Robotics II

July 7, 2010

Consider the one-dimensional mass/spring/damper scheme of a robot in contact with a compliant workpiece and with a force sensing cell in between, as shown in Fig. 1. The robot and the workpiece are represented by the mass $m_r > 0$, with position x_r, and by the mass $m_w > 0$, with position x_w, respectively. The rest positions of the springs modeling the stiffness $k_s > 0$ of the force sensor and the stiffness $k_w > 0$ of the workpiece are $x_r = x_w = 0$. Viscous damping is included between the robot mass m_r and the ground, between the workpiece mass m_w and the ground, and between the two masses, with positive coefficients b_r, b_w, and b_s, respectively. An input force F is applied on the robot mass.

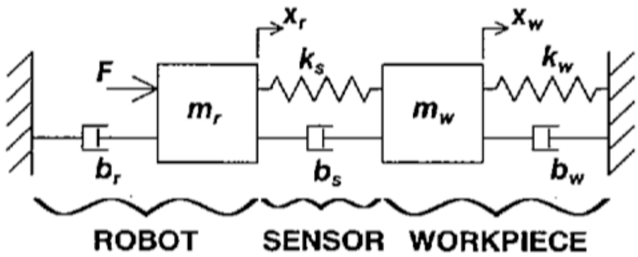

Figure 1: Robot contact model, with two masses, springs and dampers

1. Determine the differential equations of motion for this system.
2. For a contact force control problem, let the input F be specified by a proportional control law of the form $F = k_f(F_d - F_c)$, with gain k_f and desired contact force F_d, and where $F_c = k_s(x_r - x_w)$ is the force measured across the sensor. Determine the unique closed-loop equilibrium position $x_{r,e}$ and $x_{w,e}$ of the two masses m_r and m_w and show that a contact force error with respect to the desired value F_d is present at the equilibrium.
3. Which control actions should be considered in order to reduce or eliminate the presence of a steady-state contact force error?
4. *Optional.* Prove that the closed-loop equilibrium state $x_r = x_{r,e}$, $x_w = x_{w,e}$, $\dot{x}_r = \dot{x}_w = 0$ is exponentially stable for any *positive* value of the gain k_f *(Hint: Use a root locus analysis and/or the Routh criterion in the Laplace domain).*

[120 minutes (150 minutes including the optional item); open books]

Solution

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The differential equations of motion for the system are:

$$\begin{aligned} m_r\ddot{x}_r + b_r\dot{x}_r + b_s(\dot{x}_r - \dot{x}_w) + k_s(x_r - x_w) &= F \\ m_w\ddot{x}_w + b_w\dot{x}_w + b_s(\dot{x}_w - \dot{x}_r) + k_s(x_w - x_r) + k_w x_w &= 0. \end{aligned} \tag{1}$$

Setting $F = k_f(F_d - F_c) = k_f(F_d - k_s(x_r - x_w))$ and evaluating the system at an equilibrium (i.e., setting $\dot{x}_r = \dot{x}_w = \ddot{x}_r = \ddot{x}_w = 0$), gives

$$\begin{aligned} k_s(x_r - x_w) &= k_f(F_d - k_s(x_r - x_w)) \\ k_s(x_w - x_r) + k_w x_w &= 0. \end{aligned}$$

It is easy to see that the unique solution is

$$x_{r.e} = \frac{k_s + k_w}{k_s k_w}\frac{k_f}{1+k_f}F_d \qquad x_{w.e} = \frac{1}{k_w}\frac{k_f}{1+k_f}F_d,$$

that provides also the contact force at the equilibrium

$$F_{c,e} = k_s(x_{r,e} - x_{w,e}) = \frac{k_f}{1+k_f}F_d.$$

Therefore, there will be an error on the desired force given by

$$e_F = F_d - F_{c,e} = \frac{1}{1+k_f}F_d.$$

This error can be reduced by amplifying the gain k_f, but cannot be eliminated when using a simple proportional controller. This situation typically asks for the introduction of an integral action of the type

$$F = k_f(F_d - F_c) + k_i\int_0^t (F_d - F_c)d\tau = k_f(F_d - k_s(x_r - x_w)) + k_i\int_0^t (F_d - k_s(x_r - x_w))d\tau.$$

