Robotics 2

Dynamic model of robots:
Algorithm for computing kinetic energy

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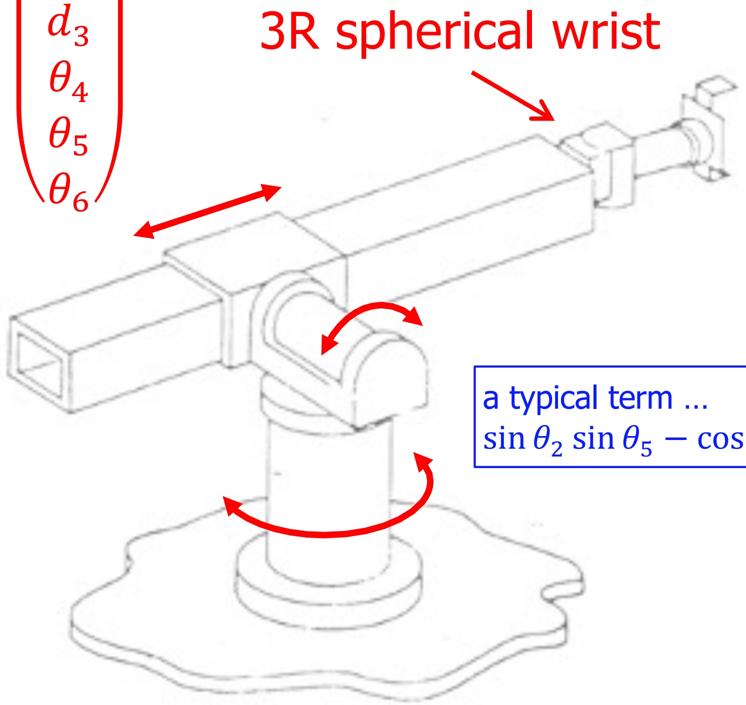
Complexity of robot inertia terms
element $m_{11}(q)$ of Stanford arm

$$q = \begin{pmatrix} \theta_1 \\ \theta_2 \\ d_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix}$$



