Dynamic model of robots: Lagrangian approach

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Dynamic model

- provides the relation between

  generalized forces $u(t)$ acting on the robot

  robot motion, i.e.,
  assumed configurations $q(t)$ over time

\[ \Phi(q, \dot{q}, \ddot{q}) = u \]

a system of 2$^{nd}$ order differential equations
Direct dynamics

- **direct relation**

  \[
  u(t) = \begin{bmatrix}
  u_1 \\
  \vdots \\
  u_N
  \end{bmatrix} \quad \rightarrow \quad q(t) = \begin{bmatrix}
  q_1 \\
  \vdots \\
  q_N
  \end{bmatrix}
  \]

  input for \( t \in [0,T] \) + \( q(0),\dot{q}(0) \)

  **initial state** at \( t = 0 \)

- **experimental solution**
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time \( T_c \))

- **solution by simulation**
  - use dynamic model and integrate numerically the differential equations (with simulation step \( T_s \leq T_c \))

\[\Phi(q,\dot{q},\ddot{q}) = u\]
Inverse dynamics

- inverse relation

\[ q_d(t), \dot{q}_d(t), \ddot{q}_d(t) \rightarrow u_d(t) \]

- experimental solution
  - repeated motion trials of direct dynamics using \( u_k(t) \), with iterative learning of nominal torques updated on trial \( k+1 \) based on the error in \([0,T]\) measured in trial \( k \): \( u_k(t) \Rightarrow u_d(t) \)

- analytic solution
  - use dynamic model and compute algebraically the values \( u_d(t) \) at every time instant \( t \)

\[ \Phi(q, \dot{q}, \ddot{q}) = u \]
Approaches to dynamic modeling

Euler-Lagrange method
(energy-based approach)
- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

Newton-Euler method
(balance of forces/torques)
- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d’Alembert, of Hamilton, of virtual works, ...
Euler-Lagrange method (energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies** (+ possibly, **concentrated elasticity** at the joints)

\( q \in \mathbb{R}^N \)  

**generalized coordinates** (e.g., joint variables, but not only!)

**Lagrangian**  
\[
L(q, \dot{q}) = T(q, \dot{q}) - U(q)
\]

kinetic energy – potential energy

- **least action principle of Hamilton**
- **virtual works principle**

**Euler-Lagrange equations**  
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \quad i = 1, \ldots, N
\]

non-conservative (external or dissipative)  
**generalized forces performing work on** \( q_i \)
Dynamics of actuated pendulum

a first example

\[ \dot{\theta}_m = n \dot{\theta} \]
\[ \theta_m = n \theta + \theta_{m0} \]
\[ \tau = n \tau_m \]
\[ q = \theta \quad \text{(or } q = \theta_m) \]
\[ T = T_m + T_\ell \]

\[ T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \]
\[ T_\ell = \frac{1}{2} \left( I_\ell + m d^2 \right) \dot{\theta}^2 \]

**Motor inertia** (around its spinning axis)

**Link inertia** (around the z-axis through its center of mass)

**Kinetic energy**

\[ T = \frac{1}{2} \left( I_\ell + m d^2 + n^2 I_m \right) \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2 \]
Dynamics of actuated pendulum (cont)

\[ U = U_0 - mg_0 d\cos \theta \]

potential energy

\[ L = T - U = \frac{1}{2} I \ddot{\theta}^2 + mg_0 d\cos \theta - U_0 \]

\[ \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta} \]

\[ \frac{\partial L}{\partial \theta} = -mg_0 d\sin \theta \]

\[ u = n\tau_m - b_\ell \dot{\theta} - nb_m \dot{\theta}_m + J_x^T F_x = n\tau_m - (b_\ell + n^2 b_m) \dot{\theta} + \ell \cos \theta \cdot F_x \]

applied or dissipated torques on motor side are multiplied by \( n \) when moved to the link side

equivalent joint torque due to force \( F_x \) applied to the tip at point \( p_x \)

