Robotics II

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Huge portal robots are used in the aeronautical industry for moving and reorienting large surface plates of aircraft bodies. Figure 1 shows a subassembly of one such robots, having three actuated joints and three passive joints (shown in different colors).

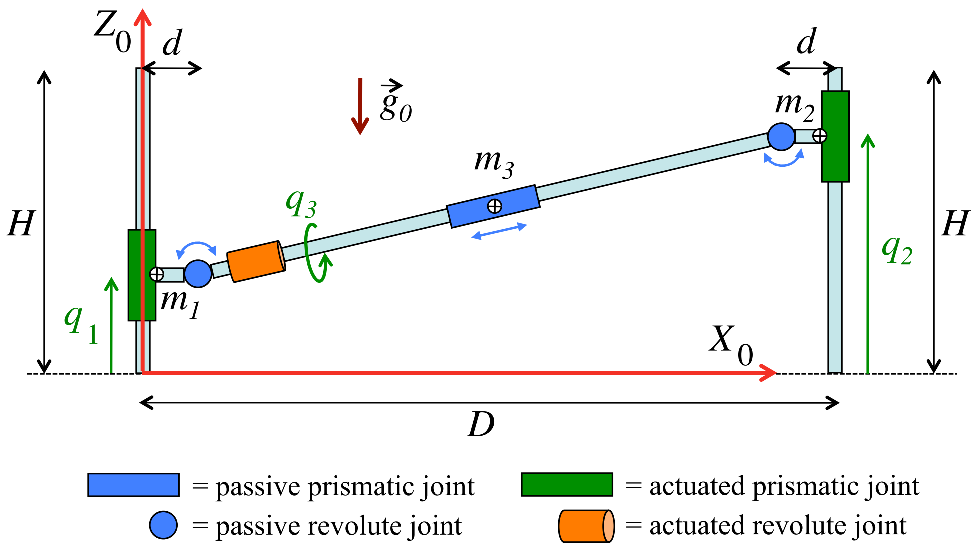

Figure 1: A portal robot used in the aeronautic industry

Two vertical bars of height H are placed at a distance D. Along the vertical bars, two actuated prismatic joints (with variables $q_1 \in [0, H]$ and $q_2 \in [0, H]$) are used to modify the position and the orientation of a connecting bar in the vertical plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$. The two passive revolute joints, placed at a distance $d << D$ from the vertical bars, transform the linear motion of q_1 and q_2 in a linear and/or angular motion of the connecting bar. The passive prismatic joint along the connecting bar accommodates itself so that the bar changes length consistently with the positions q_1 and q_2. Furthermore, the connecting bar can be rotated by an actuated revolute joint (with angle q_3) along its main geometric axis. The available actuating force/torque $\boldsymbol{u} = (F_1, F_2, \tau_3)$ performs work on the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$. From an operational point of view, curved plates of the aircraft body are placed and fixed on the connecting bar. Then, the motion of the portal robot can change the orientation of the normals to the plate surface to be worked (e.g., for riveting).

Figure 1 shows also the location of the center of masses of the three main moving bodies (of masses m_1, m_2, and m_3) of the portal robot. The following assumptions are made.

- The mechanism associated to the passive prismatic joint is such that the center of mass of the connecting bar is always kept on the vertical line passing through the center of the portal (i.e., at $x_0 = D/2$), despite the extension or retraction of the connecting bar itself.
- The connecting bar has a diagonal inertia matrix

$$^b\boldsymbol{I}_b = \text{diag}\{I_x, I_y, I_z\}, \qquad \text{with} \quad I_y = I_z$$

when expressed in a frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ attached to the bar, with origin at its center of mass and principal axes along its geometric axes of symmetry.

- The values of the scalar inertias I_x and $I_y = I_z$ change with the length of the connecting bar. Since the distance D is much larger than the actual excursion of q_1 and q_2, this effect can be neglected if needed. In this case, for dynamic modeling purposes the bar will be considered of constant length $D - 2d$ (as when it is horizontal or close to this situation).

With the above description in mind, and considering the previous assumptions:

1. Derive the dynamic model of this portal robot using the generalized coordinates $\boldsymbol{q}$ in a Lagrangian approach.
2. Define the simplest control law for $\boldsymbol{u}$ that is able to regulate the robot to a desired constant equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, defining conditions on the control parameters that guarantee global asymptotic stability.
3. *Bonus, or in alternative to 2.* Show that $\boldsymbol{q} = (q_1, q_2, q_3)$ is in fact a set of generalized coordinates for this robot, which has the structure of a closed kinematic chain (closed through the floor). In particular, show that the values of the angular position of the two passive revolute joints and the extension of the connecting bar thanks to the presence of the passive prismatic joint can be uniquely obtained from the values of q_1, q_2, and q_3.

