Robotics 2

Robust Trajectory Control

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Problem formulation

- given the real robot, modeled by

$$B(q)\ddot{q} + n(q,\dot{q}) = u$$

- assuming an estimated feedback linearization control

$$u = \hat{B}(q)a + \hat{n}(q,\dot{q})$$

- we would like to design a so as to obtain
  - asymptotic stability of the closed-loop system
  - the best possible trajectory tracking performance
- the linear feedback choice is not enough...

$$a = \ddot{q}_d + K_D(\dot{q}_d - \dot{q}) + K_P(q_d - q)$$

- questions:
  - which should be the conditions on the estimates?
  - can we guarantee stability/performance, based on known bounds on the uncertainties?

Closed-loop equations - 1

- under uncertain conditions (estimated ≠ real dynamic coefficients), feedback linearization is only approximate and the closed-loop equations are still nonlinear

$$
\begin{aligned}
\ddot{q} &= B^{-1}(q)(\hat{B}(q)a + \hat{n}(q,\dot{q}) - n(q,\dot{q})) \\
&= a + (B^{-1}(q)\hat{B}(q) - I)a \\
&\qquad + B^{-1}(q)(\hat{n}(q,\dot{q}) - n(q,\dot{q})) \\
&= a + E(q)a + B^{-1}(q)\Delta n(q,\dot{q}) \\
&= a + \eta(a,q,\dot{q})
\end{aligned}
$$

where η depends on the amount of uncertainty

$$
E(q) = B^{-1}(q)\Delta B(q) = B^{-1}(q)(\hat{B}(q) - B(q))
$$

$$
\Delta n(q,\dot{q}) = \hat{n}(q,\dot{q}) - n(q,\dot{q})
$$

Closed-loop equations - 2

- closed-loop state equations are written as

$$\dot{x} = \mathcal{A}x + \mathcal{B}(a + \eta)$$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

- closed-loop error equations with respect to a desired $q_d(t)$ are rewritten as

$$e_1 = x_1 - x_{1d} = q - q_d \qquad e_2 = x_2 - x_{2d} = \dot{q} - \dot{q}_d$$

note that errors are defined here with opposite signs w.r.t. usual

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = \ddot{q} - \ddot{q}_d = a + \eta(a, e_1, e_2, q_d, \dot{q}_d) - \ddot{q}_d$$

$$\Rightarrow \quad \boxed{\dot{e} = \mathcal{A}e + \mathcal{B}(a + \eta - \ddot{q}_d)}$$

Solution approach

- add an external robust control term/loop
  - based on computable bounds on the uncertainties
- based on the theory of guaranteed stability for nonlinear uncertain system
- Lyapunov-based analysis
- a discontinuous control law will result
  - difficult to implement because of chattering effects
  - smoothed version with only uniformly ultimate boundedness (u.u.b. stability) of the tracking error

Working assumptions

1. bound on the desired trajectory

$$\sup_{t\geq 0} \|\ddot{q}_d\| < Q_{\max} < \infty$$

2. bound on the estimate of the robot inertia matrix

$$\|E(q)\| = \|B^{-1}(q)\hat{B}(q) - I\| \leq \alpha < 1$$

   with $\alpha \geq 0$, holding for all configurations q

3. bound on the estimate of nonlinear dynamic terms

$$\|\Delta n(q, \dot{q})\| \leq \phi(e, t)$$

   with a known function ϕ , bounded for all t

   - as a general rule, exploiting the model structure (e.g., its linear parameterization) may lead to more stringent bounds

Bound on the inertia matrix

- assumption 2. can always be satisfied, knowing some upper and lower bounds (that always exist due to the positive definiteness) on the inverse of the inertia matrix

$$0 < m \leq \|B^{-1}(q)\| \leq M < \infty$$

- it is then sufficient to choose as estimate

$$\hat{B} = \frac{1}{c} I \qquad \text{with} \qquad c = \frac{M+m}{2}$$

- in fact, using the SVD factorization of the inverse inertia matrix, it can be shown that (see Appendix A)

$$\|B^{-1}\hat{B} - I\| \leq \frac{M-m}{M+m} = \alpha < 1$$

Control design – step 1

- linear control law with an added robust term

$$a = \ddot{q}_d - K_P e_1 - K_D e_2 + \Delta a$$

where the PD gains are diagonal and positive matrices

- we obtain

$$\dot{e} = \bar{\mathcal{A}} e + \mathcal{B}(\Delta a + \bar{\eta})$$

being

$$\bar{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{K} \qquad \mathcal{K} = [\,K_P \quad K_D\,]$$