… (derived by hand) in JPL Tech. Memo. 33-669, 1974

$$
\begin{aligned}
m_{11} = {} & m_1 k_{122}^2 \\
& + m_2 \left[k_{211}^2 s^2\theta_2 + k_{233}^2 c^2\theta_2 + r_2(2\bar{y}_2 + r_2)\right] \\
& + m_3 \left[k_{322}^2 s^2\theta_2 + k_{333}^2 c^2\theta_2 + r_3(2\bar{z}_3 + r_3)s^2\theta_2 + r_2^2\right] \\
& + m_4 \Big\{\tfrac{1}{2} k_{411}^2 \left[s^2\theta_2(2s^2\theta_4 - 1) + s^2\theta_4\right] + \tfrac{1}{2} k_{422}^2(1 + c^2\theta_2 + s^2\theta_4) \\
& \quad + \tfrac{1}{2} k_{433}^2 \left[s^2\theta_2(1 - 2s^2\theta_4) - s^2\theta_4\right] + r_3^2 s^2\theta_2 + r_2^2 - 2\bar{y}_4 r_3 s^2\theta_2 + 2\bar{z}_4(r_2 s\theta_4 + r_3 s\theta_2 c\theta_2 c\theta_4)\Big\} \\
& + m_5 \Big\{\tfrac{1}{2}(-k_{511}^2 + k_{522}^2 + k_{533}^2)\left[(s\theta_2 s\theta_5 - c\theta_2 s\theta_4 c\theta_5)^2 + c^2\theta_4 c^2\theta_5\right] \\
& \quad + \tfrac{1}{2}(k_{511}^2 - k_{522}^2 + k_{533}^2)(s^2\theta_4 + c^2\theta_2 c^2\theta_4) \\
& \quad + \tfrac{1}{2}(k_{511}^2 + k_{522}^2 - k_{533}^2)\left[(s\theta_2 c\theta_5 + c\theta_2 s\theta_4 s\theta_5)^2 + c^2\theta_4 s^2\theta_5\right] + r_3^2 s^2\theta_2 + r_2^2 \\
& \quad + 2\bar{z}_5\left[r_3(s^2\theta_2 c\theta_5 + s\theta_2 s\theta_4 c\theta_4 s\theta_5) - r_2 c\theta_4 s\theta_5\right]\Big\} \\
& + m_6 \Big\{\tfrac{1}{2}(-k_{611}^2 + k_{622}^2 + k_{633}^2)\left[(s\theta_2 s\theta_5 c\theta_6 - c\theta_2 s\theta_4 c\theta_5 c\theta_6 - c\theta_2 c\theta_4 s\theta_6)^2 + (c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6)^2\right] \\
& \quad + \tfrac{1}{2}(k_{611}^2 - k_{622}^2 + k_{633}^2)\left[(c\theta_2 s\theta_4 c\theta_5 s\theta_6 - s\theta_2 s\theta_5 s\theta_6 - c\theta_2 c\theta_4 c\theta_6)^2 + (c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6)^2\right] \\
& \quad + \tfrac{1}{2}(k_{611}^2 + k_{622}^2 - k_{633}^2)\left[(c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5)^2 + c^2\theta_4 s^2\theta_5\right] \\
& \quad + \left[r_6 c\theta_2 s\theta_4 s\theta_5 + (r_6 c\theta_5 + r_3)s\theta_2\right]^2 + (r_6 c\theta_4 s\theta_5 - r_2)^2 \\
& \quad + 2\bar{z}_6\Big[r_6(s^2\theta_2 c^2\theta_5 + c^2\theta_4 s^2\theta_5 + c^2\theta_2 s^2\theta_4 s^2\theta_5 + 2s\theta_2 c\theta_2 s\theta_4 s\theta_5 c\theta_5) \\
& \qquad + r_3(s\theta_2 c\theta_2 s\theta_4 s\theta_5 + s^2\theta_2 c\theta_5) - r_2 c\theta_4 s\theta_5\Big]\Big\}
\end{aligned}
$$

a typical term …
$\sin\theta_2 \sin\theta_5 - \cos\theta_2 \sin\theta_4 \cos\theta_5$

radius of gyration factors k_{ijj}^2 are being used here

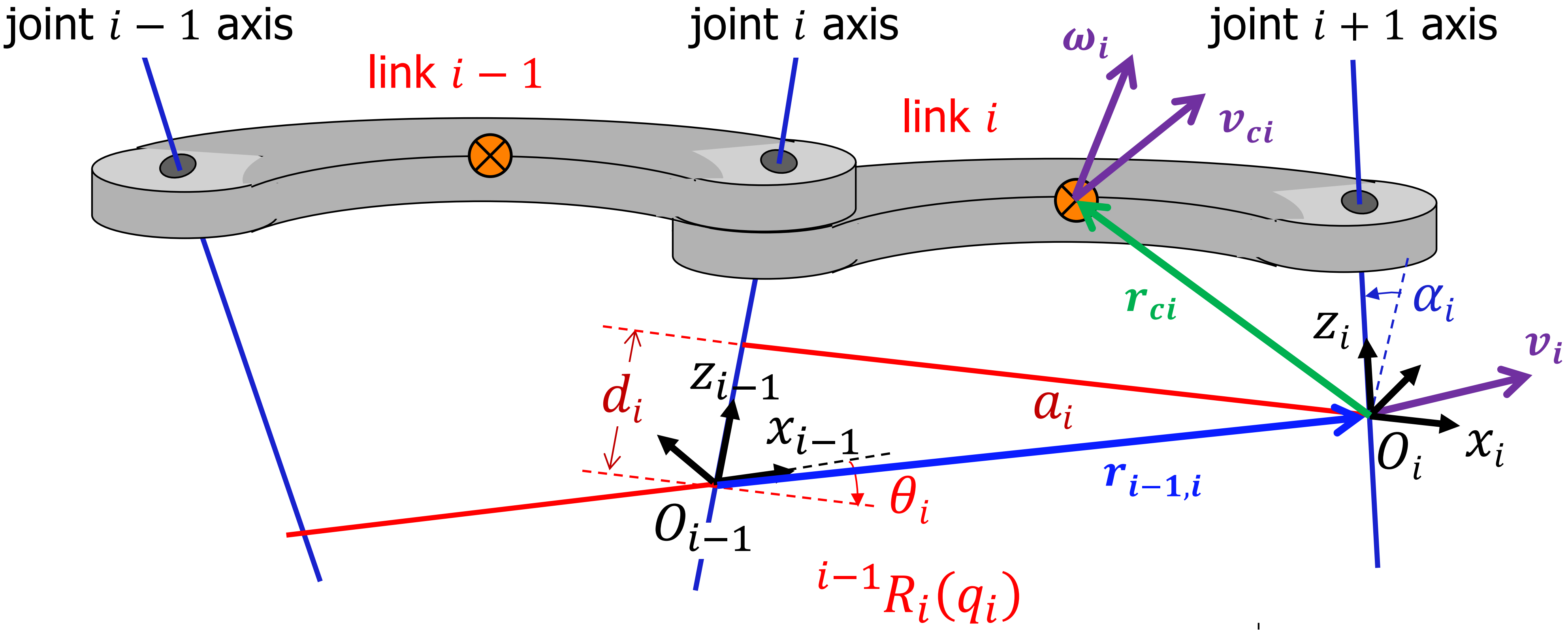
for a body of mass m_i and moment of inertia I_j with respect to an axis z_j, the radius of gyration k_{ijj} is the distance of the mass m_i from the same axis, such that $I_j = m_i k_{ijj}^2$