[240 minutes; open books]

Solution

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Following the Lagrangian approach, we compute the kinetic energy and the potential energy due to gravity for the robot bodies based on the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

The kinetic energy for the first two bodies (moving only up and down) is simply

$$T_1 = \frac{1}{2}\, m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2}\, m_2 \dot{q}_2^2.$$

For the third body (the connecting bar), it is useful to introduce the variable

$$\alpha = \arctan\left(\frac{q_1 - q_2}{D - 2d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1}$$

that represents the angle of the bar w.r.t. to the horizontal (positive counterclockwise around the axis $\boldsymbol{y}_0$). For evaluating the linear contribution to the kinetic energy of this bar

$$T_{3,l} = \frac{1}{2}\, m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3},$$

we compute the position of its center of mass and then differentiate w.r.t. time:

$$\boldsymbol{p}_{c3} = \begin{pmatrix} \dfrac{D}{2} \\ 0 \\ \dfrac{q_1 + q_2}{2} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\dot{q}_1 + \dot{q}_2}{2} \end{pmatrix}.$$

Therefore,

$$T_{3,l} = \frac{1}{2}\, m_3 \left(\frac{\dot{q}_1 + \dot{q}_2}{2}\right)^2.$$

For evaluating the angular contribution to the kinetic energy

$$T_{3,a} = \frac{1}{2}\, \boldsymbol{\omega}_3^T \boldsymbol{I}_b \boldsymbol{\omega}_3 = \frac{1}{2}\, {}^b\boldsymbol{\omega}_3^T\, {}^b\boldsymbol{I}_b\, {}^b\boldsymbol{\omega}_3$$

it will be convenient to work in the frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ attached to the center of mass and having the $\boldsymbol{x}_b$ along the principal direction of the bar (rotated thus w.r.t. $\boldsymbol{x}_0$ by the angle α around $\boldsymbol{y}_0$), in which the inertia matrix is diagonal. Note that this frame has $\boldsymbol{y}_b$ aligned with $\boldsymbol{y}_0$ when $q_3 = 0$. Its absolute orientation is expressed by the rotation matrix

$${}^0\boldsymbol{R}_b = {}^0\boldsymbol{R}_d(\alpha)\, {}^d\boldsymbol{R}_b(q_3) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos q_3 & -\sin q_3 \\ 0 & \sin q_3 & \cos q_3 \end{pmatrix} = \begin{pmatrix} c_\alpha & s_\alpha s_3 & s_\alpha c_3 \\ 0 & c_3 & -s_3 \\ -s_\alpha & c_\alpha s_3 & c_\alpha c_3 \end{pmatrix}.$$

We will first compute the angular velocity ${}^0\boldsymbol{\omega}_3$, expressed in the base frame, by superposition of the contributions of $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ (just as in the derivation of the geometric Jacobian). Consider the single contribution of $\dot{q}_1$, while $\dot{q}_2 = \dot{q}_3 = 0$. When the coordinate q_1 moves (say, upward, i.e., with $\dot{q}_1 > 0$), the same linear velocity is applied to the left end of the connecting bar. This can be decomposed in a velocity component along the bar, which will slightly extend it, and a velocity component along the normal to the bar, which is responsible for its rotation (clockwise when looking at the structure from the front side, and thus counterclockwise when seen from $\boldsymbol{y}_0$). When the bar is tilted by α, the resulting angular velocity contribution due to $\dot{q}_1$ will be

$$ {}^0\boldsymbol{\omega}_{3|\dot{q}_1} = \begin{pmatrix} 0 \\ \dfrac{\cos\alpha}{\delta} \\ 0 \end{pmatrix} \dot{q}_1, \qquad \text{with } \delta = \sqrt{(q_1 - q_2)^2 + (D - 2d)^2}, \tag{2} $$

being δ the actual length of the connecting bar. Indeed, as mentioned in the text, we could neglect the (small) contribution by $(q_1 - q_2)^2$ and have simply $\delta = D - 2d > 0$. Proceeding in the general case, the following relation holds from trigonometry:

$$ \begin{pmatrix} q_1 - q_2 \\ D - 2d \end{pmatrix} = \delta \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}. \tag{3} $$