where $\bar{\mathcal{A}}$ has all eigenvalues with negative real part, and

$$\bar{\eta} = E(\ddot{q}_d - \mathcal{K}e + \Delta a) + B^{-1}\Delta n$$

Control design – step 2

- (same) bound on nonlinear terms and added robust term

$$\|\bar{\eta}\| < \rho(e,t) \qquad \|\Delta a\| < \rho(e,t)$$

- we can use the previous data and implicitly define the bound $\rho(e,t)$ from

$$\begin{aligned}\|\bar{\eta}\| &= \|E\Delta a + E(\ddot{q}_d - \mathcal{K}e) + B^{-1}\Delta n\| \\ &\le \alpha\rho(e,t) + \alpha(Q_{\max} + \|\mathcal{K}\| \cdot \|e\|) + M\phi(e,t) \\ &=: \rho(e,t)\end{aligned}$$

yielding the well-defined (since $0 < \alpha < 1$), limited and possibly time-varying function

$$\boxed{\rho(e,t)} = \frac{1}{1-\alpha}\left[\alpha(Q_{\max} + \|\mathcal{K}\| \cdot \|e\|) + M\phi(e,t)\right]$$

Control design – step 3

- solve an associated (linear) Lyapunov equation, for any given symmetric $Q > 0$ matrix

$$\bar{A}^T P + P\bar{A} + Q = 0$$

finding the unique (symmetric) solution matrix $P > 0$

- finally, define the discontinuous robust term as

$$\Delta a = \begin{cases} -\rho(e,t)\dfrac{\mathcal{B}^T P e}{\|\mathcal{B}^T P e\|} & \text{if } \|\mathcal{B}^T P e\| \neq 0 \\ 0 & \text{if } \|\mathcal{B}^T P e\| = 0 \end{cases}$$

that also satisfies, by its own structure, $\|\Delta a\| < \rho(e,t)$

Solving the Lyapunov equation

- in general, using `lyap` in Matlab (only once, in advance)
- closed-form solution in an interesting scalar case (one robot joint/link), to get a "feeling"...

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathcal{K} = \begin{bmatrix} k_P & k_D \end{bmatrix} \quad \bar{\mathcal{A}} = \mathcal{A} - \mathcal{B}\mathcal{K} = \begin{bmatrix} 0 & 1 \\ -k_P & -k_D \end{bmatrix} \quad k_P > 0, k_D > 0$$

choose, e.g., $Q = q \cdot I_{2\times 2} > 0$ $\Rightarrow$ find $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \underbrace{> 0}_{\text{test!}}$

$$\bar{\mathcal{A}}^T P + P \bar{\mathcal{A}} + q \cdot I =$$

$$= \begin{bmatrix} -2p_{12}k_P + q & p_{11} - p_{12}k_D - p_{22}k_P \\ \text{symm} & 2(p_{12} - k_D p_{22}) + q \end{bmatrix} = 0$$

$$p_{12} = \frac{q}{2k_P} \qquad p_{22} = \frac{q}{2}\left(\frac{1+k_P}{k_P k_D}\right) > 0$$

$$p_{11} = \frac{q}{2}\left(\frac{k_D}{k_P} + \frac{1+k_P}{k_D}\right) > 0$$

$$\Rightarrow \quad p_{11}p_{22} - p_{12}^2 > 0$$

so that (also in the n-dof case) $\boxed{\mathcal{B}^T P e} = \text{block}\left\{\frac{q}{2}\left(\frac{e_1}{k_P} + \frac{1+k_P}{k_P}\frac{e_2}{k_D}\right)\right\}$

Stability analysis

Theorem 1

Defining $V(e) = e^T P e$, the presented robust control law with the discontinuous term is such that $\dot{V}(e) < 0$ along the trajectories of the closed-loop error system

Proof

$$\begin{aligned}\dot{V}(e) &= \dot{e}^T P e + e^T P \dot{e} \\ &= e^T \left(\bar{A}^T P + P\bar{A}\right) e + 2e^T P\mathcal{B}(\Delta a + \bar{\eta}) \\ &= -e^T Q e + 2e^T P\mathcal{B}(\Delta a + \bar{\eta}) \\ &= -e^T Q e + 2w^T(\Delta a + \bar{\eta})\end{aligned}$$

if $w = 0 \quad \Rightarrow \quad \dot{V} = -e^T Q e < 0$

if $w \neq 0 \quad \Rightarrow \quad \Delta a = -\rho w / \|w\| \quad \Rightarrow$

$$\begin{aligned} w^T(-\rho\frac{w}{\|w\|} + \bar{\eta}) &= -\rho\frac{w^T w}{\|w\|} + w^T\bar{\eta} \\ &\leq -\rho\|w\| + \|w\| \cdot \|\bar{\eta}\| \\ &= \|w\|(-\rho + \|\bar{\eta}\|) \leq 0 \end{aligned}$$

$\Rightarrow \quad \dot{V} < 0$

note: because of the discontinuity we cannot directly conclude on the (global) asymptotic stability of e=0

