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## *Robotics 2*

# Regulation in the Joint Space

(with an introduction to stability)

Prof. Alessandro De Luca

DIPARTIMENTO DI INGEGNERIA INFORMATICA  
AUTOMATICA E GESTIONALE ANTONIO RUBERTI



SAPIENZA  
UNIVERSITÀ DI ROMA



# Equilibrium states of a robot

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

$$\begin{aligned} \rightarrow \dot{x} &= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1, x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u \\ \uparrow & \\ = 0 & \quad = f(x) + G(x_1)u \end{aligned}$$

$$x_e \text{ unforced equilibrium} \quad (u = 0) \quad \rightarrow \quad f(x_e) = 0 \quad \rightarrow \quad \begin{cases} x_{e2} = 0 \\ g(x_{e1}) = 0 \end{cases}$$

$$x_e \text{ forced equilibrium} \quad (u = u(x)) \quad \rightarrow \quad f(x_e) + G(x_{e1})u(x_e) = 0 \quad \rightarrow \quad \begin{cases} x_{e2} = 0 \\ u(x_e) = g(x_{e1}) \end{cases}$$

all equilibrium states of mechanical systems have zero velocity!

joint torques must balance gravity at the equilibrium!

# Stability of dynamical systems

## definitions - 1



$$\dot{x} = f(x)$$

e.g., a closed-loop system  
(under feedback control)

$$x_e \text{ equilibrium: } f(x_e) = 0$$

(sometimes we consider as equilibrium state  
 $x_e = 0$ , e.g., when using errors as variables)

stability of  $x_e$

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \|x(t_0) - x_e\| < \delta_\varepsilon \implies \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$$

asymptotic stability of  $x_e$

stability +

$$\exists \delta > 0: \|x(t_0) - x_e\| < \delta \implies \|x(t) - x_e\| \rightarrow 0, \text{ for } t \rightarrow \infty$$

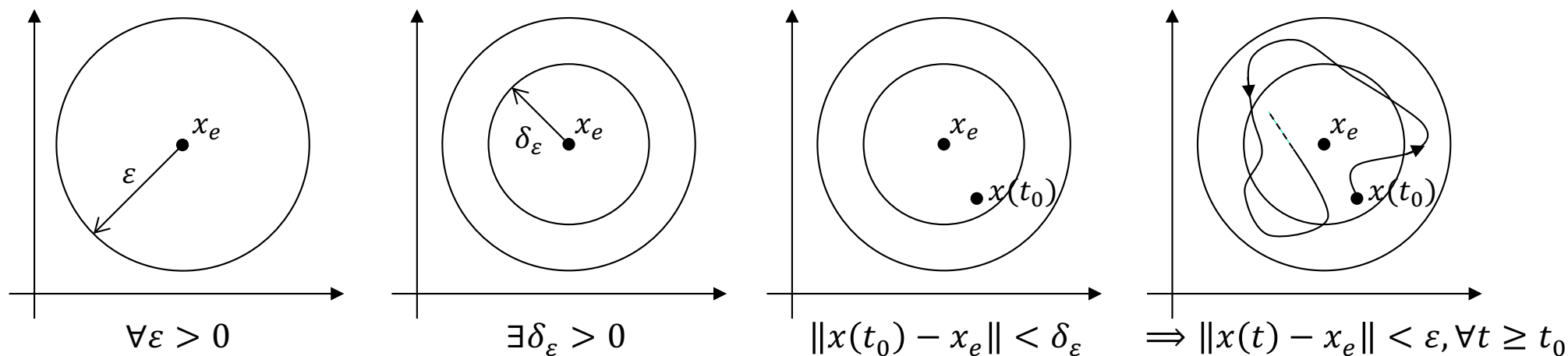
asymptotic stability may become **global** ( $\forall \delta > 0$ , finite)

**note:** these are definitions of stability "in the sense of Lyapunov"



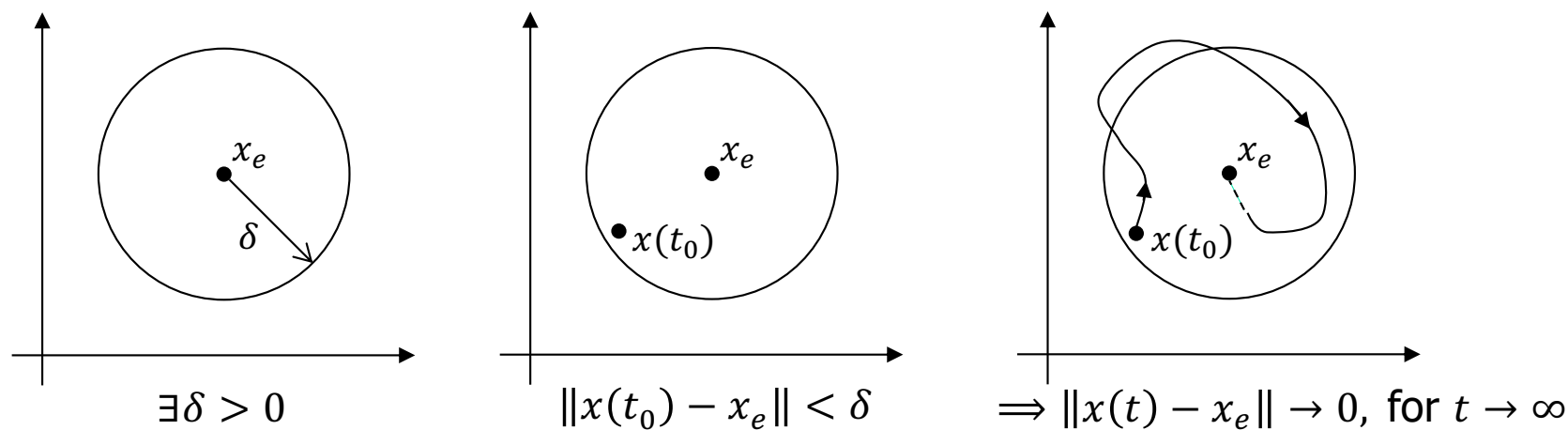
# Stability vs. asymptotic stability

whiteboard...



equilibrium state  $x_e$  is **stable**

+



equilibrium state  $x_e$  is **asymptotically stable**



# Stability of dynamical systems

## definitions - 2

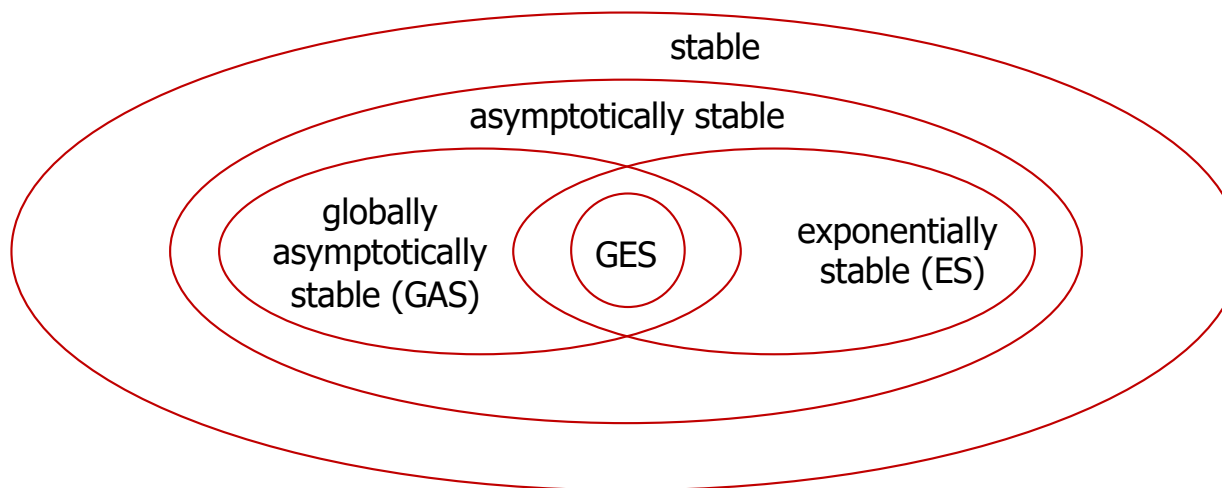
### exponential stability of $x_e$

exponential rate  $\lambda$

$$\exists \delta, c, \lambda > 0: \|x(t_0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| \leq c e^{-\lambda(t-t_0)} \|x(t_0) - x_e\|$$

- allows to estimate the time needed to "approximately" converge: for  $c = 1$ , in  $t - t_0 = 3 \times$  the **time constant**  $\tau = 1/\lambda$ , the initial error is reduced to 5%
- typically, this is a **local** property only (within some maximum **finite** radius  $\delta$ )  
 $\Rightarrow$  such "domain of attraction" is hard to be estimated accurately

taxonomy  
of stability  
definitions



a **necessary**  
condition for  
 $x_e$  to be GAS  
is that  
it is the **only**  
equilibrium state  
of the system



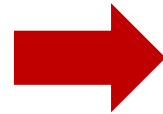
# The need for analysis and criteria

whiteboard...