Therefore, we can rewrite the contribution of $\dot{q}_1$ to ${}^0\boldsymbol{\omega}_3$ as

$$ {}^0\boldsymbol{\omega}_{3|\dot{q}_1} = \begin{pmatrix} 0 \\ \dfrac{\delta\cos\alpha}{\delta^2} \\ 0 \end{pmatrix} \dot{q}_1 = \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_1. $$

The same argument can be used for the contribution of $\dot{q}_2$, except that an opposite sign will result. Adding also the contribution of $\dot{q}_3$ to ${}^0\boldsymbol{\omega}_3$, we obtain finally

$$ {}^0\boldsymbol{\omega}_3 = \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_1 - \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_2 + {}^0\boldsymbol{R}_b \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{q}_3. $$

Therefore, in the frame attached to the bar, it is

$$ \begin{aligned} {}^b\boldsymbol{\omega}_3 = {}^0\boldsymbol{R}_b^T\,{}^0\boldsymbol{\omega}_3 &= {}^0\boldsymbol{R}_b^T \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} (\dot{q}_1 - \dot{q}_2) + \begin{pmatrix} \dot{q}_3 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \dot{q}_3 \\ \dfrac{(D - 2d)(\dot{q}_1 - \dot{q}_2)}{(q_1 - q_2)^2 + (D - 2d)^2} \cos q_3 \\ -\dfrac{(D - 2d)(\dot{q}_1 - \dot{q}_2)}{(q_1 - q_2)^2 + (D - 2d)^2} \sin q_3 \end{pmatrix}. \end{aligned} \tag{4} $$

By neglecting instead the presence of $(q_1 - q_2)^2$, we have the *reduced* expression

$$ {}^b\boldsymbol{\omega}_{3,r} = \begin{pmatrix} \dot{q}_3 \\ \dfrac{\dot{q}_1 - \dot{q}_2}{D - 2d} \cos q_3 \\ -\dfrac{\dot{q}_1 - \dot{q}_2}{D - 2d} \sin q_3 \end{pmatrix}. \tag{5} $$

When using eq. (4), the angular kinetic energy $T_{3,a}$ of the bar will be

$$\begin{aligned} T_{3,a} = \frac{1}{2}\,{}^b\boldsymbol{\omega}_3^T\,{}^b\boldsymbol{I}_b\,{}^b\boldsymbol{\omega}_3 &= \frac{1}{2}\left(I_x\dot{q}_3^2 + \left(\frac{(D-2d)(\dot{q}_1-\dot{q}_2)}{(q_1-q_2)^2+(D-2d)^2}\right)^2\left(I_y\cos^2 q_3 + I_z\sin^2 q_3\right)\right) \\ &= \frac{1}{2}\left(I_x\dot{q}_3^2 + I_y\left(\frac{D-2d}{(q_1-q_2)^2+(D-2d)^2}\right)^2(\dot{q}_1-\dot{q}_2)^2\right), \end{aligned}$$

where the last expression has been simplified using the assumption $I_y = I_z$ (indeed, a very relevant inertial property!). If eq. (5) is used instead, the (reduced) angular kinetic energy $T_{3,a|r}$ of the bar will be

$$T_{3,a|r} = \frac{1}{2}\,{}^b\boldsymbol{\omega}_{3,r}^T\;{}^b\boldsymbol{I}_b\,{}^b\boldsymbol{\omega}_{3,r} = \frac{1}{2}\left(I_x\dot{q}_3^2 + \frac{I_y}{(D-2d)^2}\,(\dot{q}_1-\dot{q}_2)^2\right).$$

At this stage, the total kinetic energy of the robot takes the usual form

$$T = T_1 + T_2 + (T_{3,l} + T_{3,a}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

with the robot inertia matrix given by

$$\boldsymbol{B}(\boldsymbol{q}) = \boldsymbol{B}' + \boldsymbol{B}''(q_1,q_2) = \begin{pmatrix} m_1 + \frac{m_3}{4} & \frac{m_3}{2} & 0 \\ \frac{m_3}{2} & m_2 + \frac{m_3}{4} & 0 \\ 0 & 0 & I_x \end{pmatrix} + \begin{pmatrix} I_y\,b(q_1,q_2) & -I_y\,b(q_1,q_2) & 0 \\ -I_y\,b(q_1,q_2) & I_y\,b(q_1,q_2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (6)$$

with the notation

$$b(q_1,q_2) = \left(\frac{D-2d}{(q_1-q_2)^2+(D-2d)^2}\right)^2.$$

We note that only the (symmetric) matrix term $\boldsymbol{B}''$ of the positive definite robot inertia matrix is configuration dependent (and depends only on the difference $q_1 - q_2$).