A smoother robust controller

- for any given (small) $\epsilon > 0$, define the continuous robust term as

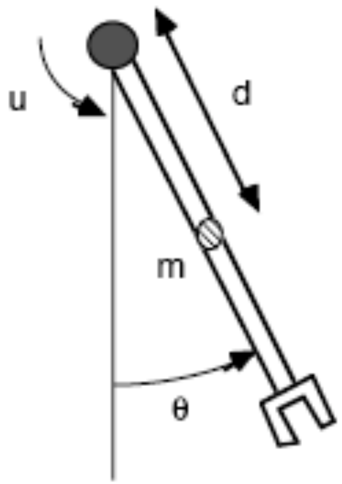
$$\Delta a = \begin{cases} -\rho(e,t)\dfrac{\mathcal{B}^T P e}{\| \mathcal{B}^T P e\|} & \text{if } \|\mathcal{B}^T P e\| \geq \epsilon \\ -\dfrac{\rho(e,t)}{\epsilon} \mathcal{B}^T P e & \text{if } \|\mathcal{B}^T P e\| < \epsilon \end{cases}$$

Theorem 2

With the above continuous robust control law, any solution $e(t)$, with $e(0) = e_0$, of the closed-loop error system is uniformly ultimately bounded with respect to a suitable set S (a neighborhood of the origin)

Proof in Appendix B

Case study: Single-link under gravity



model $I\ddot{\theta} + mgd\sin\theta = u$ (no friction)

error equations

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = \frac{1}{I}u - \frac{mgd}{I}\sin\theta - \ddot{\theta}_d$$

$$= \frac{1}{I}[\hat{I}(a+\Delta a) + \widehat{mgd}\sin\theta] - \frac{mgd}{I}\sin\theta - \ddot{\theta}_d$$

$$e_1 = \theta - \theta_d$$

$$e_2 = \dot{\theta} - \dot{\theta}_d$$

$$= a + \Delta a + \left(\frac{\hat{I}}{I} - 1\right)(a+\Delta a) + \frac{\Delta mgd}{I}\sin\theta - \ddot{\theta}_d$$

known bounds for control design

$$5 \le I \le 10 \qquad 5 \le mgd \le 7$$

Calculations for robust control

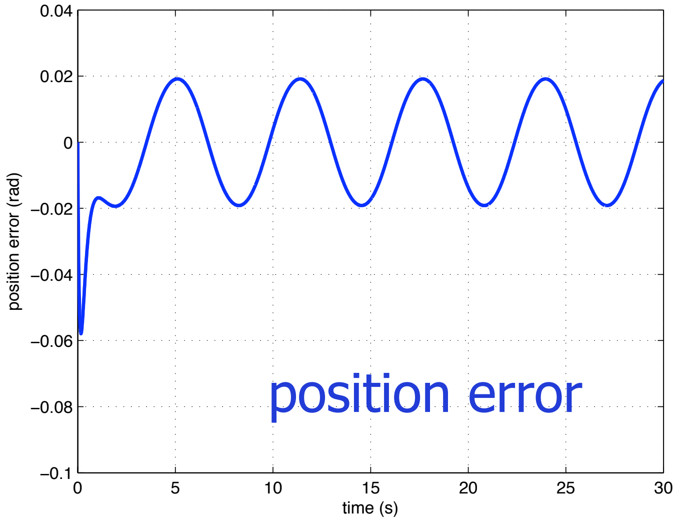
```
% real robot
I=5; mgd=7;
% initial robot state
th0=0;thp0=0;
% range of uncertainties
I_min=5; I_max=10;
mgd_min=5;mgd_max=7;
% linear tracking stabilizer gains
kp=25; kd=10; % two poles in -5

% robust control part
% Lyapunov matrix P and b^T P term
A=[0 1; -kp -kd];
q=1; Q=q*eye(2);
P=lyap(A',Q); % solve A'*P+P*A+Q=0
b=[0 1];
bP=b*P; % =[0.02  0.052]
```

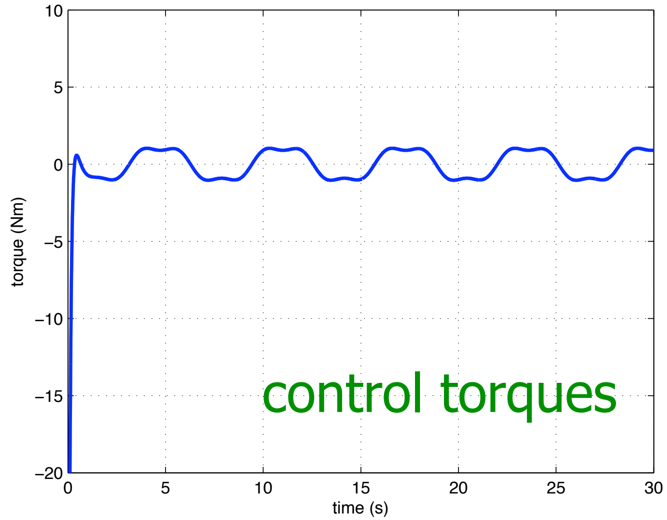
```
% bounding dynamic terms
% inertia
m=1/I_max; M=1/I_min;
c=(M+m)/2;
alpha=(M-m)/(M+m);
Ihat=1/c; % =6.6667
% nonlinear terms (only gravity)
Mphi=M*(mgd_max-mgd_min);
mgdhat=5;
% overall bounding
rho0=Mphi/(1-alpha)  % =0.6
rho1=alpha/(1-alpha) % =0.5
% smoothed version
epsilon=5*10^-4;
```