At steady-state ($t \to \infty$), the argument of the integral term should vanish. Therefore,

$$\lim_{t\to\infty}\int_0^t (F_d - k_s(x_r - x_w))d\tau = constant \quad \Rightarrow \quad \lim_{t\to\infty} k_s(x_r - x_w) = F_d.$$

The modified equilibrium (denoted with a prime) will be

$$x'_{r.e} = \frac{k_s + k_w}{k_s k_w}F_d \qquad x'_{w.e} = \frac{1}{k_w}F_d,$$

and the associated contact force will be desired one,

$$F'_{c,e} = k_s(x'_{r,e} - x'_{w,e}) = F_d,$$

as expected. The same result is obtained if we combine the feedback control action with a constant feedforward term F_d, i.e.

$$F = F_d + k_f(F_d - F_c).$$

The equilibrium configuration should satisfy then

$$\begin{aligned} k_s(x_r - x_w) &= F_d + k_f(F_d - k_s(x_r - x_w)) \\ k_s(x_w - x_r) + k_w x_w &= 0, \end{aligned}$$

leading again to $F'_{c,e} = F_d$ and the same previous positions $x'_{r.e}$ and $x'_{w,e}$.

Indeed, no matter if we keep the proportional control law or add also the integral action or the feedforward term, we need to show that the associated equilibrium is asymptotically stable for the closed-loop system (otherwise it would never be reached from a generic initial state). Since the system is linear, asymptotic stability is equivalent to exponential stability. Moreover, whenever it holds, this result is global.

Considering again the case of a proportional force controller, the easiest way to prove asymptotic stability of the closed-loop state $x_r = x_{r,e}$, $x_w = x_{w,e}$, $\dot{x}_r = \dot{x}_w = 0$ is to use Laplace transform and the root locus method, thanks to the linearity of the system. From eq. (1) we have

$$\begin{aligned} \left(m_r s^2 + (b_r + b_s)s + k_s\right) X_r(s) &= F(s) + (b_s s + k_s) X_w(s) \\ \left(m_w s^2 + (b_w + b_s)s + (k_w + k_s)\right) X_w(s) &= (b_s s + k_s) X_r(s), \end{aligned}$$

where $X_r(s)$, $X_w(s)$, and $F(s)$ are the Laplace transforms of $x_r(t)$, $x_w(t)$, and $F(t)$. Defining for compactness

$$N_s(s) = b_s s + k_s \quad D_r(s) = \left(m_r s^2 + (b_r + b_s)s + k_s\right) \quad D_w(s) = \left(m_w s^2 + (b_w + b_s)s + (k_w + k_s)\right),$$

we can solve for

$$\frac{X_r(s)}{F(s)} = \frac{D_w(s)}{D_r(s)D_w(s) - N_s^2(s)} \qquad \frac{X_w(s)}{F(s)} = \frac{N_s(s)}{D_r(s)D_w(s) - N_s^2(s)} \qquad \frac{X_w(s)}{X_r(s)} = \frac{N_s(s)}{D_w(s)}.$$

Being the output defined as the contact force $F_c(s) = k_s(X_r(s) - X_w(s))$, the transfer function of the (open-loop) system is

$$P(s) = \frac{F_c(s)}{F(s)} = k_s \, \frac{D_w(s) - N_s(s)}{D_r(s)D_w(s) - N_s^2(s)}.$$

Closing the feedback loop with $F(s) = k_f(F_d(s) - F_c(s))$, where $F_d(s)$ is the Laplace transform of the step reference input $F_d(t) = F_d \cdot \delta_{-1}(t)$, we obtain the transfer function

$$W(s) = \frac{F_c(s)}{F_d(s)} = \frac{k_f P(s)}{1 + k_f P(s)} = \frac{k_f k_s (D_w(s) - N_s(s))}{(D_r(s)D_w(s) - N_s^2(s)) + k_f\,(k_s(D_w(s) - N_s(s)))}.$$