a nonlinear system  $\dot{x} = f(x)$  in  $\mathbb{R}^2$

two equilibria  $f(x_e) = 0$

$$\begin{cases} \dot{x}_1 = 1 - x_1^3 \\ \dot{x}_2 = x_1 - x_2^2 \end{cases}$$

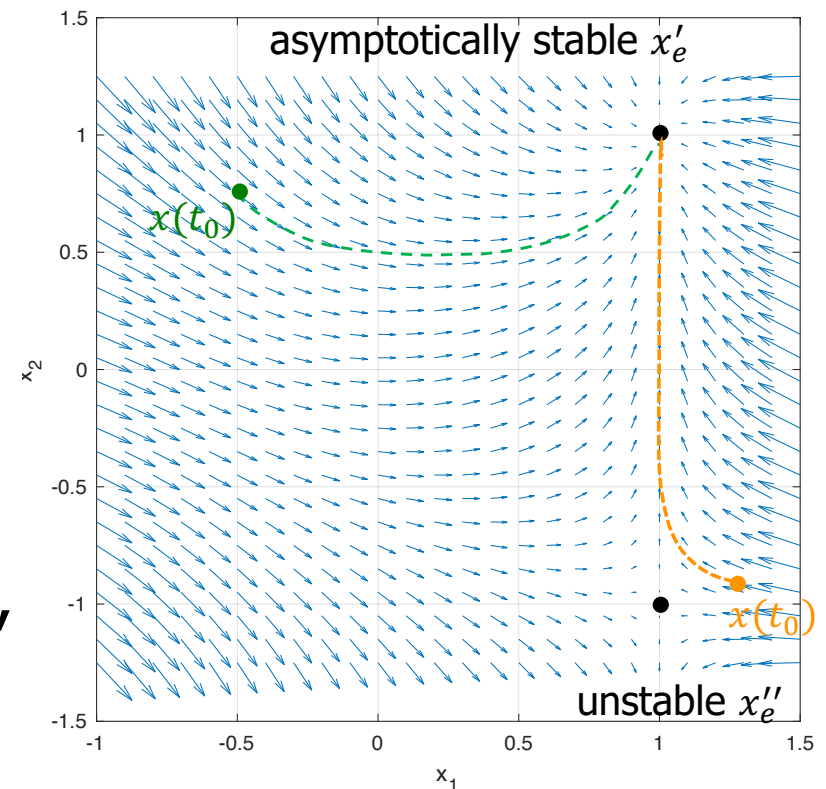


$$x'_e = (1, 1), \quad x''_e = (1, -1)$$

to assess (asymptotic) stability [or not] of equilibria, do we need to compute all system trajectories, starting from all possible initial states  $x(t_0)$ ?



rather, we may be able to just look at the time evolution of **a scalar function  $V$** , evaluated **analytically** along the state trajectories of the system (even in  $\mathbb{R}^n$ !)



# Stability of dynamical systems

## results - 1



### Lyapunov candidate

$V(x): \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$V(x_e) = 0, V(x) > 0, \forall x \neq x_e$$

positive  
definite  
function

typically, **quadratic** (e.g.,  $\frac{1}{2}(x - x_e)^T P(x - x_e)$  with level surfaces = ellipsoids)  
may also be a **local** candidate only ( $\forall x \neq x_e: \|x - x_e\| < \delta$ )

### sufficient condition of stability

$\exists V$  candidate:  $\dot{V}(x) \leq 0$ , along the trajectories of  $\dot{x} = f(x)$

negative  
semi-definite  
function

### sufficient condition of asymptotic stability

$\exists V$  candidate:  $\dot{V}(x) < 0$ , along the trajectories of  $\dot{x} = f(x)$

negative  
definite  
function

### sufficient condition of instability

$\exists V$  candidate:  $\dot{V}(x) > 0$ , along the trajectories of  $\dot{x} = f(x)$



# Stability of dynamical systems

results - 2

## LaSalle Theorem

if  $\exists V$  candidate:  $\dot{V}(x) \leq 0$  along the trajectories of  $\dot{x} = f(x)$



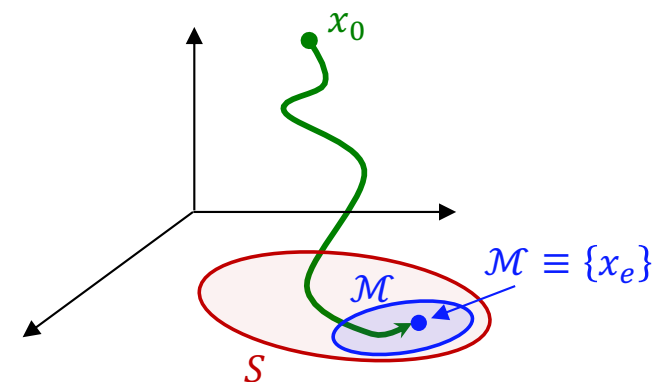
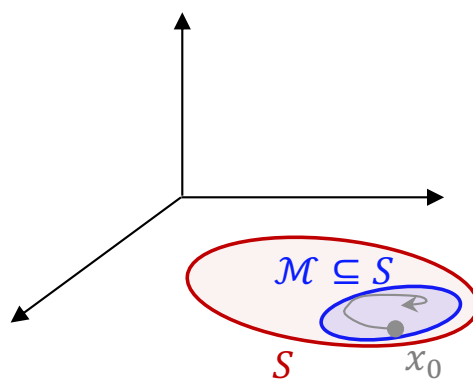
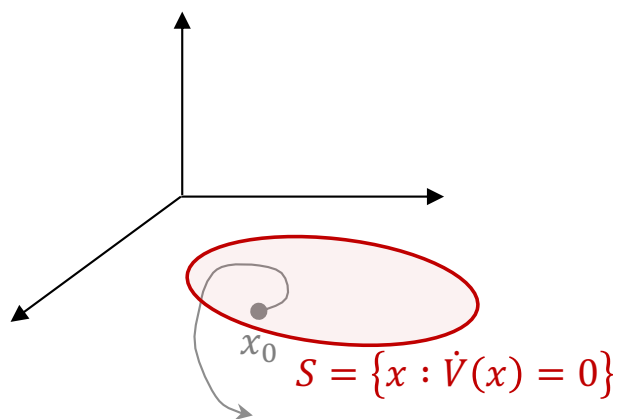
then system trajectories asymptotically converge to the

largest invariant set  $\mathcal{M} \subseteq S = \{x \in \mathbb{R}^n: \dot{V}(x) = 0\}$

$\mathcal{M}$  is invariant if  $x(t_0) \in \mathcal{M} \Rightarrow x(t) \in \mathcal{M}, \forall t \geq t_0$

## Corollary

$\mathcal{M} \equiv \{x_e\} \Rightarrow$  asymptotic stability





# Bird-eye view on Lyapunov analysis

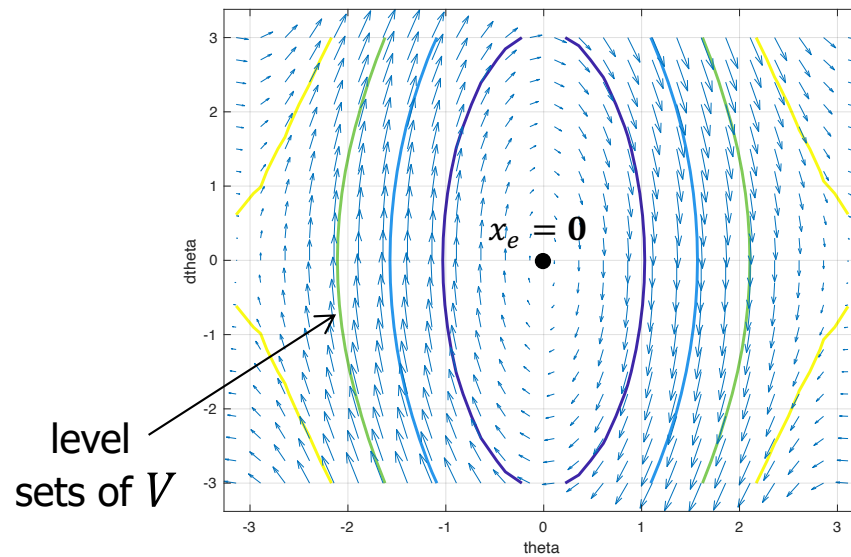
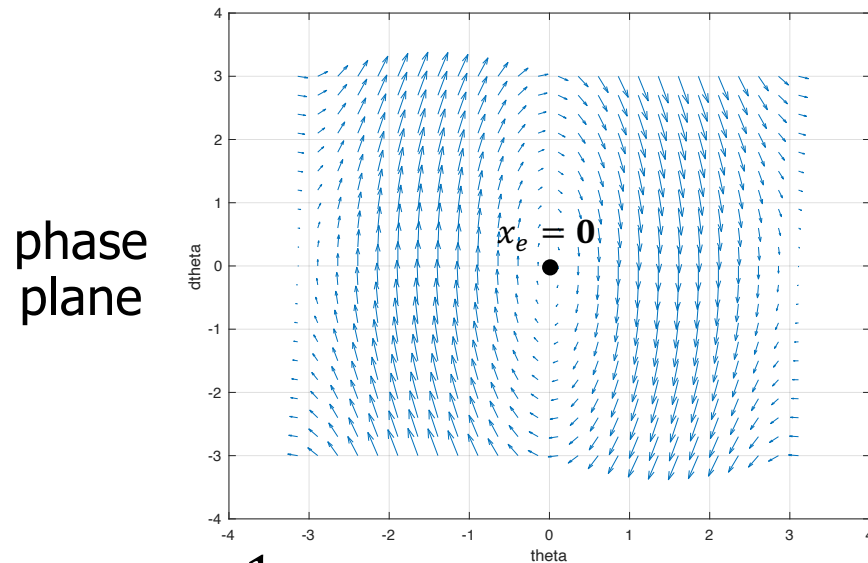
## whiteboard...



a mass  $m$  at the end of an unforced (passive) pendulum of length  $l$

$$ml^2\ddot{\theta} + b\dot{\theta} + mlg_0 \sin \theta = 0 \Rightarrow x = (x_1, x_2) \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\frac{g_0}{l}\right) \sin x_1 - \left(\frac{b}{ml^2}\right) x_2 \end{cases}$$

lower equilibrium at  $\theta_e = 0$   $\Rightarrow$   $= (\theta, \dot{\theta}) \in \mathbb{R}^2$



$$V = E = \frac{1}{2} ml^2 \dot{\theta}^2 + mlg_0 (1 - \cos \theta) \geq 0 \quad V = 0 \Leftrightarrow x_e = (\theta_e, \dot{\theta}_e) = (0, 0)$$

$$\dot{V} = \dot{\theta}(ml^2\ddot{\theta} + mlg_0 \sin \theta) = -b\dot{\theta}^2 \leq 0 \Rightarrow \text{stability of equilibrium } x_e = 0 \text{ (... at least!)}$$

$$\Rightarrow \text{use LaSalle: } \dot{V} = 0 \Leftrightarrow \dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\left(\frac{g_0}{l}\right) \sin \theta \neq 0 \text{ unless } \theta = \theta_e = 0 \text{ (or } \pi\text{!)}$$