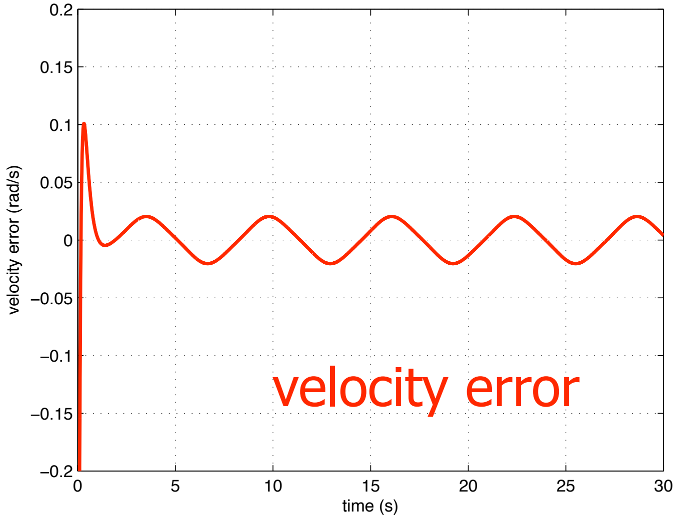
red values are used in Simulink

Results

first trajectory – feedback linearization, no robust loop







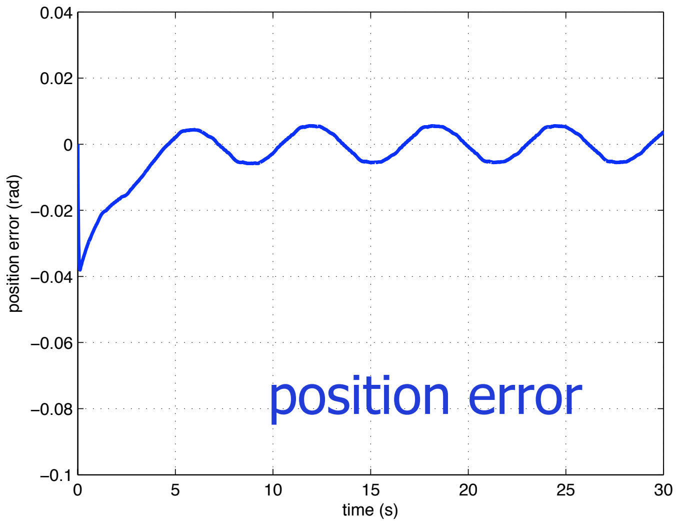
$$\theta_d(t) = -\sin t$$

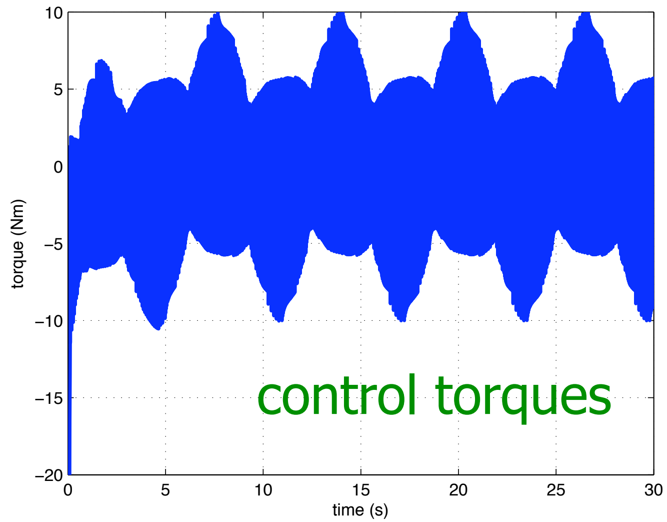
$$\theta(0) = 0, \ \dot{\theta}(0) = 0$$