Expression of v_{ci} and ω_i

- v_{ci} and ω_i can be written using the relations of the robot differential kinematics (partial Jacobians)
- it is useful however to operate in a **recursive** way, expressing each vector quantity related to link i in the "**moving**" **frame** RF$_i$ attached to link i (with the notation ${}^{i}\mathbf{vector}_i$)
  - particularly convenient when using algebraic/symbolic manipulation languages (Matlab Symbolic Toolbox, Maple, Mathematica, …) for computing the kinetic energy of a (open chain) robot arm, when the number of joints increases (e.g., for $N \geq 4$)

→ Moving Frames

Recall: D-H frames



$$
{}^{i-1}A_i(q_i) = \left(\begin{array}{ccc|c} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right)
$$

Moving Frames algorithm

$$\boxed{v_{ci} = v_i + \omega_i \times r_{ci}}$$

velocity of center of mass of link i

velocity of O_i (origin of RF$_i$)

angular velocity of link i

position of center of mass of link i w.r.t. O_i

set σ_i = $\begin{cases} 0 & \text{revolute joint} \\ 1 & \text{prismatic joint} \end{cases}$

$$^{i}\omega_i = {}^{i-1}R_i^T(q_i)\left[{}^{i-1}\omega_{i-1} + (1-\sigma_i)\dot{q}_i\,{}^{i-1}z_{i-1}\right] = {}^{i-1}R_i^T(q_i)\,{}^{i-1}\omega_i$$