$\Rightarrow$  local asymptotic stability

# Stability of dynamical systems

## results - 3



- previous results are also valid for **periodic** time-varying systems

$$\dot{x} = f(x, t) = f(x, t + T_p) \Rightarrow V(x, t) = V(x, t + T_p)$$

- for general **time-varying** systems (e.g., in robot **trajectory tracking** control)

$$\dot{x} = f(x, t)$$

### Barbalat Lemma

if i) a function  $V(x, t)$  is lower bounded

ii)  $\dot{V}(x, t) \leq 0$

then  $\Rightarrow \exists \lim_{t \rightarrow \infty} V(x, t)$  (but this does **not** imply that  $\lim_{t \rightarrow \infty} \dot{V}(x, t) = 0$ )

if in addition iii)  $\ddot{V}(x, t)$  is bounded

then  $\Rightarrow \lim_{t \rightarrow \infty} \dot{V}(x, t) = 0$

### Corollary

if a Lyapunov candidate  $V(x, t)$  satisfies Barbalat Lemma along the trajectories of  $\dot{x} = f(x, t)$ , **then** the conclusions of LaSalle Theorem hold



# Stability of dynamical systems

additional definition and result (for robust control)

## “practical” stability of a set $S$

$$\exists T(x(t_0), S) \in \mathbb{R}: x(t) \in S, \forall t \geq t_0 + T(x(t_0), S)$$

a finite time

also known as **u.u.b. stability**

⇒ trajectories  $x(t)$  are “uniformly ultimately bounded” (use in **robust control**)

## sufficient condition of u.u.b. stability of a set $S$

∃  $V$  candidate: **i)**  $S$  is a level set of  $V$  for a given  $c_0$

$$S = S(c_0) = \{x \in \mathbb{R}^n: V(x) \leq c_0\}$$

**ii)**  $\dot{V}(x) < 0$  along trajectories of  $\dot{x} = f(x), x \notin S$

# Stability of linear systems

time-invariant case



$$\dot{x} = Ax$$

$x_e = 0$  is always an equilibrium state

- I. asymptotic stability
- II. global asymptotic stability
- III. exponential stability
- IV.  $\sigma(A) \subset \mathbb{C}^-$  (all eigenvalues of  $A$  have negative real part)
- V.  $\forall Q > 0$  (positive definite),  $\exists! P > 0: A^T P + PA = -Q$   
Lyapunov equation  $\Rightarrow \frac{1}{2} x^T P x$  is a Lyapunov candidate

**ALL EQUIVALENT !!**

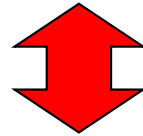
if  $x_e = 0$  is an asymptotically stable equilibrium,  
then it is necessarily the **unique equilibrium**



# Stability of the linear approximation

Let  $\Delta x = x - x_e$  and let  $\Delta \dot{x} = \frac{df}{dx} \Big|_{x=x_e} (x - x_e) = A \Delta x$  be the linear approximation of  $\dot{x} = f(x)$  around the equilibrium  $x_e$

$A$  asymptotically stable ( $\sigma(A) \subset \mathbb{C}^-$ )



the original nonlinear system is **exponentially** stable at the origin

this is only a **local** result  
(used also to estimate the domain of attraction)



# PD control

(proportional + derivative action on the error)

robot  $M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$

**goal:** asymptotic stabilization (= **regulation**)  
of the closed-loop equilibrium state

$$q = q_d, \dot{q} = 0$$

possibly obtained from kinematic inversion:  $q_d = f^{-1}(r_d)$

control law  $u = K_P(q_d - q) - K_D\dot{q}$

$K_P > 0, K_D > 0$  (positive definite), symmetric



# Asymptotic stability with PD control

## Theorem 1

In the absence of gravity ( $g(q) \equiv 0$ ), the robot state  $(q_d, 0)$  under the given PD joint control law is **globally asymptotically stable**

## Proof

let  $e = q_d - q$  ( $q_d$  constant)

Lyapunov candidate

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e \geq 0$$

$$V = 0 \Leftrightarrow e = \dot{e} = 0$$

$$\begin{aligned} \dot{V} &= \dot{q}^T M \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M} \dot{q} - e^T K_P \dot{q} = \dot{q}^T \left( u - \underbrace{S \dot{q} + \frac{1}{2} \dot{M} \dot{q}}_{= 0, \text{ due to energy conservation}} \right) - e^T K_P \dot{q} \\ &= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} - \cancel{e^T K_P \dot{q}} = -\dot{q}^T K_D \dot{q} \leq 0 \quad (K_D > 0, \text{ symmetric}) \end{aligned}$$

up to here, we proved stability only

but  $\dot{V} = 0 \Leftrightarrow \dot{q} = 0$  continues ...  $\rightarrow$



# Asymptotic stability with PD control

$\dot{V} = 0 \Leftrightarrow \dot{q} = 0$  ➔ LaSalle system trajectories converge to the largest **invariant set** of states  $\mathcal{M}$  where  $\dot{q} \equiv 0$  (that is  $\dot{q} = \ddot{q} = 0$ )

$$\dot{q} = 0 \quad \color{red}{\Rightarrow} \quad \underbrace{M(q)\ddot{q} = K_P e}_{\text{closed-loop dynamics}} \quad \color{red}{\Rightarrow} \quad \ddot{q} = \underbrace{M^{-1}(q)K_P e}_{\text{invertible}}$$

$$\dot{q} = 0, \ddot{q} = 0 \Leftrightarrow e = 0$$

➔ the only invariant state in  $\dot{V} = 0$  is given by  $q = q_d, \dot{q} = 0$

**note:** typically,  $K_P = \text{diag}\{k_{Pi}\}$ ,  $K_D = \text{diag}\{k_{Di}\}$ ,  
➔ **decentralized linear control** (**local** to each joint)



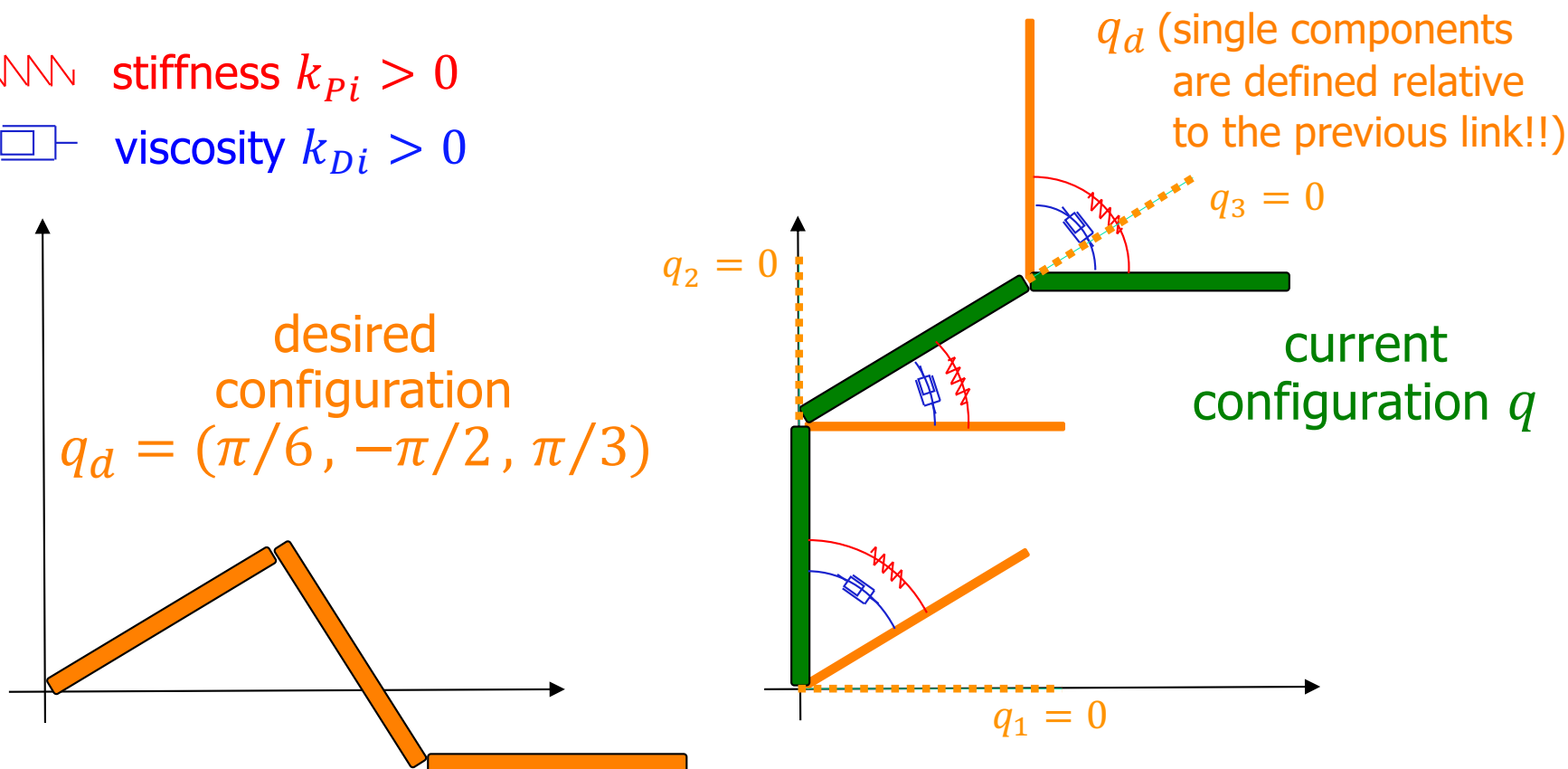


# Mechanical interpretation

- for **diagonal** positive definite gain matrices  $K_P$  and  $K_D$  (thus, with **positive** diagonal elements), such values correspond to stiffness of “virtual” **springs** and viscosity of “virtual” **dampers** placed at the joints