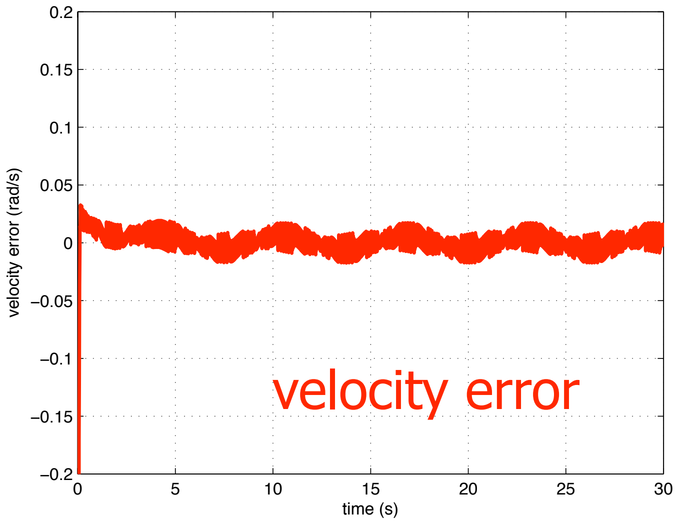
non-zero initial error
on velocity

Results

first trajectory – discontinuous robust control





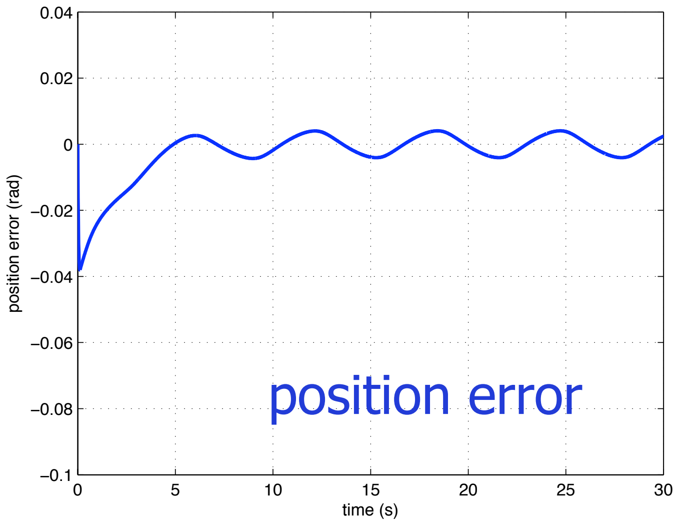


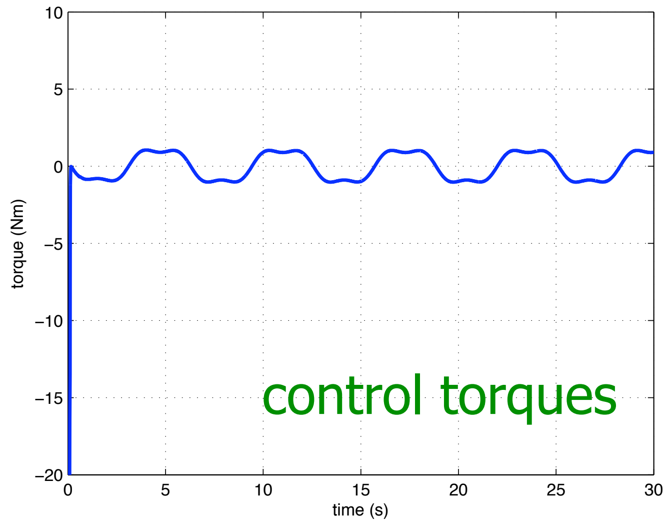
$$\theta_d(t) = -\sin t$$

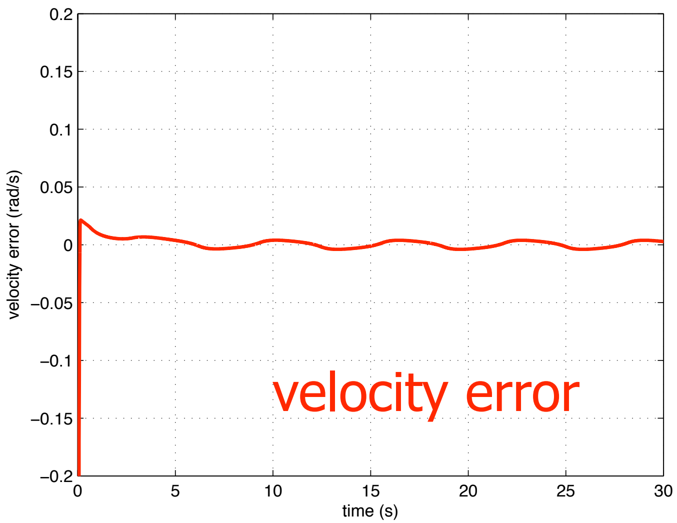
position and velocity errors are largely reduced, but control chattering at high frequency (when error is close to zero)