Asymptotic stability of the closed-loop system depends on the location on the complex plane s of the poles of $W(s)$, i.e., of the roots of the polynomial equation

$$\left(D_r(s)D_w(s) - N_s^2(s)\right) + k_f\,(k_s(D_w(s) - N_s(s))) = A(s) + k_f B(s) = 0. \tag{2}$$

By varying k_f in (2), we can explore the root locus. For $k_f = 0$, the (four) poles coincide with the poles of the open-loop system $P(s)$, i.e., with the roots of

$$\begin{aligned} A(s) &= D_r(s)D_w(s) - N_s^2(s) \\ &= m_r m_w s^4 + ((b_w + b_s)m_r + (b_r + b_s)m_w)\, s^3 + ((k_s + k_w)m_r + k_s m_w + b_r b_s + b_s b_w + b_r b_w)\, s^2 \\ &\quad + ((b_r + b_w)k_s + (b_r + b_s)k_w)\, s + k_s k_w = 0. \end{aligned}$$

Applying the Routh criterion to the fourth-order polynomial $A(s)$ and using the positivity of all physical coefficients in (1), it is straightforward to check that the four poles of $P(s)$ have all negative real parts (the process itself is asymptotically stable). Moreover, by the standard rules of the root locus, when k_f is increased toward $+\infty$, two of the closed-loop poles converge to the open-loop zeros, i.e., to the roots of

$$B(s) = k_s \left(D_w(s) - N_s(s)\right) = k_s \left(m_w s^2 + b_w s + k_w\right) = 0,$$

which are in the left-hand side of the complex plane. The two other poles will approach the asymptotes of the positive root locus, which are vertical (since the pole-zero excess of $P(s)$ is $n - m = 2$) and are located in the left-hand side of the complex plane s. As a result, the closed-loop system is asymptotically stable for any value of $k_f \geq 0$.

Figure 2 shows the complete positive root locus obtained with Matlab, for some arbitrary (positive) values of the model parameters, confirming the above analysis.

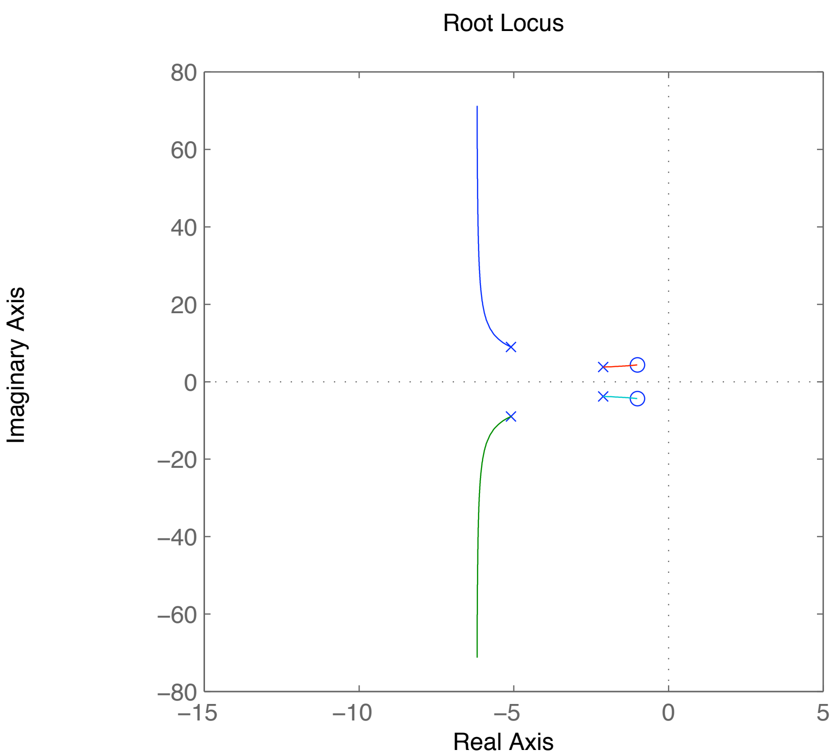

Figure 2: A typical root locus of eq. (2), when varying $k_f \in [0, +\infty)$

Indeed, the same asymptotic stability result could have been proven by applying the Routh criterion to check the locations of roots of the whole polynomial on the left-hand side of (2) (but this would have been more tedious).

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