z-axis of RF$_{i-1}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

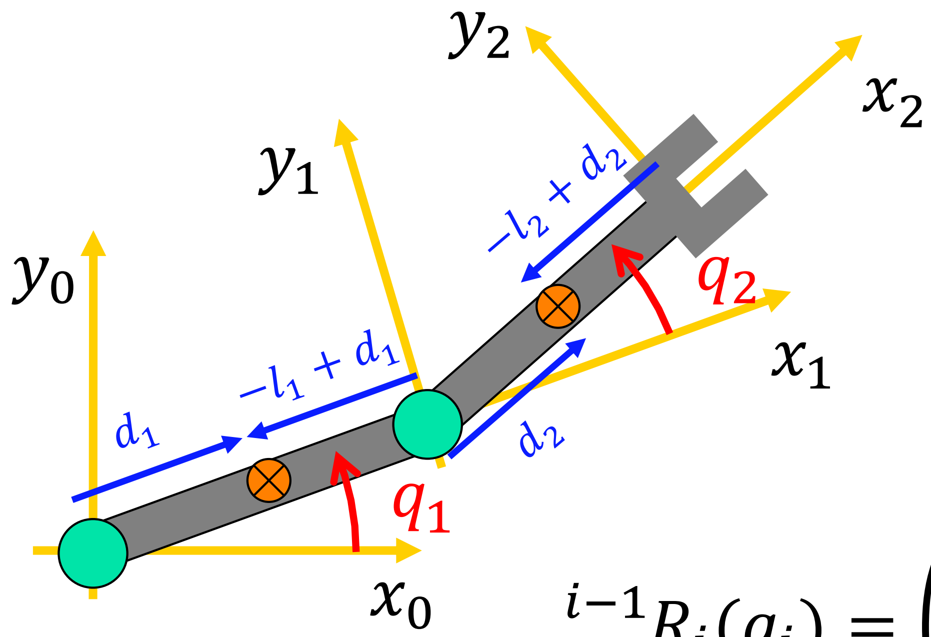
$$^{i}v_i = {}^{i-1}R_i^T(q_i)\left[{}^{i-1}v_{i-1} + \sigma_i\,\dot{q}_i\,{}^{i-1}z_{i-1} + {}^{i-1}\omega_i \times {}^{i-1}r_{i-1,i}\right]$$

… = $^{i}\omega_i$ already computed

… = $^{i}r_{i-1,i}$ (constant, if joint i is revolute!)

Dynamic model of a 2R robot

application of the algorithm



$\sigma_i = 0, \qquad i = 1,2$
(revolute joints)

$$g = \begin{pmatrix} 0 \\ -g_0 \\ 0 \end{pmatrix}$$

moving in the vertical plane

$$g_0 = 9.81$$

$${}^{i-1}R_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad {}^{i}r_{ci} = \begin{pmatrix} -l_i + d_i \\ 0 \\ 0 \end{pmatrix}$$

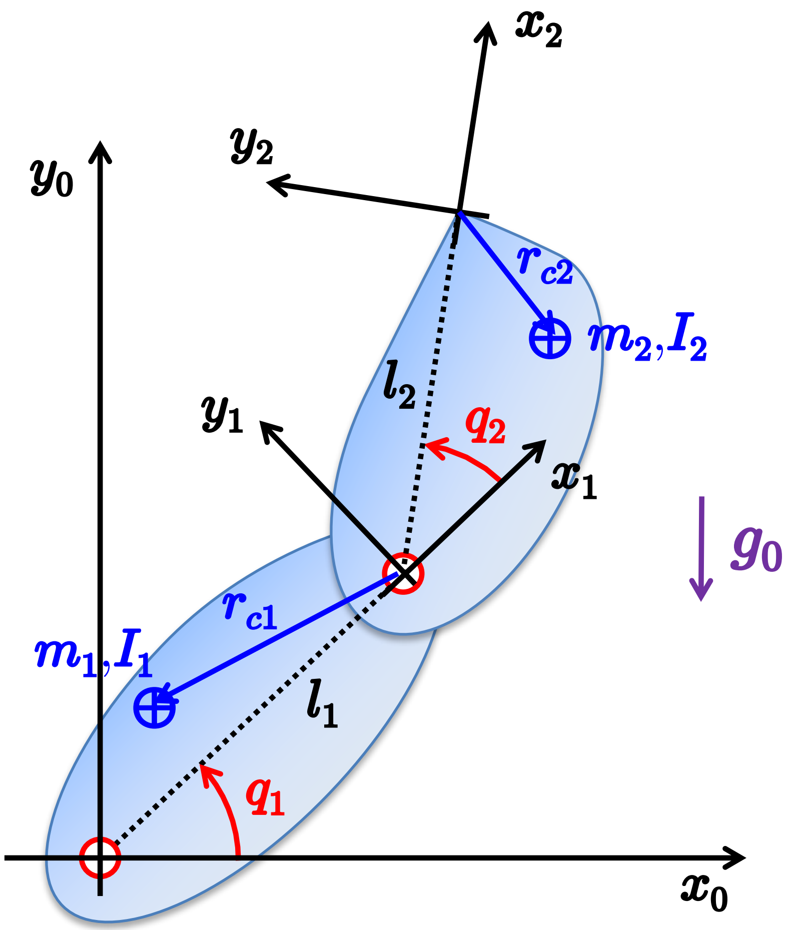
assumption: center of mass of each link is on its kinematic axis