Results

first trajectory – smoothed robust control







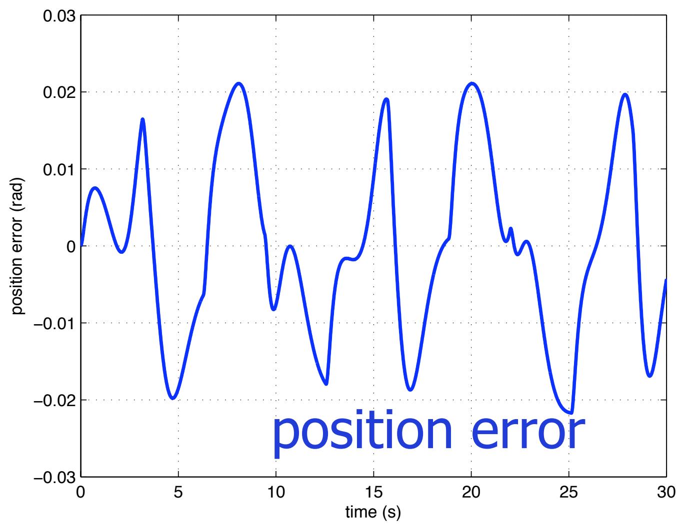
$$\theta_d(t) = -\sin t$$

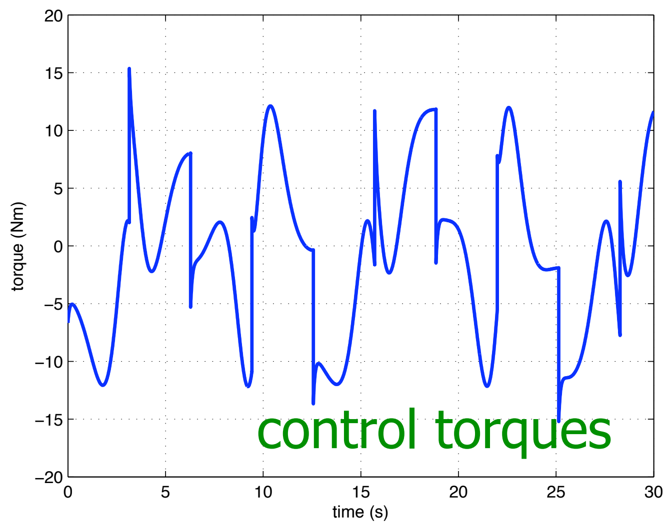
position and velocity errors are similarly reduced, without control chattering

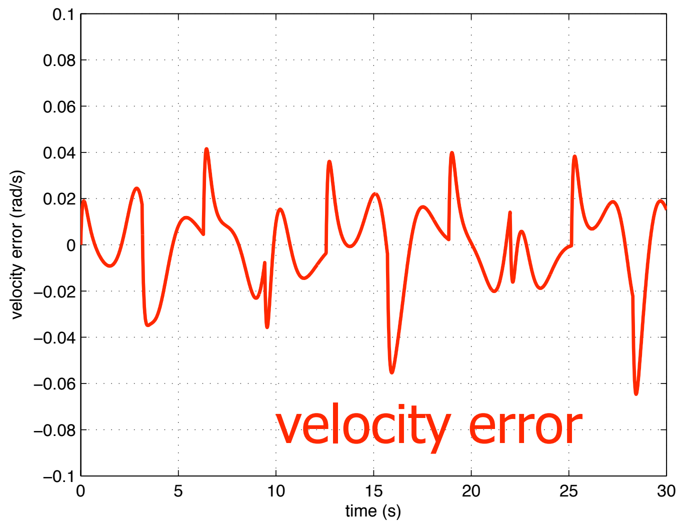
(using here $\epsilon > 0$)