 stiffness  $k_{Pi} > 0$

 viscosity  $k_{Di} > 0$



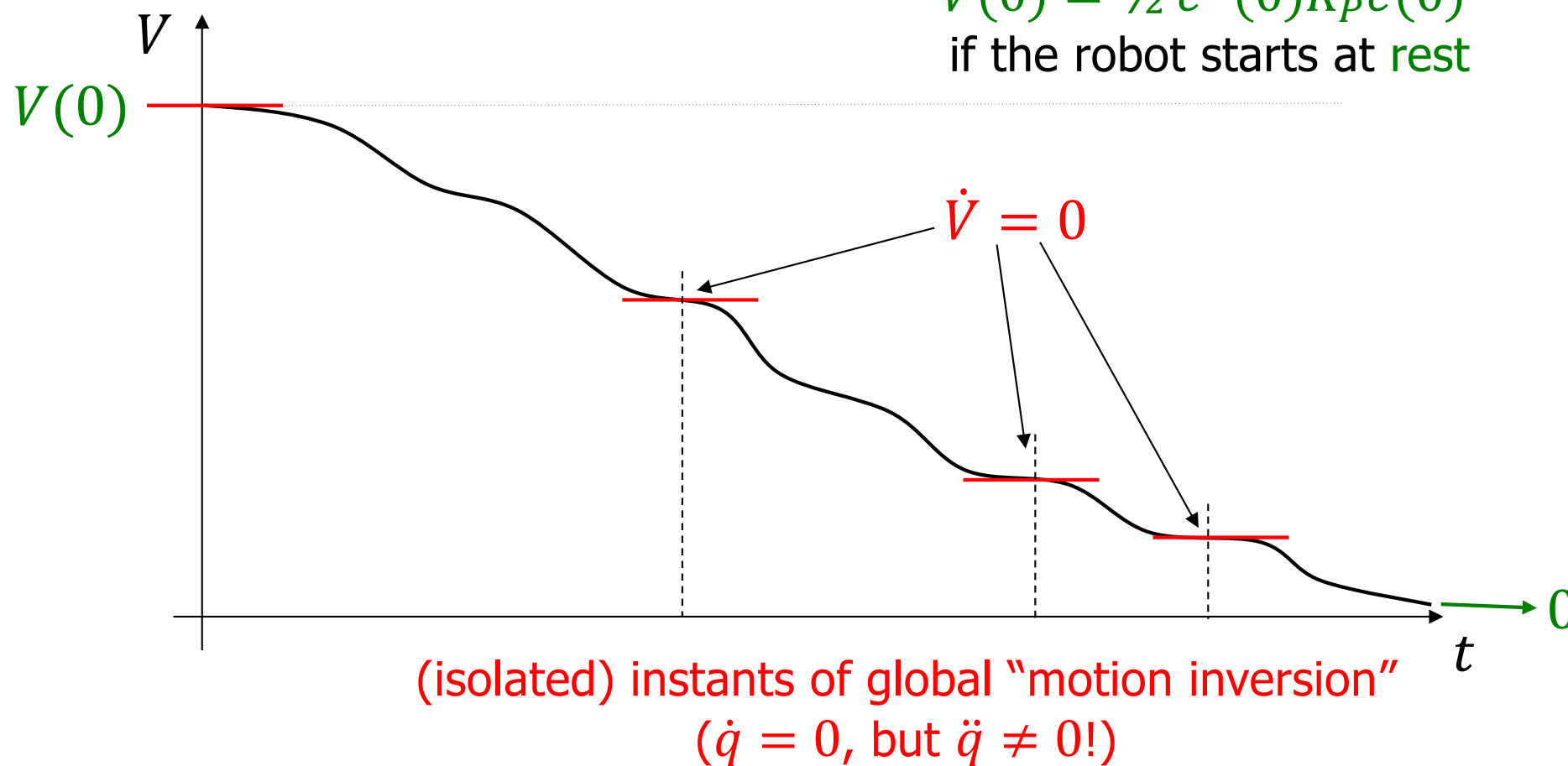


# Plot of the Lyapunov function $V$

- time evolution of the Lyapunov candidate

$$V(0) = \frac{1}{2} e^T(0) K_P e(0)$$

if the robot starts at rest





# Comments on PD control - 1

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- **choice of control gains** affects robot evolution during transients and practical settling times
  - hard to define values that are “optimal” in the whole workspace
  - “full”  $K_P$  and  $K_D$  gain matrices allow to assign desired eigenvalues to the **linear approximation** of the robot dynamics around the final desired state  $(q_d, 0)$
- when (joint) **viscous friction** is present, the derivative term in the control law is not strictly necessary
  - $-F_V\dot{q}$  in the robot model acts similarly to  $-K_D\dot{q}$  in the control law, but the latter can be modulated at will
- in the absence of tachometers, the actual realization of the derivative term in the feedback law requires some processing of **joint position data measured** by digital encoders (or analog resolvers/potentiometers)



# Comments on PD control - 2

- **analog** or **digital** implementation of derivative action in the control law when **only position is measured** at the joints (e.g., through **encoders**)

continuous-time control law (design)

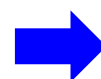
$$u(t) = K_P e(t) + K_D \dot{e}(t)$$

$$e = q_d - q, \dot{e} = -\dot{q}$$

representation in the Laplace domain

$$u(s) = (K_P + K_D s) e(s)$$

**not realizable** as such (non-proper transfer function)



$$u(s) = \left( K_P + \frac{K_D s}{1 + \tau s} \right) e(s)$$

derivative action **limited in bandwidth** (up to  $\omega \leq 1/\tau$ )

transformation in the Zeta-domain (e.g., via **backward differentiation** rule on samples, every  $T_c$  sec)

$$u(z) = \left( K_P + K_D \frac{1 - z^{-1}}{T_c} \right) e(z)$$

$$u(z) = \left( K_P + K_D \frac{\frac{1 - z^{-1}}{T_c}}{1 + \tau \frac{1 - z^{-1}}{T_c}} \right) e(z)$$

discrete-time implementations

$$u_k = K_P e_k + K_D \frac{e_k - e_{k-1}}{T_c}$$

**both realizable**

$$u_k = K_P e_k + \frac{K_D}{\tau + T_c} (e_k - e_{k-1}) + \frac{\tau}{\tau + T_c} (u_{k-1} - K_P e_{k-1})$$



# Inclusion of gravity

- in the presence of gravity, the same previous arguments (and proof) show that the control law

$$u = K_P(q_d - q) - K_D\dot{q} + g(q)$$

$$K_P > 0, K_D > 0$$

will make the equilibrium state  $(q_d, 0)$  globally asymptotically stable (**nonlinear cancellation of gravity**)

- if gravity is **not** cancelled or only **approximately** cancelled

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q)$$

$$\hat{g}(q) \neq g(q)$$

it is  $q \rightarrow q^* \neq q_d, \dot{q} \rightarrow 0$ , with a **steady-state** position error

- $q^*$  is not unique in general, except when  $K_P$  is chosen large enough
- explanation in terms of linear systems: there is **no integral action before** the point of access of the **constant "disturbance"** acting on the system



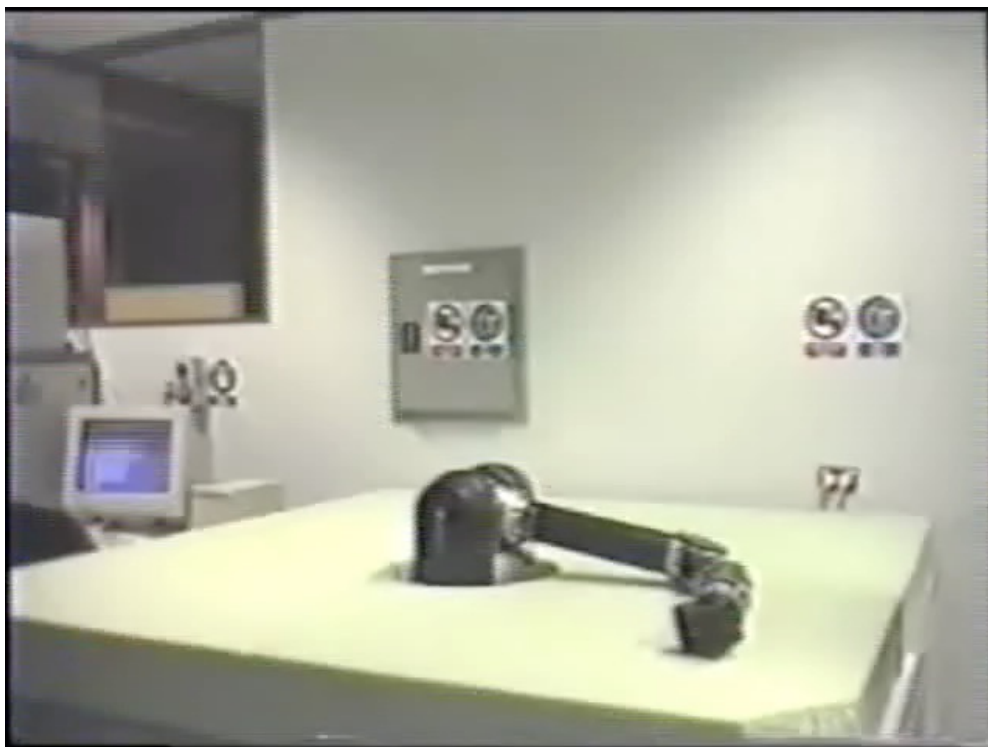
# Approximate cancellation of gravity

WAM Barrett  
(with some viscous friction)

$$u = \hat{g}(q)$$

$$\hat{g}(q) = g(q)$$

$$\hat{g}(q) \neq g(q)$$



two-part video

<http://handbookofrobotics.org/view-chapter/69/videodetails/611>



# PD control + constant gravity compensation

if the robot potential energy  $U(q)$  is bounded for all  $q$ , then its partial derivative  $g(q)$  is also **bounded** everywhere and the following **structural property** holds

finite  $\curvearrowright$

$$\exists \alpha > 0: \left\| \frac{\partial^2 U}{\partial q^2} \right\| = \left\| \frac{\partial g}{\partial q} \right\| \leq \alpha, \forall q$$

consequence  $\rightarrow$

$$\|g(q) - g(q_d)\| \leq \alpha \|q - q_d\|$$

note: induced  
norm of  
a matrix

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)} \triangleq A_M \geq A_m \triangleq \sqrt{\lambda_{\min}(A^T A)}$$

**LINEAR CONTROL law**

$$u = K_P(q_d - q) - K_D \dot{q} + g(q_d)$$

$K_P, K_D > 0$   
symmetric

linear feedback + **constant** feedforward



# More on the basic assumption ...

(in PD control + gravity compensation)

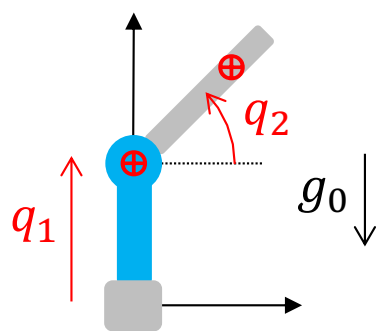
when is the (non-zero) gravity term  $g(q)$  bounded for all  $q$ ?

