$$k_e(x - d)|_{x=x_E} = k_p(x_d - x)|_{x=x_E} \qquad \Rightarrow \qquad x_E = \frac{k_e d + k_p x_d}{k_e + k_p} = d + \frac{k_p}{k_e(k_e + k_p)}F_d \neq x_d,$$

where (1) has been used. Accordingly, the steady-state contact force is

$$F_E = k_e(x_E - d) = \frac{k_p}{k_e + k_p}\, F_d,$$

which is never equal to the desired one, even for arbitrary large but finite values of k_p.

A better control design is achieved by adding to the law (2) a term that nominally cancels the interaction force, i.e.,

$$u = k_p(x_d - x) - k_d\dot{x} + k_e(x - d), \qquad \text{with } k_p > 0 \text{ and } k_d > 0. \tag{3}$$

When in contact, the associated closed-loop equation is

$$m\ddot{x} + k_d\dot{x} + k_p x = k_p x_d,$$

which is an asymptotically stable system driven by the constant signal $k_p x_d$, converging at steady state to the desired position $x = x_d$, thus with $F_c = F_d$.

Unfortunately, an uncertain value of the environment stiffness (which appears twice in the control law, namely in the definition of x_d in (1) and in the term compensating the interaction force in (3)) would lead to a residual force error at steady state. In practice, we can implement only

$$u = k_p(\hat{x}_d - x) - k_d\dot{x} + \hat{k}_e(x - d), \qquad \text{with } k_p > 0 \text{ and } k_d > 0. \tag{4}$$

where $\hat{k}_e$ is an estimate of k_e and $\hat{x}_d = d + (F_d/\hat{k}_e)$. For the time being, assume that d is accurately known, so that the last term in (4) is present only when $x \geq d$ (otherwise is zero). When in contact, the closed-loop equation becomes

$$m\ddot{x} + k_d\dot{x} + \left(k_p + k_e - \hat{k}_e\right)x = k_p\hat{x}_d + \left(k_e - \hat{k}_e\right)d = \left(k_p + k_e - \hat{k}_e\right)d + \frac{k_p}{\hat{k}_e}\, F_d. \tag{5}$$

Provided that the proportional gain k_p in the control law (4) is sufficiently large, i.e.,

$$k_p \geq \alpha > \left|k_e - \hat{k}_e\right| \geq 0, \tag{6}$$

it is easy to see that system (5) will remain asymptotically stable. Its position and contact force converge respectively to

$$x_E = d + \frac{k_p}{\hat{k}_e\left(k_p + k_e - \hat{k}_e\right)}\, F_d \qquad \text{and} \qquad F_E = \left(\frac{k_e}{\hat{k}_e}\right)\left(\frac{k_p}{k_p + k_e - \hat{k}_e}\right) F_d \neq F_d.$$

The last formula shows that by increasing $k_p \to \infty$ (i.e., to very large values), one can certainly neglect the effect of the estimation error $k_e - \hat{k}_e$, but still *not* the error due to the scaling factor $k_e/\hat{k}_e \neq 1$.

The analysis of the case when the position d of the environment is not accurately known is more complex. Again, the estimate $\hat{d}$ enters twice in place of d within the control law (4), explicitly in the last term and implicitly through the definition of x_d. Not having an accurate estimate of d may lead to a wrong activation/deactivation of the term $\hat{k}_e(x - \hat{d})$, possibly in an inconsistent way with respect to the true contact/no contact situation. A chattering behavior may thus result in the proximity of the contact point. While a more detailed analysis of such a situation is possible, this is out of our scope here.

On the other hand, the above control laws are fully independent from the knowledge of the mass value m, which only affects the behavior of the system during transient phases.

Let us turn the attention to the case when a force sensor is available, whose measure F_m ideally provides the contact force, or

$$F_m = F_c = \begin{cases} k_e(x-d), & x \geq d, \\ 0, & x < d. \end{cases}$$

Then, we can implement a *force control* law of the form

$$u = F_d + k_f(F_d - F_m) - k_d\dot{x}, \qquad \text{with } k_f > 0 \text{ and } k_d > 0. \tag{7}$$

with a constant feedforward, a term proportional to the force error, and a velocity damping term (as before). When in contact, the resulting closed-loop equation is

$$m\ddot{x} + k_d\dot{x} + (1+k_f)\,k_e(x-d) = (1+k_f)\,F_d. \tag{8}$$

Equation (8) represents again an asymptotically stable system, whose position and contact force converge to their desired values, i.e.,

$$x_E = x_d = d + \frac{1}{k_e}\,F_d \qquad \text{and} \qquad F_E = F_d.$$

Note that the same control law (7) can be applied also during a phase of no contact (when $F_m = 0$). In that case, the closed-loop equation takes the form

$$m\ddot{x} + k_d\dot{x} = (1+k_f)\,F_d,$$

and the (approaching) velocity of the peg will converge to the constant value

$$\dot{x}_E = \frac{1+k_f}{k_d}F_d > 0,$$

and proceed in this way until contact is established.

An alternative to the addition of the constant force feedforward F_d in (7) is the use of a PI control law on the force error $e_f = F_d - F_m$ (with $F_m = F_c$), still complemented by a velocity damping term, i.e.,

$$u(t) = k_f(F_d - F_m(t)) + k_i\int_0^t (F_d - F_m(\tau))d\tau - k_d\dot{x}(t), \quad \text{with } k_d > 0, k_i > 0, \text{ and (at least) } k_f > 0. \tag{9}$$

It can be shown that the control scheme obtained by using (9) is equivalent to closing in a unitary feedback loop the transfer function

$$G(s) = \frac{F_m(s)}{e_f(s)} = \frac{k_e\,(k_f s + k_i)}{s\,(ms^2 + k_d s + k_e)}, \tag{10}$$

yielding the closed-loop system $W(s) = F_m(s)/F_d(s) = G(s)/(1+G(s))$. The denominator of $W(s)$ is

$$\text{den } W(s) = ms^3 + k_d s^2 + (1+k_f)k_e s + k_i k_e$$

and, by the Routh criterion, its roots are all in the open left-hand side of the complex plane if and only if

$$k_d > 0, \qquad k_i > 0, \qquad k_f > m\,\frac{k_i}{k_d} - 1.$$

Under these conditions, the closed-loop system is asymptotically stable and the force error e_f will converge to zero.

As a matter of fact, both force control laws (7) and (9) achieve always their target, without the need to know any of the parameters k_e, m, or d. The cost of including a force sensor is paid back by the achieved robustness of the control laws that can be designed with the force measurements.

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