Results

second trajectory – fbk linearization, no robust loop





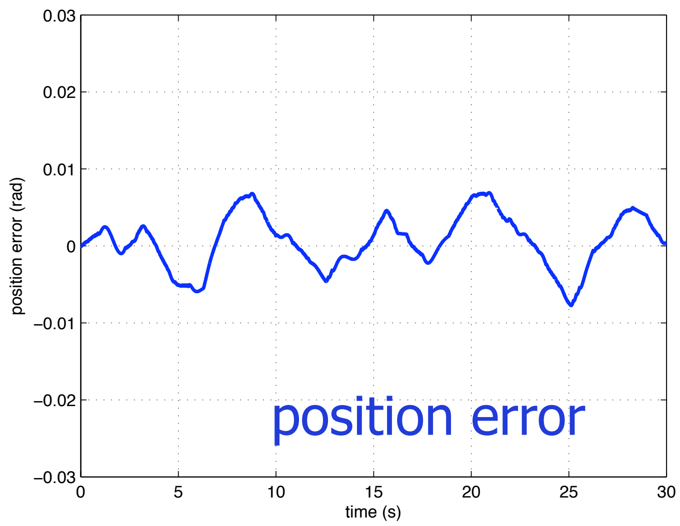


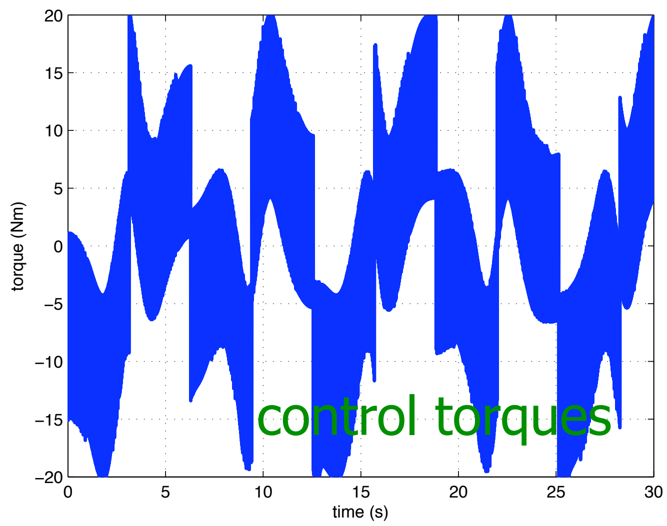
bang-bang acceleration profile at 1 rad/s frequency and with Q_{max} = 1 rad/s^2

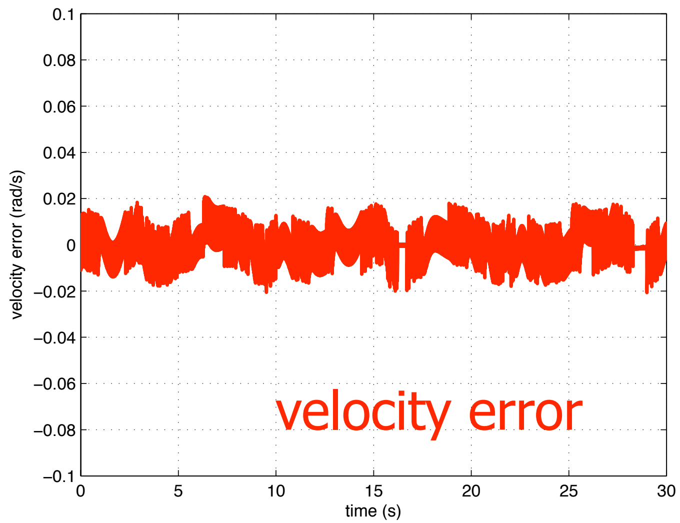
zero initial tracking error (matching state conditions)

