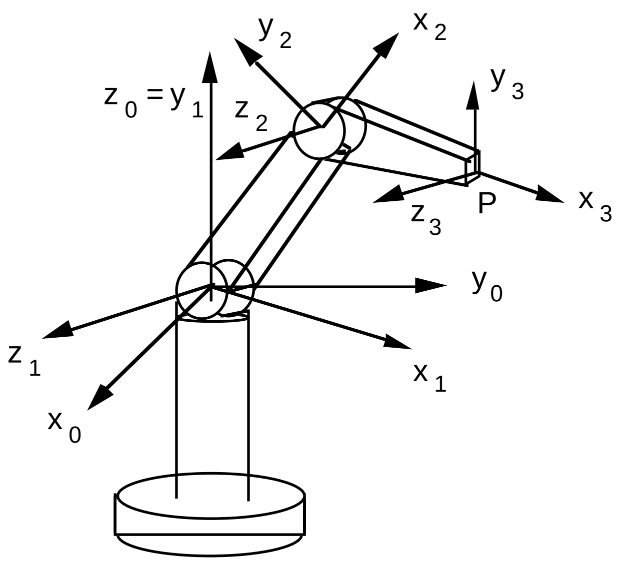
Compito di Robotica II

Origine: Robotica Industriale, Autovalutazione 11 Aprile 1997

Si consideri il seguente robot antropomorfo, a tre gradi di libertà rotatori, con le relative terne di riferimento assegnate secondo il formalismo di Denavit-Hartenberg.



[1] Determinare il modello dinamico di tale robot nella forma

$$B(q)\ddot{q} + c(q, \dot{q}) + g(q) = u, \tag{1}$$

assumendo che:

a) la posizione del baricentro del braccio $i$esimo sia lungo l'asse x_i, per $i = 1, 2, 3$;

b) la matrice di inerzia baricentrale del braccio $i$esimo sia diagonale, per $i = 1, 2, 3$;

c) l'asse z_0 sia verticale.

[2] Ricavare una fattorizzazione dei termini in velocità nella dinamica (1)

$$c(q, \dot{q}) = S(q, \dot{q})\dot{q},$$

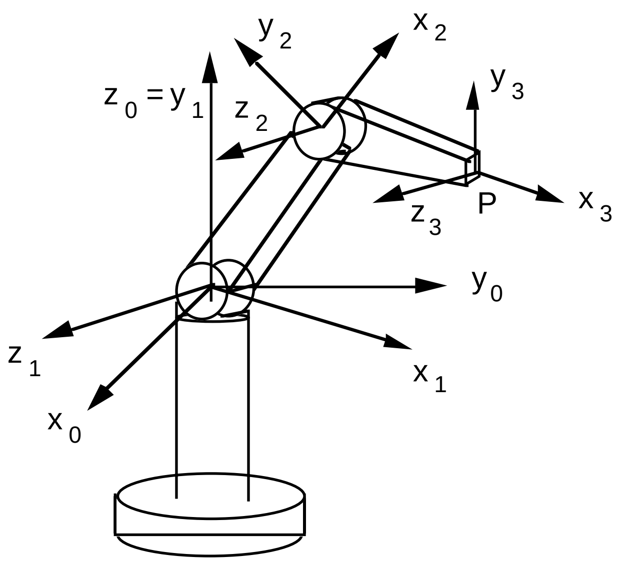
tale per cui $\dot{B} - 2S$ sia una matrice antisimmetrica.

[3] Ricavare le espressioni di una fattorizzazione lineare della dinamica (1) in termini di un vettore di coefficienti dinamici a nella forma

$$Y(q, \dot{q}, \ddot{q})\, a = u.$$

[210 minuti di tempo; libri aperti]

Soluzione



Matrici e assi di rotazione

$$R_1 = \begin{bmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{bmatrix} \qquad z_0 = z_{0|0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad z_1 = R_1 z_{1|1} = R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad z_2 = (R_1 R_2) z_{2|2} = (R_1 R_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

notazione: $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $s_{ij} = \sin(\theta_i + \theta_j)$, $c_{ij} = \cos(\theta_i + \theta_j)$