The associated Coriolis and centrifugal term

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix}$$

is computed as usual through the Christoffel symbols

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\left[\left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right)\right]\dot{\boldsymbol{q}}, \qquad i = 1,2,3,$$

being $\boldsymbol{b}_i(\boldsymbol{q})$ the ith column of matrix $\boldsymbol{B}(\boldsymbol{q})$. Thus

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\begin{pmatrix} I_y\,c(q_1,q_2) & -I_y\,c(q_1,q_2) & 0 \\ -I_y\,c(q_1,q_2) & I_y\,c(q_1,q_2) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$c(q_1, q_2) = \frac{\partial b(q_1, q_2)}{\partial q_1} = -\frac{4(D-2d)^2(q_1-q_2)}{\left((q_1-q_2)^2+(D-2d)^2\right)^3}$$

and where we used the specular dependence of b on q_1 and q_2. Similarly, it is easy to see that $\boldsymbol{C}_2(\boldsymbol{q}) = -\boldsymbol{C}_1(\boldsymbol{q})$. Moreover, $\boldsymbol{C}_3 = \boldsymbol{O}$. As a result, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\begin{pmatrix} I_y\, c(q_1, q_2)\, (\dot{q}_1 - \dot{q}_2)^2 \\ -I_y\, c(q_1, q_2)\, (\dot{q}_1 - \dot{q}_2)^2 \\ 0 \end{pmatrix}. \tag{7}$$

If we had used instead the reduced expression for the kinetic energy of the bar, then

$$T = T_1 + T_2 + (T_{3,l} + T_{3,a|r}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{B}\,\dot{\boldsymbol{q}}, \tag{8}$$

and the robot inertia matrix would be constant

$$\boldsymbol{B} = \begin{pmatrix} m_1 + \dfrac{m_3}{4} + \dfrac{I_y}{(D-2d)^2} & \dfrac{m_3}{2} - \dfrac{I_y}{(D-2d)^2} & 0 \\ \dfrac{m_3}{2} - \dfrac{I_y}{(D-2d)^2} & m_2 + \dfrac{m_3}{4} + \dfrac{I_y}{(D-2d)^2} & 0 \\ 0 & 0 & I_x \end{pmatrix}. \tag{9}$$

Accordingly, $\boldsymbol{c} = \boldsymbol{0}$ would follow (no Coriolis and centrifugal terms).

We turn now to the computation of the potential energy due to gravity, $U(\boldsymbol{q}) = U_1 + U_2 + U_3$. For this, since ${}^0\boldsymbol{g} = \begin{pmatrix} 0 & 0 & -g_0 \end{pmatrix}^T$ with $g_0 = 9.81$ [m/s^2], we just need to evaluate the height of the centers of masses. Thus

$$U_1 = m_1 g_0 q_1, \qquad U_2 = m_2 g_0 q_2,$$

while, in view of the assumption on the position of the center of mass of the connecting bar,

$$U_3 = m_3 g_0\, \frac{q_1 + q_2}{2}.$$

Therefore,

$$\boldsymbol{g} = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} \left(m_1 + \dfrac{m_3}{2}\right) g_0 \\ \left(m_2 + \dfrac{m_3}{2}\right) g_0 \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix}. \tag{10}$$

The gravity term is constant, and so $\partial \boldsymbol{g}/\partial \boldsymbol{q} = \boldsymbol{0}$.

Summarizing, in the general case the robot dynamic model takes the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g} = \boldsymbol{u}, \tag{11}$$

with inertia matrix, Coriolis and centrifugal term, and gravity term given respectively by eqs. (6), (7), and (10). In the reduced case, the dynamic model simplifies to

$$\boldsymbol{B}\,\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \tag{12}$$

with constant inertia matrix given by (9) and gravity term as in (10). The dynamic equations (12) are fully linear.