Results

second trajectory – discontinuous robust control





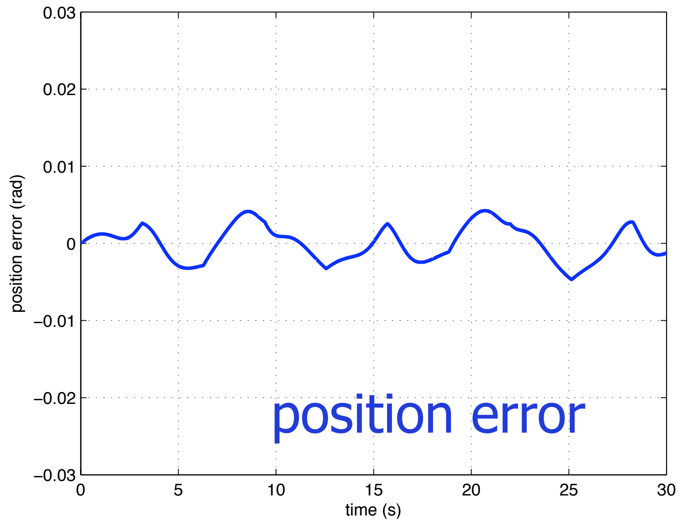


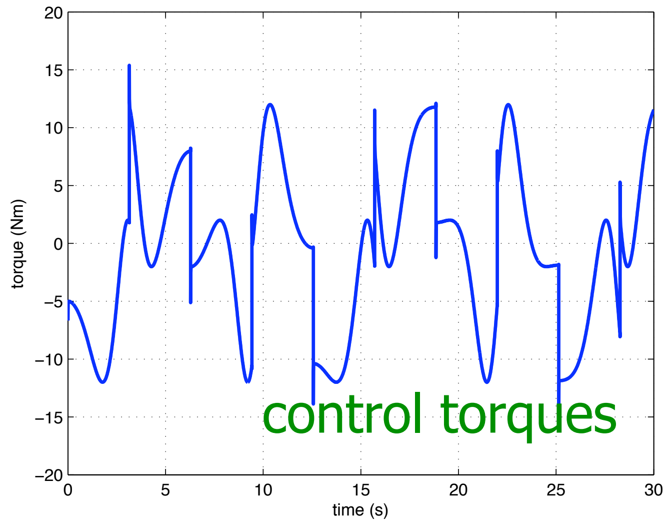
bang-bang acceleration profile

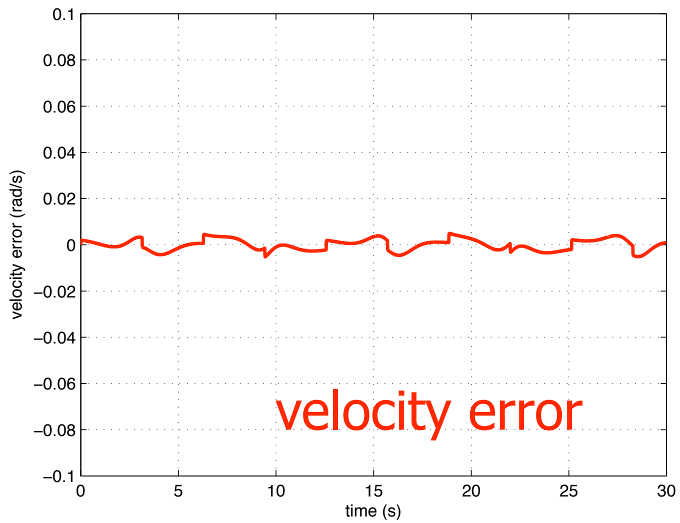
position and velocity errors
again largely reduced,
but control chattering
and larger effort

Results

second trajectory – smoothed robust control







bang-bang acceleration profile

position and velocity errors are further reduced, without control chattering and same control effort as without robustifying term

(using here $\epsilon > 0$)

Appendix A
Proof of bounds on the inertia matrix

- the SVD factorization of the (symmetric) inverse inertia matrix is

$$B^{-1} = U\,\Sigma^{-1}\,U^T = U\,\mathrm{diag}\{\frac{1}{\sigma_1},\ldots,\frac{1}{\sigma_n}\}\,U^T \qquad m \le \frac{1}{\sigma_n} \le \ldots \le \frac{1}{\sigma_1} \le M$$

so that, with the choice made for its estimate, it follows that

$$\begin{aligned}
\|B^{-1}\hat{B} - I\| &= \|U\,\Sigma^{-1}\,U^T\hat{B} - I\| \\
&= \|U\,\Sigma^{-1}\,U^T \cdot (\frac{1}{c}\,I) - I\| \\
&= \|U\,(\Sigma^{-1}\cdot\frac{1}{c} - I)\,U^T\| \\
&\le \|U\|\cdot\|\,\Sigma^{-1}\cdot\frac{1}{c} - I\,\|\cdot\|U^T\| \\
&= \|\,\Sigma^{-1}\cdot\frac{1}{c} - I\,\| \le \frac{M}{c} - 1 \\
&= \frac{M-c}{c} = \frac{M-m}{M+m} = \alpha < 1
\end{aligned}$$

Appendix B
Proof of Theorem 2

- setting $w = \mathcal{B}^T P e$, note that for the robust term in the control law it is

$$\|\Delta a\| = \begin{cases} \rho & \text{if } \|w\| \geq \epsilon \\ (\rho/\epsilon)\|w\| < \rho & \text{if } \|w\| < \epsilon \end{cases}$$

- defining as before $V(e) = e^T P e$, we have

$$\dot{V}(e) = -e^T Q e + 2w^T(\Delta a + \bar{\eta})$$
$$\leq -e^T Q e + 2w^T \left(\Delta a + \rho \frac{w}{\|w\|}\right)$$

having used the chain of inequalities

$$w^T \bar{\eta} \leq \|w^T \bar{\eta}\| \leq \|w\| \cdot \|\bar{\eta}\| \leq \|w\| \rho = w^T \rho \frac{w}{\|w\|}$$

- if $\|w\| \geq \epsilon$, the rest of the proof is the same as in Theorem 1

%

Appendix B
Proof of Theorem 2 (cont)

- if $\|w\| < \epsilon$, the second term in the derivative of V is

$$2w^T\left(-\frac{\rho}{\epsilon}w + \rho\frac{w}{\|w\|}\right) = 2\rho\left(-\frac{\|w\|^2}{\epsilon} + \|w\|\right)$$

with a maximum value $\rho\frac{\epsilon}{2}$ attained for $\|w\| = \frac{\epsilon}{2}$

- therefore, it is

$$\dot{V}(e) \leq -e^T Q e + \rho\,\frac{\epsilon}{2} \leq -\lambda_{\min}(Q)\,\|e\|^2 + \rho\,\frac{\epsilon}{2} < 0$$

provided that

$$\|e\| \geq \left[\frac{\rho\,\epsilon}{2\lambda_{\min}(Q)}\right]^{1/2} := \omega$$

- if S is the *smallest* level set of $V = e^T P e$ (an ellipsoid) containing the hyper-sphere of radius ω, then

$e \notin S \quad \Longrightarrow \quad \dot{V}(e) < 0$ and u.u.b. is obtained for S

(an upper bound for the time needed to reach S can be given)