- the robot has **all revolute** joints
  - all terms in  $U(q)$  and thus in  $g(q)$  are trigonometric (bounded)
- the robot has both types of joints, but **no prismatic variables in  $g(q)$** 
  - potential energy  $U(q)$  may still be unbounded!
- all **prismatic** joints of the robot have a **limited range**
  - ... ok, but one should take these limits into account in the control analysis

$$U = g_0(m_1 q_1 + m_2(q_1 + d_{c2} \sin q_2)) + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 + m_2) \\ m_2 d_{c2} \cos q_2 \end{pmatrix}$$

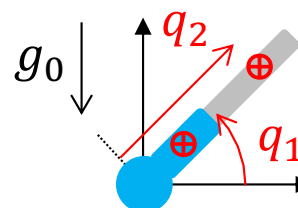
$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha = m_2 d_{c2} g_0$$



**PR robot**

$$U = g_0(m_1 d_{c1} + m_2 q_2) \sin q_1 + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 d_{c1} + m_2 q_2) \cos q_1 \\ m_2 \sin q_1 \end{pmatrix}$$



$$\left\| \frac{\partial g}{\partial q} \right\| \leq ??$$

**RP robot**



# PD control + constant gravity compensation

## stability analysis



### Theorem 2

If  $K_{P,m} > \alpha$ , the state  $(q_d, 0)$  of the robot under joint-space PD control + constant gravity compensation at  $q_d$  is **globally asymptotically stable**

### Proof

1.  $(q_d, 0)$  is the unique closed-loop equilibrium state

in fact, for  $\dot{q} = 0$  and  $\ddot{q} = 0$ , it is  $K_P e = g(q) - g(q_d)$  which can hold only for  $q = q_d$ , because when  $q \neq q_d$

$$\|K_P e\| \geq K_{P,m} \|e\| > \alpha \|e\| \geq \|g(q) - g(q_d)\|$$

$$M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = K_P e - K_D \dot{q} + g(q_d)$$



# PD control + constant gravity compensation

## stability analysis

with  $e = q_d - q$ ,  $g(q) = \left(\frac{\partial U}{\partial q}\right)^T$ , consider as Lyapunov candidate

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

2.  $V$  is convex in  $\dot{q}$  and  $e$ , and zero only for  $e = \dot{q} = 0$

$$\left(\frac{\partial V}{\partial \dot{q}}\right)^T = M(q) \dot{q} = 0 \text{ only for } \dot{q} = 0$$

$$\frac{\partial^2 V}{\partial \dot{q}^2} = M(q) > 0$$

$(q_d, 0)$  is a  
global minimum  
of  $V \geq 0$

$$\left(\frac{\partial V|_{\dot{q}=0}}{\partial e}\right)^T = K_P e - \left(\frac{\partial U}{\partial q}\right)^T + g(q_d) = K_P e + g(q_d) - g(q) = 0$$

$$\frac{\partial e}{\partial q} = -I$$

only for  $q = q_d$

$$\frac{\partial^2 V|_{\dot{q}=0}}{\partial e^2} = K_P + \frac{\partial^2 U}{\partial q^2} > 0, \text{ since } \|K_P\| = K_{P,M} \geq K_{P,m} > \alpha$$



# PD control + constant gravity compensation

## stability analysis

differentiating

$$V = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \frac{1}{2} e^T K_P e + U(q) - U(q_d) + e^T g(q_d)$$

$$\begin{aligned} \dot{V} &= \dot{q}^T \left( M(q) \ddot{q} + \frac{1}{2} \dot{M}(q) \dot{q} \right) - e^T K_P \dot{q} + \frac{\partial U(q)}{\partial q} \dot{q} - \dot{q}^T g(q_d) \\ &= \dot{q}^T \left( \underbrace{u - S(q, \dot{q}) \dot{q} + \frac{1}{2} \dot{M}(q) \dot{q} - g(q)}_{= 0} \right) - e^T K_P \dot{q} + \dot{q}^T (g(q) - g(q_d)) \\ &= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} + \dot{q}^T (g(q_d) - g(q)) - \cancel{e^T K_P \dot{q}} + \dot{q}^T (g(q) - g(q_d)) \\ &= -\dot{q}^T K_D \dot{q} \leq 0 \end{aligned}$$

for  $\dot{V} = 0$  ( $\Leftrightarrow \dot{q} = 0$ ), we have in the **closed-loop** system

$$M(q) \ddot{q} + g(q) = K_P e + g(q_d)$$

$$\Rightarrow \ddot{q} = M^{-1}(q) (K_P e + g(q_d) - g(q)) = 0 \Leftrightarrow e = 0$$

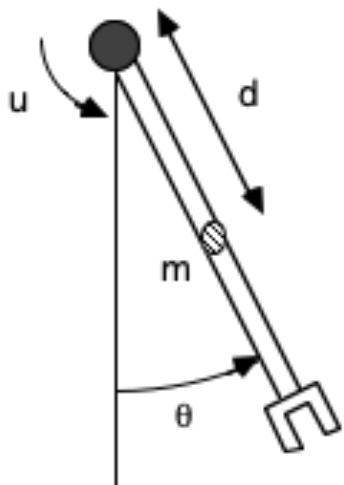
by LaSalle Theorem, the thesis follows





# Example of a single-link robot

## stability analysis



**task:** regulate the link position to the **upward equilibrium**

$$\theta_d = \pi \rightarrow g(\theta_d) = 0$$

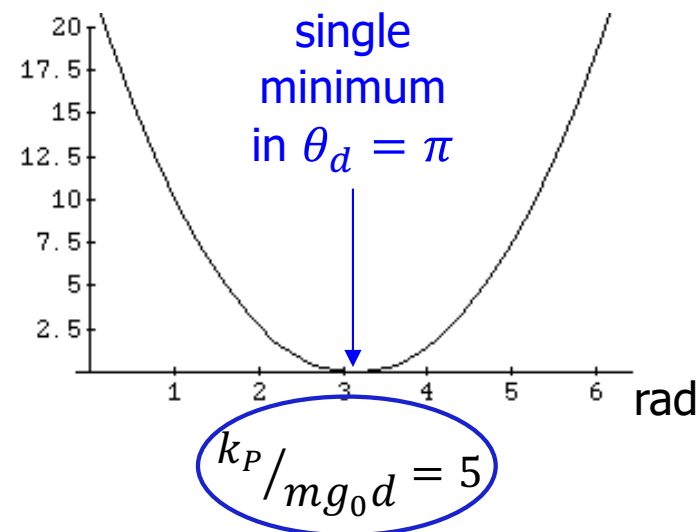
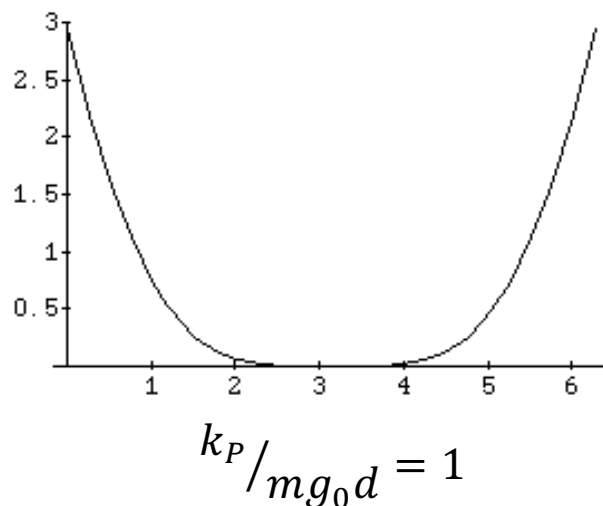
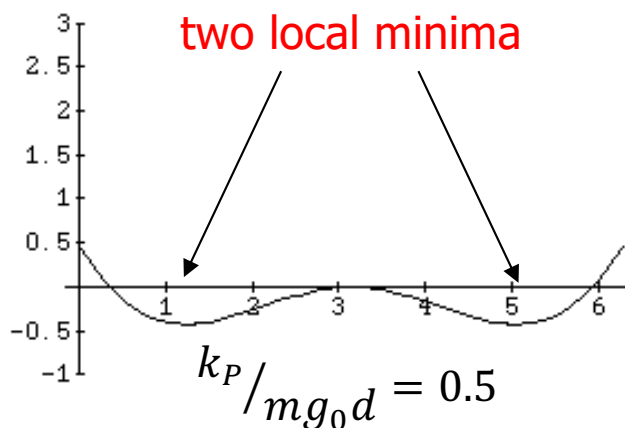
PD control + constant gravity compensation (here, **zero!**)

$$u = k_P(\pi - \theta) - k_D\dot{\theta}$$

by Theorem 2, it is **sufficient** (here, also **necessary\***) to choose

$$k_P > \alpha = mg_0d, \quad k_D > 0$$

$$I\ddot{\theta} + mg_0d \sin \theta = u$$



plots of  $V(\theta)$  (for  $\dot{\theta} = 0$ )