initialization: $i = 0$ $\quad$ ${}^{0}\omega_0 = 0$ $\quad$ ${}^{0}v_0 = 0$

Dynamic model of a 2R robot

what if the CoM is not on the kinematic axis …

see **Robotics 2 Midterm 2021** (14 April)



$$ {}^{i-1}R_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{i}r_{ci} = \begin{pmatrix} r_{ci,x} \\ r_{ci,y} \\ 0 \end{pmatrix} $$

may also work in 2D (planar case) ${}^{i-1}\bar{R}_i(q_i)$, ${}^{i}\bar{r}_{ci}$

$$ T_1 = \frac{1}{2} m_1 \|\boldsymbol{v}_{c1}\|^2 + \frac{1}{2} \boldsymbol{\omega}_1^T \boldsymbol{I}_1 \boldsymbol{\omega}_1 $$

$$ = \frac{1}{2} m_1 \left((l_1 + r_{c1,x})^2 + r_{c1,y}^2 \right) \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 $$

$$ {}^{0}\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + {}^{0}\bar{\boldsymbol{R}}_2(q_1, q_2) \begin{pmatrix} l_2 + r_{c2,x} \\ r_{c2,y} \end{pmatrix} $$

$$ {}^{0}\bar{\boldsymbol{R}}_2(q_1, q_2) = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} $$

etc …

First step (link 1)

$i = 1$

$$\boxed{{}^1\omega_1 = {}^0R_1^T(q_1)\left[{}^0\omega_0 + \dot{q}_1\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}}$$

$${}^1v_1 = {}^0R_1^T(q_1)\left[{}^0v_0 + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}l_1c_1\\l_1s_1\\0\end{pmatrix}\right] = {}^0R_1^T(q_1)\begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times{}^0R_1^T(q_1)\begin{pmatrix}l_1c_1\\l_1s_1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}l_1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\l_1\dot{q}_1\\0\end{pmatrix}$$

$$\boxed{{}^1v_{c1} = \begin{pmatrix}0\\l_1\dot{q}_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix}\times\begin{pmatrix}-l_1+d_1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\d_1\dot{q}_1\\0\end{pmatrix}}$$

Kinetic energy of link 1

$$\boxed{{}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix}} \qquad \boxed{{}^1v_{c1} = \begin{pmatrix} 0 \\ d_1\dot{q}_1 \\ 0 \end{pmatrix}}$$

$$\Downarrow$$

$$T_1 = \frac{1}{2} m_1 d_1^2 \dot{q}_1^2 + \frac{1}{2} I_{c1,zz} \dot{q}_1^2 = \frac{1}{2} \underbrace{\left(I_{c1,zz} + m_1 d_1^2\right)} \dot{q}_1^2$$

the actual inertia around the rotation axis
of the first joint (parallel axis theorem)

Second step (link 2)

$i = 2$

$$\boxed{{}^2\omega_2 = {}^1R_2^T(q_2)\left[{}^1\omega_1 + \dot{q}_2\begin{pmatrix}0\\0\\1\end{pmatrix}\right] = \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}}$$

$${}^2v_2 = {}^1R_2^T(q_2)\left[{}^1v_1 + \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}\times\begin{pmatrix}l_2c_2\\l_2s_2\\0\end{pmatrix}\right] = \begin{pmatrix}l_1s_2\dot{q}_1\\l_1c_2\dot{q}_1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\dot{q}_1+\dot{q}_2\end{pmatrix}\times\begin{pmatrix}l_2\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}l_1s_2\dot{q}_1\\l_1c_2\dot{q}_1 + l_2(\dot{q}_1+\dot{q}_2)\\0\end{pmatrix}$$