In order to solve the regulation problem for a desired equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \mathbf{0})$ as requested, we can use a decentralized PD control law with *constant* compensation of gravity (which is anyway constant for all configurations!). Therefore, we have

$$\boldsymbol{u} = \boldsymbol{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_d\dot{\boldsymbol{q}} + \boldsymbol{g} = \begin{pmatrix} k_{p1}(q_{d,1} - q_1) - k_{d1}\dot{q}_1 + g_1 \\ k_{p2}(q_{d,2} - q_2) - k_{d2}\dot{q}_2 + g_2 \\ k_{p3}(q_{d,3} - q_3) - k_{d3}\dot{q}_3 \end{pmatrix}. \tag{13}$$

Since the Hessian of the gravitational potential energy is identically zero, there will be no strictly positive lower bound for the (diagonal) elements of $\boldsymbol{K}_p$ in this control law: in order to guarantee global asymptotic stability, the sufficient conditions for the control gains are only $\boldsymbol{K}_p > 0$ and $\boldsymbol{K}_d > 0$. Note that this applies to both dynamic models (11) and (12). The only difference is that for (12), global *exponential* stability will be further obtained since the system dynamics is linear. In this case, the following simple modification of (13)

$$\boldsymbol{u} = \boldsymbol{B}\left(\boldsymbol{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_d\dot{\boldsymbol{q}}\right) + \boldsymbol{g},$$

i.e., with non-diagonal but still constant PD gains $\boldsymbol{B}\boldsymbol{K}_p$ and $\boldsymbol{B}\boldsymbol{K}_d$, will also provide a fully decoupled dynamics of the joint errors in the closed-loop system.

For the bonus (or alternative to the above) question, consider the sketch in Fig. 2 where, beside the variables q_1, q_2 and q_3 related to the actuated joints, we have assigned also variables β and γ to the two passive revolute joints[1], and δ to the passive prismatic joint. The purpose of our analysis is to show that the set of variables (β, γ, δ) can always be expressed as a function of (q_1, q_2) —this is similar to what we have already shown in eq. (1) for the angle α. Therefore, the robot configuration can be fully described by the minimal set (q_1, q_2, q_3), which is indeed the chosen set of configuration variables used for the Lagrangian dynamics. Because the closed kinematic chain lies always in the plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$, the variable q_3 plays no role in the analysis, and will then no longer be considered.

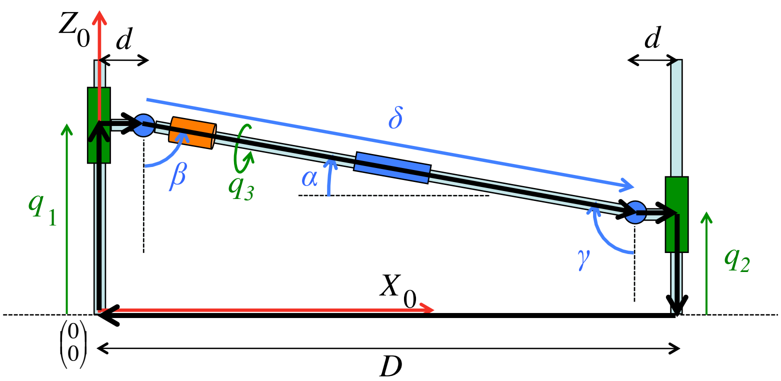

Figure 2: The portal robot as a closed kinematic chain in the plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$. The passive joints are equipped with the additional variables β, γ, and δ

[1] Note that all angles are conveniently shown in Fig. 2 with an arrow indicating their positive rotation.

The presence of a ‘loop closure’ (through the floor) in this robotic structure imposes the following kinematic constraints (refer to the vectors shown in black in Fig. 2):

$$\begin{pmatrix} 0 \\ q_1 \end{pmatrix} + \begin{pmatrix} d \\ 0 \end{pmatrix} + \delta \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} + \begin{pmatrix} d \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -q_2 \end{pmatrix} + \begin{pmatrix} -D \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

Moreover, the following relations hold among angles:

$$\beta + \gamma = \pi, \qquad \alpha + \beta = \frac{\pi}{2}. \tag{15}$$

From these, both β and γ can be expressed as a function of α and then, via eq. (1), as a function of $q_1 - q_2$. Equation (14) can be rearranged as

$$\begin{pmatrix} D - 2d \\ q_1 - q_2 \end{pmatrix} = \delta \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} = \delta \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \tag{16}$$

where the last equality follows from the second relation in (15). This is exactly the trigonometric relation (3). Moreover, squaring and adding the two equations in (16) yields

$$\delta = \sqrt{(q_1 - q_2)^2 + (D - 2d)^2} > 0,$$

which is the variable already defined in (2). Finally, dividing the second equation in (16) by the first one, we obtain

$$\tan\alpha = \frac{q_1 - q_2}{D - 2d},$$

recovering thus the original expression for α introduced in (1).

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