\* by a local analysis of the linear approximation at  $\pi$



# Example of a single-link robot

simulations with data:  $I = 0.9333$ ,  $mg_0d = 19.62 (= \alpha)$

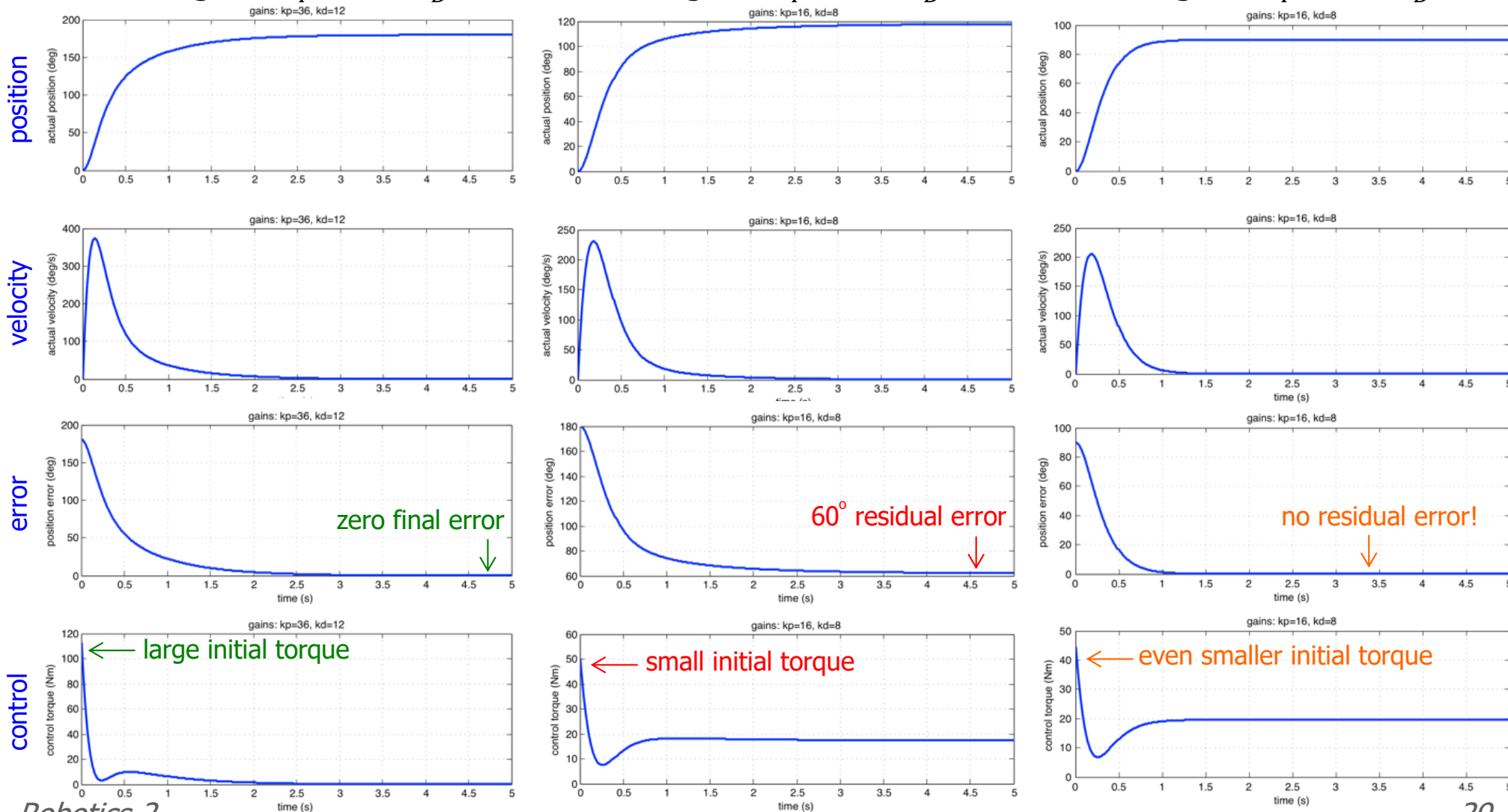
$\theta_d = 180^\circ \rightarrow g(\theta_d) = 0$

$\theta_d = 90^\circ \rightarrow g(\theta_d) = mg_0d$

sufficient P gain:  $k_P = 36, k_D = 12$

low P gain:  $k_P = 16, k_D = 8$

low P gain:  $k_P = 16, k_D = 8$





# Approximate gravity compensation

the **approximate** control law

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q_d)$$

leads, under similar hypotheses, to a closed-loop equilibrium  $q^*$

- its uniqueness is not guaranteed (unless  $K_P$  is large enough)
- for  $K_P \rightarrow \infty$ , one has  $q^* \rightarrow q_d$

**conclusion:** in the presence of gravity, the previous regulation control laws require an **accurate knowledge** of the **gravity term** in the dynamic model to guarantee the zeroing of the position error (since we can only use “finite” control gains  $\Rightarrow$  in practice, not too large)



# PID control

- in linear systems, the addition of an integral control action is used to eliminate a constant error in the step response at steady state
- in robots, a PID may be used to recover such a position error due to an incomplete (or absent) gravity compensation/cancellation

➔ the control law 
$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t (q_d - q(\tau)) d\tau - K_D \dot{q}(t)$$

- is independent from any robot dynamic model term
- if the desired closed-loop equilibrium is asymptotically stable under PID control, the integral term is “loaded” at steady state to the value

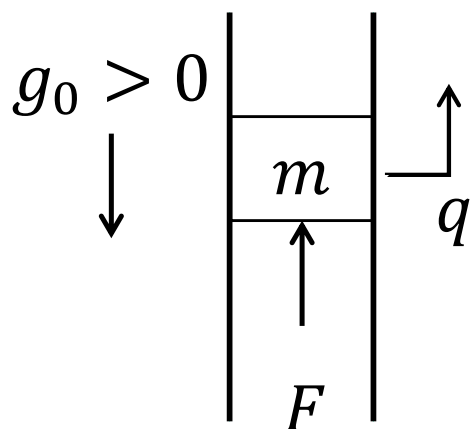
$$K_I \int_0^{\infty} (q_d - q(\tau)) d\tau = g(q_d)$$

- however, one can show only local asymptotic stability of this law, i.e., for  $q(0) \in \Delta(q_d)$ , under complex conditions on  $K_P, K_I, K_D$  and  $e(0)$



# Linear example with PID control

whiteboard...



$$m\ddot{q} + mg_0 = F \quad (\text{no friction})$$

$$e(t) = q_d - q(t)$$

$$\dot{e}(t) = -\dot{q}(t)$$

$$F = k_P(q_d - q) - k_D\dot{q}$$

(**PD**  $\Rightarrow$  steady-state error  $e = q_d - \bar{q}$ , with  $\bar{q} = q_d - \frac{mg_0}{k_P}$ )

$$F = k_P(q_d - q) - k_D\dot{q} + mg_0$$

(**PD + gravity cancellation**  $\Rightarrow$  regulation  $\forall k_P > 0, k_D > 0$ )

$$F = k_P(q_d - q) - k_D\dot{q} + k_I \int_0^t (q_d - q(\tau)) d\tau$$

(**PID**  $\Rightarrow$  regulation  $\forall k_I > 0, k_D > 0, k_P > \frac{mk_I}{k_D} > 0$ )

with global exponential stability!

Laplace domain analysis:  $e(s) = \mathcal{L}[e(t)]$ ,  $d(s) = \mathcal{L}[mg_0]$  + **Routh** criterion

$$\frac{e(s)}{d(s)} = W_d(s) = \frac{s}{ms^3 + k_Ds^2 + k_Ps + k_I}$$

|   |                       |       |
|---|-----------------------|-------|
| 3 | $m$                   | $k_P$ |
| 2 | $k_D$                 | $k_I$ |
| 1 | $(k_Dk_P - mk_I)/k_D$ |       |
| 0 | $k_I$                 |       |





# Saturated PID control

- more in general, one can prove **global** asymptotic stability of  $(q_d, 0)$ , under **lower bound limitations** for  $K_P, K_I, K_D$  (depending on suitable “bounds” on the terms in the dynamic model), for a **nonlinear PID law**

$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t \Phi(q_d - q(\tau)) d\tau - K_D \dot{q}$$

where  $\Phi(q_d - q)$  is a **saturation-type** function, such as

$$\Phi(x) = \begin{cases} \sin x, & |x| \leq \pi/2 \\ 1, & x > \pi/2 \\ -1, & x < -\pi/2 \end{cases} \quad \text{or} \quad \Phi(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(see paper by R. Kelly, IEEE TAC, 1998; available as extra material on the course web)



# Limits of robot regulation controllers

---

- **response times** needed for reaching the desired steady state are **not** easily **predictable** in advance
  - depend heavily on robot dynamics, on PD/PID gains, on the required total displacement, and on the interested area of the robot workspace
  - integral term (when present) needs some time to “unload” itself from the error history accumulated during transients
    - large initial errors are stored in the integral term
    - anti-windup schemes stop the integration when commands saturate
    - ... an intuitive explanation for the success of “saturated” PID law
- **control efforts in the few first instants** of motion typically exceed by far those required at steady state
  - especially for high positional gains
  - may lead to saturation (hard nonlinearity) of robot actuators