Kinetic energy of link 2

$i = 2$

$$\boxed{{}^2v_{c2}} = {}^2v_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -l_2 + d_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{q}_1 \\ l_1 c_2 \dot{q}_1 + d_2(\dot{q}_1 + \dot{q}_2) \\ 0 \end{pmatrix}$$

$$T_2 = \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 + d_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 l_1 d_2 c_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \right)$$
$$+ \frac{1}{2} I_{c2,zz} (\dot{q}_1 + \dot{q}_2)^2$$

Robot inertia matrix $M(q)$

$$T = T_1 + T_2 = \frac{1}{2}\begin{pmatrix}\dot{q}_1 & \dot{q}_2\end{pmatrix}^T \begin{pmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22} \end{pmatrix}\begin{pmatrix}\dot{q}_1 \\ \dot{q}_2\end{pmatrix}$$

$$m_{11}(q) = I_{c1,zz} + m_1 d_1^2 + I_{c2,zz} + m_2 d_2^2 + m_2 l_1^2 + 2 m_2 l_1 d_2 c_2$$
$$= a_1 + 2 a_2 \cos q_2$$

$$m_{12}(q) = m_{21}(q) = I_{c2,zz} + m_2 d_2^2 + m_2 l_1 d_2 c_2 = a_3 + a_2 \cos q_2$$

$$m_{22} = I_{c2,zz} + m_2 d_2^2 = a_3$$

NOTE: introduction of **dynamic coefficients** a_i is a convenient **regrouping** of the dynamic parameters (more on this later → linear parametrization of dynamics)

Centrifugal and Coriolis terms

$$C_1(q) = \frac{1}{2}\left(\frac{\partial M_1}{\partial q} + \left(\frac{\partial M_1}{\partial q}\right)^T - \frac{\partial M}{\partial q_1}\right) = \frac{1}{2}\left(\begin{pmatrix} 0 & -2a_2s_2 \\ 0 & -a_2s_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2s_2 & -a_2s_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 0 & -a_2s_2 \\ -a_2s_2 & -a_2s_2 \end{pmatrix} \Rightarrow \boxed{c_1(q,\dot q) = -a_2s_2(\dot q_2^2 + 2\dot q_1\dot q_2)}$$

$$C_2(q) = \frac{1}{2}\left(\frac{\partial M_2}{\partial q} + \left(\frac{\partial M_2}{\partial q}\right)^T - \frac{\partial M}{\partial q_2}\right) = \cdots = \begin{pmatrix} a_2s_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{c_2(q,\dot q) = a_2s_2\dot q_1^2}$$

Gravity terms

$$U_1 = -m_1 g^T r_{0,c1} = -\, m_1 (0 \quad -g_0 \quad 0) \begin{pmatrix} * \\ d_1 s_1 \\ * \end{pmatrix} = m_1 g_0\, d_1 s_1$$

$$U_2 = -\, m_2 g^T r_{0,c2} = m_2 g_0\, (l_1 s_1 + d_2 s_{12})$$

$$U = U_1 + U_2$$

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T = \begin{pmatrix} g_0 (m_1 d_1 c_1 + m_2 l_1 c_1 + m_2 d_2 c_{12}) \\ g_0 m_2 d_2\, c_{12} \end{pmatrix} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix}$$

Dynamic model of a 2R robot

$$(a_1 + 2a_2c_2)\ddot{q}_1 + (a_2c_2 + a_3)\ddot{q}_2 - a_2s_2(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) + a_4c_1 + a_5c_{12} = u_1$$

$$(a_2c_2 + a_3)\ddot{q}_1 + a_3\ddot{q}_2 + a_2s_2\dot{q}_1^2 + a_5c_{12} = u_2$$

Q1: is it $a_2 = 0$ possible? …physical interpretation? …consequences?

Q2: is it $a_4 = a_5 = 0$ possible as well? …physical interpretation?

Q3: based on the expressions of the dynamic coefficients a_1, a_2, a_3, check that the robot inertia matrix is always positive definite, and in particular that the diagonal elements are always positive ($\forall q$)

Q4: provide two different matrices S' and S'' for the factorization of the quadratic velocity terms, respectively satisfying and not satisfying the skew-symmetry of $\dot{M} - 2S$