Calcolo ricorsivo velocità angolari e lineari (moving frames)

$$\omega_{0|0} = 0 \qquad v_{0|0} = 0$$

i=1

$$\omega_{1|1} = R_1^T \left[\omega_{0|0} + \dot{\theta}_1 z_{0|0}\right] = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$v_{1|1} = R_1^T \left[v_{0|0} + (\omega_{0|0} + \dot{\theta}_1 z_{0|0}) \times r_{0,1|0}\right] = 0$$

$$r_{1,c1|1} = \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{c1|1} = v_{1|1} + \omega_{1|1} \times r_{1,c1|1} = \begin{bmatrix} 0 \\ 0 \\ -d_1\dot{\theta}_1 \end{bmatrix}$$

i=2

$$\omega_{2|2} = R_2^T \left[\omega_{1|1} + \dot{\theta}_2 z_{1|1}\right] = \begin{bmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$v_{2|2} = R_2^T \left[v_{1|1} + (\omega_{1|1} + \dot{\theta}_2 z_{1|1}) \times r_{1,2|1}\right] = \begin{bmatrix} 0 \\ \ell_2\dot{\theta}_2 \\ -\ell_2 c_2\dot{\theta}_1 \end{bmatrix}$$

$$r_{2,c2|2} = \begin{bmatrix} -\ell_2 + d_2 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{c2|2} = v_{2|2} + \omega_{2|2} \times r_{2,c2|2} = \begin{bmatrix} 0 \\ d_2\dot{\theta}_2 \\ -d_2 c_2\dot{\theta}_1 \end{bmatrix}$$

i=3

$$\omega_{3|3} = R_3^T \left[\omega_{2|2} + \dot{\theta}_3 z_{2|2}\right] = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$$\begin{aligned} v_{3|3} &= R_3^T \left[v_{2|2} + (\omega_{2|2} + \dot{\theta}_3 z_{2|2}) \times r_{2,3|2}\right] = R_3^T v_{2|2} + \omega_{3|3} \times r_{2,3|3} \\ &= \begin{bmatrix} \ell_2 s_3 \dot{\theta}_2 \\ \ell_2 c_3 \dot{\theta}_2 \\ -\ell_2 c_2 \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \ell_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -\ell_3 c_{23}\dot{\theta}_1 \end{bmatrix} \end{aligned}$$

$$r_{3,c3|3} = \begin{bmatrix} -\ell_3 + d_3 \\ 0 \\ 0 \end{bmatrix}$$

$$v_{c3|3} = v_{3|3} + \omega_{3|3} \times r_{3,c3|3} = \begin{bmatrix} \ell_2 s_3 \dot{\theta}_2 \\ \ell_2 c_3 \dot{\theta}_2 + d_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -\ell_2 c_2 \dot{\theta}_1 - d_3 c_{23}\dot{\theta}_1 \end{bmatrix}$$

nota bene: $A(b \times c) \neq Ab \times Ac$, a meno che $A = R$ di rotazione

Energia cinetica

$$T_1 = \frac{1}{2}(I_{yy1} + m_1 d_1^2)\dot{\theta}_1^2$$

$$T_2 = \frac{1}{2}\left[I_{xx2}s_2^2 + (I_{yy2} + m_2 d_2^2)c_2^2\right]\dot{\theta}_1^2 + \frac{1}{2}(I_{zz2} + m_2 d_2^2)\dot{\theta}_2^2$$

$$\begin{aligned} T_3 = \frac{1}{2}m_3 \Big[&\ell_2^2\dot{\theta}_2^2 + d_3^2(\dot{\theta}_2 + \dot{\theta}_3)^2 + 2\ell_2 d_3 c_3 \dot{\theta}_2(\dot{\theta}_2 + \dot{\theta}_3) \\ &+ (\ell_2^2 c_2^2 + d_3^2 c_{23}^2 + 2\ell_2 d_3 c_2 c_{23})\dot{\theta}_1^2\Big] \\ &+ \frac{1}{2}(I_{xx3}s_{23}^2 + I_{yy3}c_{23}^2)\dot{\theta}_1^2 + \frac{1}{2}I_{zz3}(\dot{\theta}_2 + \dot{\theta}_3)^2 \end{aligned}$$