# Regulation in industrial robots

- in industrial robots, the planner generates a **reference trajectory**  $q_r(t)$  even when the task requires **only** positioning/regulation of the robot
  - “smooth” enough, with a user-defined **transfer time**  $T$
  - reference trajectory interpolates initial and final desired position

$$q_r(0) = q(0) \quad q_r(t \geq T) = q_d$$

- $q_r(t)$  is used within a control law of the form

$$u = K_P(q_r(t) - q) + K_D(\dot{q}_r(t) - \dot{q}) + g(q)$$

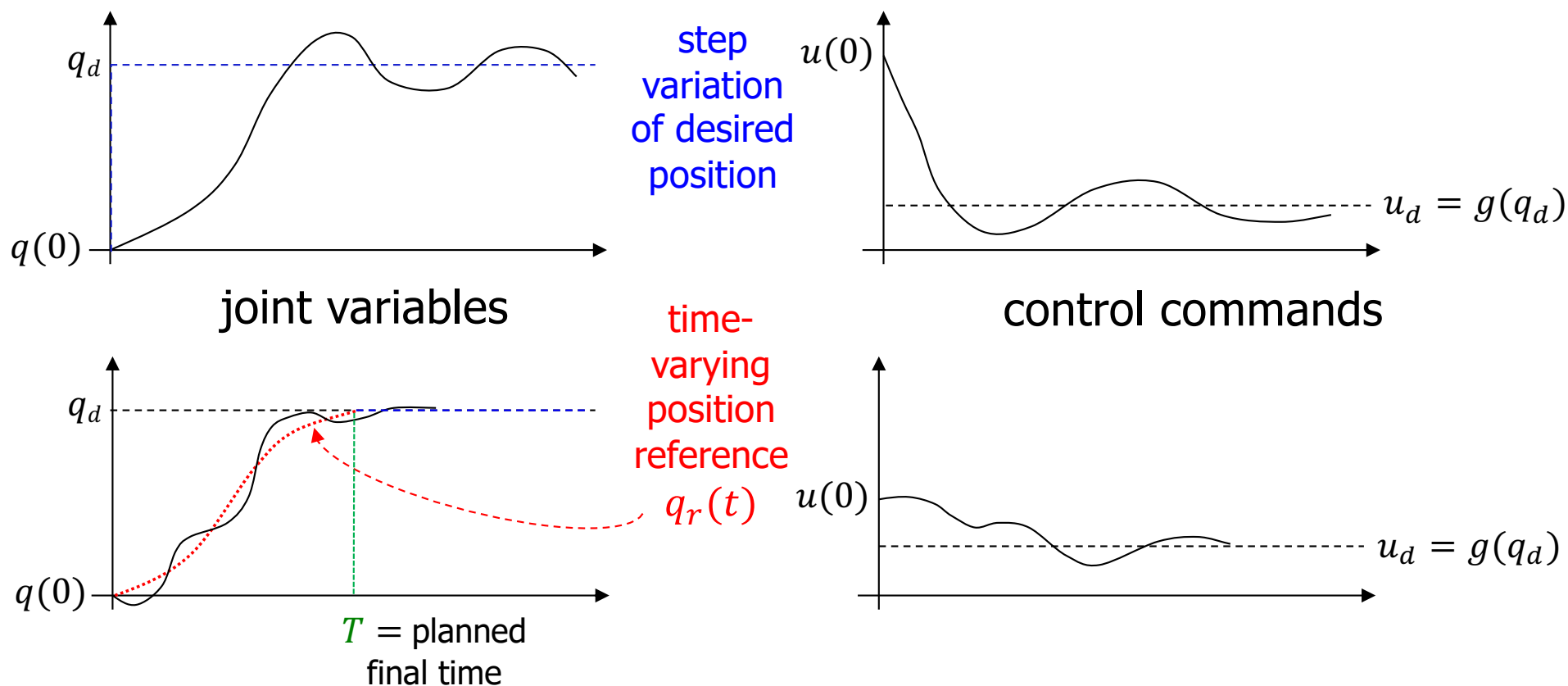
e.g., PD with  
gravity  
cancellation

↑  
often neglected

- in this way, the position error is **initially zero**
- robot motion stays only “in the vicinity” of the reference trajectory until  $t = T$ , typically with small position errors (gains can be **larger!**)
- **final** regulation is only a “local” problem ( $e(T) = q_d - q(T)$  is small)



# Qualitative comparison



- **no saturation** of commands: in principle, much larger gains can be used
- better **prediction of settling times**: local exponential convergence (designed on the linear approximation of the robot dynamics around  $(q_d, 0)$ )
- “fine tuning” of control gains made easier, but still a **tedious** and **delicate task**



# Quantitative comparison

## planar 2R arm

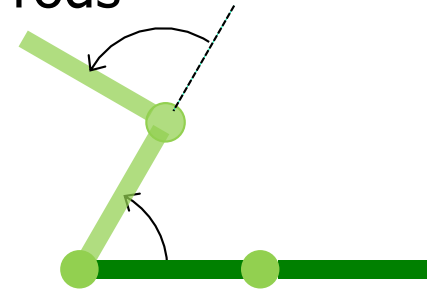
|       |                            |
|-------|----------------------------|
| $m_1$ | 10 [ kg ]                  |
| $m_2$ | 5 [ kg ]                   |
| $l_1$ | 0.5 [ m ]                  |
| $l_2$ | 0.5 [ m ]                  |
| $d_1$ | 0.25 [ m ]                 |
| $d_2$ | 0.25 [ m ]                 |
| $I_1$ | 5/24 [ kg m <sup>2</sup> ] |
| $I_2$ | 5/48 [ kg m <sup>2</sup> ] |

robot data: links are uniform thin rods

no gravity (horizontal plane)

rest-to-rest motion task:

$q(0) = (0, 0)$  (straight)  $\rightarrow q_d = (\pi/3, \pi/2)$



interpolating trajectory: cubic polynomials

### three case studies

a) low gains (overdamped)  $K_P = \text{diag}\{80, 40\}$ ,  $K_D = \text{diag}\{60, 30\}$

vs interpolating trajectory in  $T = 2$  s

b) medium gains (very overdamped)  $K_P = \text{diag}\{200, 100\}$ ,  $K_D = \text{diag}\{200, 100\}$

vs interpolating trajectory in  $T = 2$  s

c) high gains  $K_P = \text{diag}\{1250, 180\}$ ,  $K_D = \text{diag}\{200, 70\}$

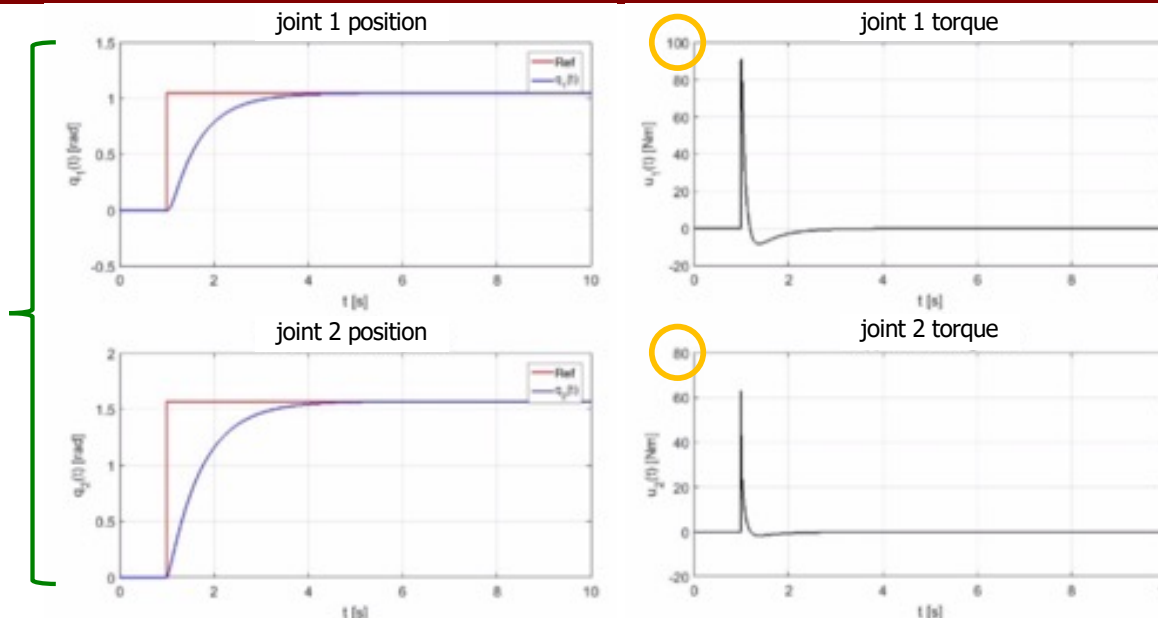
vs interpolating trajectory in  $T = 1$  s, with torque saturation  $u_{1,\max} = 30$  Nm,

$u_{2,\max} = 10$  Nm

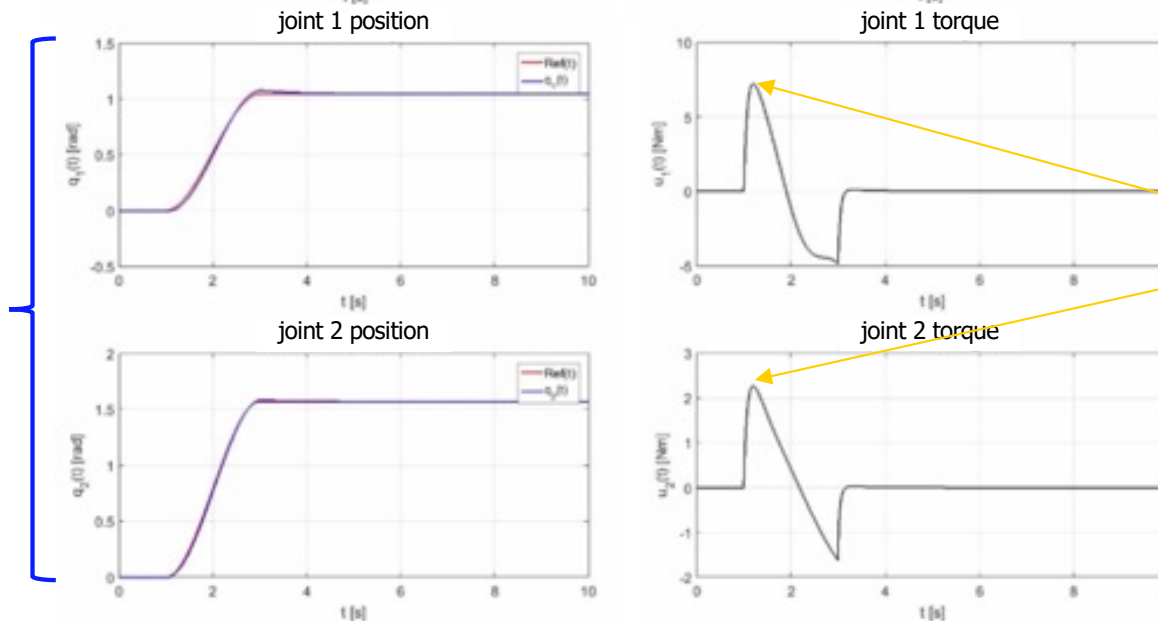


# Comparison on a planar 2R arm – case a

PD with low gains  
 $K_P = \text{diag}\{80, 40\}$   
 $K_D = \text{diag}\{60, 30\}$   
(overdamped)