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\dot{\theta}^T B(\theta)\dot{\theta}$$

Matrice di inerzia

$$B(\theta) = \begin{bmatrix} B_{11}(\theta_2, \theta_3) & 0 & 0 \\ 0 & B_{22}(\theta_3) & B_{23}(\theta_3) \\ 0 & B_{32}(\theta_3) & B_{33} \end{bmatrix}$$

$$\begin{aligned} B_{11} = &I_{yy1} + m_1 d_1^2 + I_{xx2}\sin^2\theta_2 + (I_{yy2} + m_2 d_2^2 + m_3\ell_2^2)\cos^2\theta_2 \\ &+ I_{xx3}\sin^2(\theta_2 + \theta_3) + (I_{yy3} + m_3 d_3^2)\cos^2(\theta_2 + \theta_3) \\ &+ 2m_3\ell_2 d_3 \cos\theta_2 \cos(\theta_2 + \theta_3) \end{aligned}$$

$$B_{22} = I_{zz2} + m_2 d_2^2 + I_{zz3} + m_3 d_3^2 + m_3\ell_2^2 + 2m_3\ell_2 d_3 \cos\theta_3$$

$$B_{23} = I_{zz3} + m_3 d_3^2 + m_3\ell_2 d_3\cos\theta_3$$

$$B_{32} = B_{23}$$

$$B_{33} = I_{zz3} + m_3 d_3^2$$

Fattorizzazione elementi di $B(\theta)$

$$B(\theta) = \begin{bmatrix} \begin{matrix} a_1 + a_2 c_2^2 + a_3 c_{23}^2 \\ +2a_4 c_2 c_{23} \end{matrix} & 0 & 0 \\ 0 & a_5 + 2a_4 c_3 & a_6 + a_4 c_3 \\ 0 & a_6 + a_4 c_3 & a_6 \end{bmatrix}$$

$$a_1 = I_{yy1} + m_1 d_1^2 + I_{xx2} + I_{xx3}$$

$$a_2 = I_{yy2} + m_2 d_2^2 + m_3 \ell_2^2 - I_{xx2}$$

$$a_3 = I_{yy3} + m_3 d_3^2 - I_{xx3}$$

$$a_4 = m_3 \ell_2 d_3$$

$$a_5 = I_{zz2} + m_2 d_2^2 + I_{zz3} + m_3 d_3^2 + m_3 \ell_2^2$$

$$a_6 = I_{zz3} + m_3 d_3^2$$

nota bene:

da 8 parametri “apparenti” a 6 con $s_2^2 = 1 - c_2^2$ e $s_{23}^2 = 1 - c_{23}^2$

Derivazione simboli di Christoffel

$$C_i(\theta) = \frac{1}{2}\left[\frac{\partial b_i}{\partial \theta} + \left(\frac{\partial b_i}{\partial \theta}\right)^T - \frac{\partial B}{\partial \theta_i}\right] = \{C_{ijk}\} \qquad i = 1, 2, 3$$

con b_i $i$esima colonna di $B(\theta)$

i=1

$$\frac{\partial b_1}{\partial \theta} = \begin{bmatrix} 0 & -\big(a_2 \sin(2\theta_2) + a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \sin(2\theta_2+\theta_3)\big) & -\big(a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \cos\theta_2 \sin(\theta_2+\theta_3)\big) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial B}{\partial \theta_1} = 0$$

elementi diversi da zero in $C_1(\theta)$

$$C_{112} = -\frac{1}{2}\big(a_2 \sin(2\theta_2) + a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \sin(2\theta_2+\theta_3)\big)$$

$$C_{113} = -\frac{1}{2}\big(a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \cos\theta_2 \sin(\theta_2+\theta_3)\big)$$

$$C_{121} = C_{112}$$

$$C_{131} = C_{113}$$

i=2

$$\frac{\partial b_2}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2a_4 \sin\theta_3 \\ 0 & 0 & -a_4 \sin\theta_3 \end{bmatrix}$$

$$\frac{\partial B}{\partial \theta_2} = \begin{bmatrix} -\big(a_2 \sin(2\theta_2) + a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \sin(2\theta_2+\theta_3)\big) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

elementi diversi da zero in $C_2(\theta)$

$$C_{211} = \frac{1}{2}\big(a_2 \sin(2\theta_2) + a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \sin(2\theta_2+\theta_3)\big)$$

$$C_{223} = -a_4 \sin\theta_3$$

$$C_{232} = C_{223}$$

$$C_{233} = -a_4 \sin\theta_3$$

i=3

$$\frac{\partial b_3}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_4 \sin\theta_3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial B}{\partial \theta_3} = \begin{bmatrix} -\big(a_3 \sin[2(\theta_2+\theta_3)] + 2a_4 \cos\theta_2 \sin(\theta_2+\theta_3)\big) & 0 & 0 \\ 0 & -2a_4 \sin\theta_3 & -a_4 \sin\theta_3 \\ 0 & -a_4 \sin\theta_3 & 0 \end{bmatrix}$$

elementi diversi da zero in $C_3(\theta)$

$$C_{311} = \frac{1}{2} a_3 \sin[2(\theta_2+\theta_3)] + a_4 \cos\theta_2 \sin(\theta_2+\theta_3)$$

$$C_{322} = a_4 \sin\theta_3$$

Vettore dei termini di Coriolis e centrifughi

$$c_i(\theta, \dot{\theta}) = \dot{\theta}^T C_i(\theta)\dot{\theta} \qquad i = 1, 2, 3$$

$$c(\theta, \dot{\theta}) = \begin{bmatrix} (C_{112} + C_{121})\dot{\theta}_1\dot{\theta}_2 + (C_{113} + C_{132})\dot{\theta}_1\dot{\theta}_3 \\ C_{211}\dot{\theta}_1^2 + (C_{223} + C_{232})\dot{\theta}_2\dot{\theta}_3 + C_{233}\dot{\theta}_3^2 \\ C_{311}\dot{\theta}_1^2 + C_{322}\dot{\theta}_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\big(a_2\sin(2\theta_2) + a_3\sin[2(\theta_2+\theta_3)] + 2a_4\sin(2\theta_2+\theta_3)\big)\dot{\theta}_1\dot{\theta}_2 \\ -\big(a_3\sin[2(\theta_2+\theta_3)] + 2a_4\cos\theta_2\sin(\theta_2+\theta_3)\big)\dot{\theta}_1\dot{\theta}_3 \\ \\ \frac{1}{2}\big(a_2\sin(2\theta_2) + a_3\sin[2(\theta_2+\theta_3)] + 2a_4\sin(2\theta_2+\theta_3)\big)\dot{\theta}_1^2 \\ -2a_4\sin\theta_3\dot{\theta}_2\dot{\theta}_3 - a_4\sin\theta_3\dot{\theta}_3^2 \\ \\ \big(\frac{1}{2}a_3\sin[2(\theta_2+\theta_3)] + a_4\cos\theta_2\sin(\theta_2+\theta_3)\big)\dot{\theta}_1^2 + a_4\sin\theta_3\dot{\theta}_2^2 \end{bmatrix}$$

Energia potenziale

$$U_i = -m_i g^T r_{0,ci} + U_{i,0} \qquad g = \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} \qquad (g_0 = 9.81)$$

$$\begin{aligned} r_{0,c1} &= [\, d_1 c_1 \quad d_1 s_1 \quad 0 \,]^T \\ U_1 &= U_{1,0} \end{aligned}$$

$$\begin{aligned} r_{0,c2} &= [\, * \quad * \quad d_2 s_2 \,]^T \\ U_2 &= U_{2,0} + g_0 m_2 d_2 \sin\theta_2 \end{aligned}$$

$$\begin{aligned} r_{0,c3} &= [\, * \quad * \quad \ell_2 s_2 + d_3 s_{23} \,]^T \\ U_3 &= U_{3,0} + g_0 m_3 \ell_2 \sin\theta_2 + g_0 m_3 d_3 \sin(\theta_2 + \theta_3) \end{aligned}$$

$$U = U_1 + U_2 + U_3 \qquad\qquad g(\theta) = \frac{\partial U}{\partial \theta}^T$$