PD with same gains  
on interpolating  
trajectory of  $T = 2$  s



a reduction of the  
peak control effort  
by a factor of 100  
on joint 1 &  
by a factor of 30  
on joint 2

max torques  
of 7 and 2.3 Nm



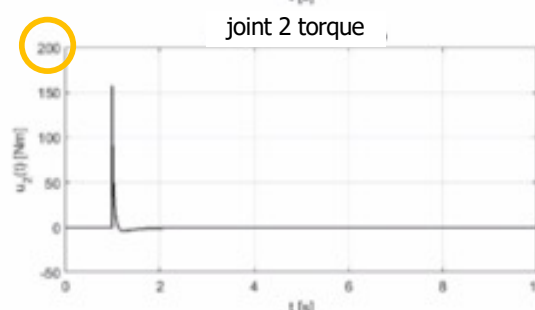
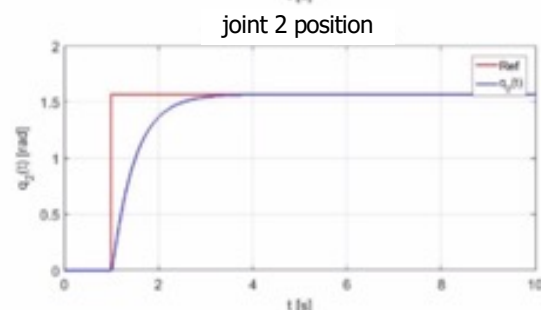
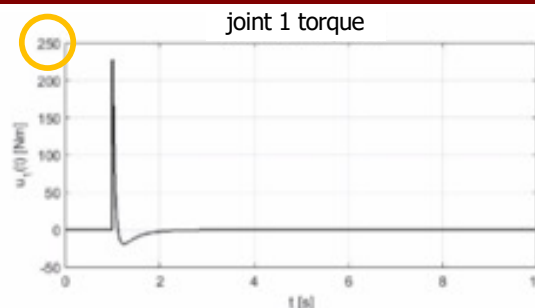
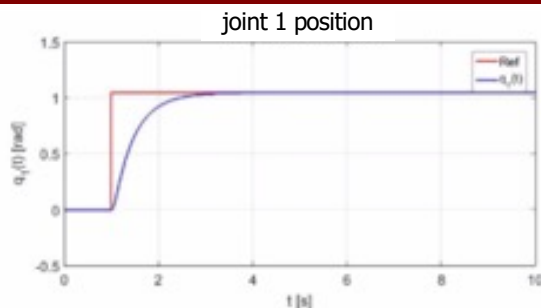
# Comparison on a planar 2R arm – case b

PD with medium gains

$$K_P = \text{diag}\{200, 100\}$$

$$K_D = \text{diag}\{200, 100\}$$

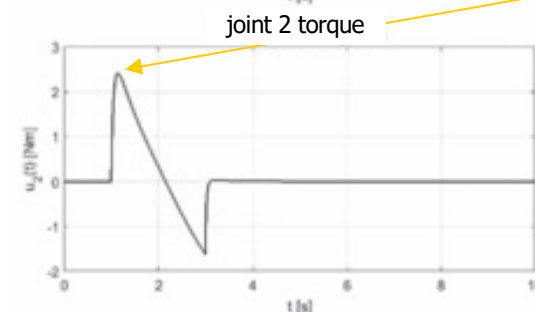
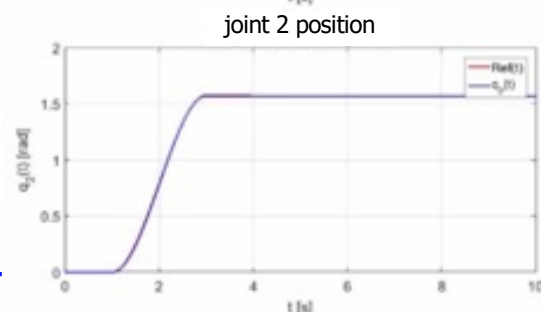
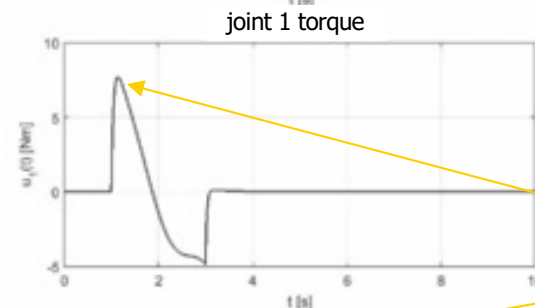
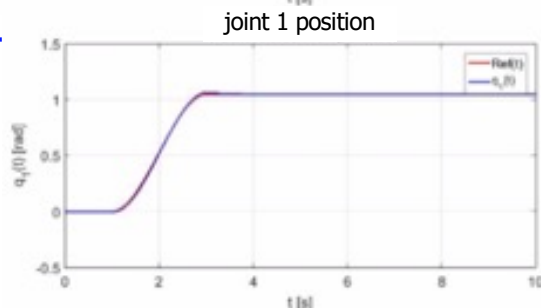
(very overdamped)



even stronger peak reduction, with similar total control effort, plus improved tracking of reference trajectory on both joints

PD with same gains

on interpolating trajectory of  $T = 2$  s



max torques of 7.5 and 2.4 Nm

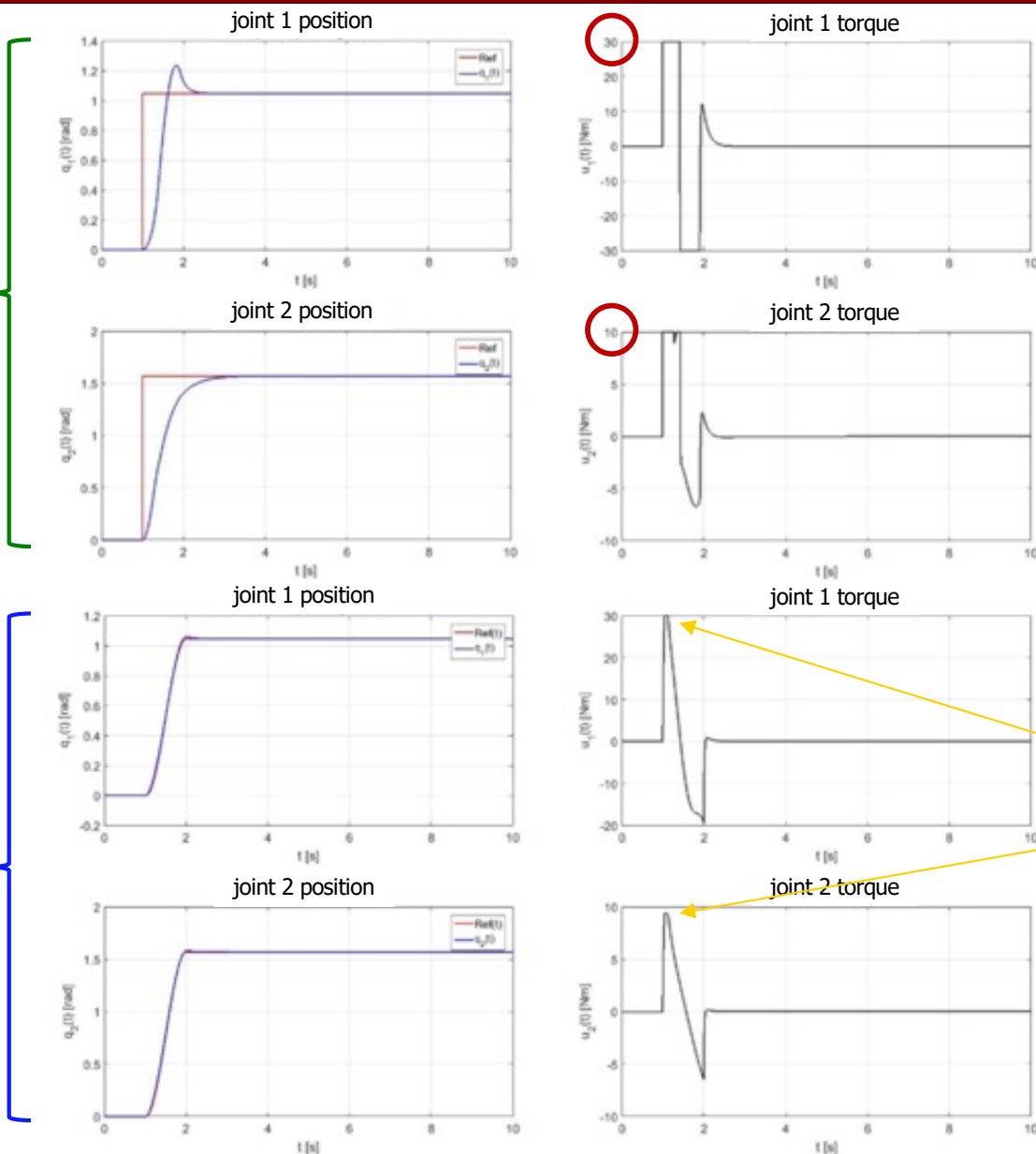


# Comparison on a planar 2R arm – case c

PD with high gains  
 $K_P = \text{diag}\{1250, 180\}$   
 $K_D = \text{diag}\{200, 70\}$

torque saturation  
 $u_{1,\max} = 30 \text{ Nm}$   
 $u_{2,\max} = 10 \text{ Nm}$

PD with same gains  
on interpolating  
trajectory of  $T = 1 \text{ s}$



position overshoot  
and long saturations  
are avoided,  
with very good  
tracking of the  
faster reference  
trajectory

max torques  
of 30 and 9.5 Nm