Vettore dei termini di gravità

$$g(\theta) = \begin{bmatrix} 0 \\ a_7 c_2 + a_8 c_{23} \\ a_8 c_{23} \end{bmatrix}$$

$$a_7 = g_0(m_2 d_2 + m_3 \ell_2)$$

$$a_8 = g_0 m_3 d_3$$

Fattorizzazione termini di velocità

$$c(\theta,\dot\theta) = S(\theta,\dot\theta)\dot\theta \quad \Longrightarrow \quad \dot B - 2S \text{ antisimmetrica}$$

$$\Downarrow$$

$$S(\theta,\dot\theta) = \begin{bmatrix} s_1^T(\theta,\dot\theta) \\ s_2^T(\theta,\dot\theta) \\ s_3^T(\theta,\dot\theta) \end{bmatrix} = \begin{bmatrix} \dot\theta^T C_1(\theta) \\ \dot\theta^T C_2(\theta) \\ \dot\theta^T C_2(\theta) \end{bmatrix}$$

con $C_i(\theta)$ matrici dei simboli di Christoffel

Parametrizzazione lineare del modello dinamico

$$B(\theta)\ddot\theta + c(\theta,\dot\theta) + g(\theta) = Y(\theta,\dot\theta,\ddot\theta)a = u$$

$$a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix}^T$$

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & 0 & 0 & 0 & 0 \\ 0 & Y_{22} & Y_{23} & Y_{24} & Y_{25} & Y_{26} & Y_{27} & Y_{28} \\ 0 & 0 & Y_{33} & Y_{34} & 0 & Y_{36} & 0 & Y_{38} \end{bmatrix}$$

elementi diversi da zero in $Y(\theta,\dot\theta,\ddot\theta)$

$$Y_{11} = \ddot\theta_1$$

$$Y_{12} = \cos^2\theta_2\,\ddot\theta_1 - \sin(2\theta_2)\dot\theta_1\dot\theta_2$$

$$Y_{13} = \cos^2(\theta_2+\theta_3)\,\ddot\theta_1 - \sin[2(\theta_2+\theta_3)]\dot\theta_1(\dot\theta_2+\dot\theta_3)$$

$$\begin{aligned} Y_{14} &= 2\cos\theta_2\cos(\theta_2+\theta_3)\,\ddot\theta_1 - 2\sin(2\theta_2+\theta_3)\dot\theta_1\dot\theta_2 \\ &\quad - 2\cos\theta_2\sin(\theta_2+\theta_3)\dot\theta_1\dot\theta_3 \end{aligned}$$

$$Y_{22} = \frac{1}{2}\sin(2\theta_2)\dot{\theta}_1^2$$

$$Y_{23} = \sin[2(\theta_2 + \theta_3)]\dot{\theta}_1^2$$

$$Y_{24} = \cos\theta_3(2\ddot{\theta}_2 + \ddot{\theta}_3) + 2\sin(2\theta_2 + \theta_3)\dot{\theta}_1^2 - \sin\theta_3(2\dot{\theta}_2 + \dot{\theta}_3)\dot{\theta}_3$$

$$Y_{25} = \ddot{\theta}_2$$

$$Y_{26} = \ddot{\theta}_3$$

$$Y_{27} = \cos\theta_2$$

$$Y_{28} = \cos(\theta_2 + \theta_3)$$

$$Y_{33} = \frac{1}{2}\sin[2(\theta_2 + \theta_3)]\dot{\theta}_1^2$$

$$Y_{34} = \cos\theta_3\,\ddot{\theta}_2 + \cos\theta_2\sin(\theta_2 + \theta_3)\dot{\theta}_1^2 + \sin\theta_3\dot{\theta}_2^2$$

$$Y_{36} = \ddot{\theta}_2 + \ddot{\theta}_3$$

$$Y_{38} = \cos(\theta_2 + \theta_3)$$

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