“sum” of non-conservative torques

Robotics 2
Dynamics of actuated pendulum (cont)

\[ I \ddot{\theta} + mg_0 d \sin \theta = n \tau_m - \left( b_\ell + n^2 b_m \right) \dot{\theta} + \ell \cos \theta \cdot F_x \]

dividing by \( n \) and substituting \( \theta = \theta_m/n \)

\[ \frac{I}{n^2} \dddot{\theta}_m + \frac{m}{n} g_0 d \sin \frac{\theta_m}{n} = \tau_m - \left( \frac{b_\ell}{n^2} + b_m \right) \dot{\theta}_m + \frac{\ell}{n} \cos \frac{\theta_m}{n} \cdot F_x \]

dynamic model in \( q = \theta_m \)
Kinetic energy of a rigid body

mass \( m = \int \rho(x, y, z) \, dx \, dy \, dz = \int dm \)

position of center of mass (CoM) \( r_c = \frac{1}{m} \int r \, dm \)

when all vectors are referred to a body frame \( RF_c \) attached to the CoM, then \( r_c = 0 \) \( \Rightarrow \int_B r \, dm = 0 \)

kinetic energy \( T = \frac{1}{2} \int_B v^T(x, y, z) v(x, y, z) \, dm \)

kinematic relation for a rigid body \( v = v_c + \omega \times r = v_c + S(\omega) r \)

skew-symmetric matrix

Robotics 2
Kinetic energy of a rigid body (cont)

\[
T = \frac{1}{2} \int_B \left[ v_c + S(\omega) r \right]^T \left[ v_c + S(\omega) r \right] dm \\
= \frac{1}{2} \int_B v_c^T v_c \, dm + \int_B v_c^T S(\omega) r \, dm + \frac{1}{2} \int_B r^T S^T(\omega) S(\omega) r \, dm \\
= \frac{1}{2} m v_c^T v_c + \int_B v_c^T S(\omega) r \, dm = 0 \\
= \frac{1}{2} \int_B \text{trace} \{ S(\omega) r \cdot r^T S^T(\omega) \} \, dm \\
= \frac{1}{2} \text{trace} \{ S(\omega) \left( \int_B r \cdot r^T \, dm \right) S^T(\omega) \} \\
= \frac{1}{2} \text{trace} \{ S(\omega) J_c S^T(\omega) \} \\
= \frac{1}{2} \omega^T I_c \omega \\
= \frac{1}{2} \text{trace} \{ \omega^T I_c \omega \} \\
= \frac{1}{2} \text{trace} \{ J_c \omega \} \\
= \frac{1}{2} \text{trace} \{ I_c \omega \} \\
= \frac{1}{2} \text{trace} \{ I_c \omega \} \\
= \frac{1}{2} \text{trace} \{ J_c \omega \} \\
= \frac{1}{2} \text{trace} \{ I_c \omega \}
\]

Ex #1: provide the expressions of the elements of Euler matrix \( J_c \).
Ex #2: prove last equality and provide the expressions of the elements of inertia matrix \( I_c \).

König theorem
Examples of body inertia matrices
homogeneous bodies of mass m, with axes of symmetry

parallelepiped with sides
a (length/height), b, c (base)

\[ I_c = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m(b^2 + c^2) \\ \frac{1}{12} m(a^2 + c^2) \\ \frac{1}{12} m(a^2 + b^2) \end{pmatrix} \]

empty cylinder with length h,
and external/internal radius a, b

\[ I_c = \begin{pmatrix} \frac{1}{2} m(a^2 + b^2) \\ \frac{1}{12} m(3(a^2 + b^2) + h^2) \\ I_{zz} \end{pmatrix} \]

\[ I_{zz} = I_{zz} + m(h/2)^2 \] (parallel) axis translation theorem

its generalization:
changes on body inertia matrix
due to a pure translation r of
the reference frame

Steiner theorem

\[ I = I_c + m(r^T r \cdot E_{3 \times 3} - r r^T) = I_c + m S^T(r) S(r) \]

body inertia matrix relative to the CoM
identity matrix

Ex #3: prove the last equality
Robot kinetic energy

\[ T = \sum_{i=1}^{N} T_i \]  N rigid bodies (+ fixed base)

\[ T_i = T_i(q_j, \dot{q}_j, j \leq i) \]  open kinematic chain

\[ T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]  König theorem

absolute velocity of the center of mass (CoM)

absolute angular velocity of whole body

i-th link (body) of the robot
Kinetic energy of a robot link

\[ T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]

\( \omega_i, I_{ci} \) should be expressed in the same reference frame, but the product \( \omega_i^T I_{ci} \omega_i \) is invariant w.r.t. any chosen frame

in frame \( RF_{ci} \) attached to (the center of mass of) link \( i \)

\[ iI_{ci} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int x y dm & -\int x z dm \\ -\int x y dm & \int (x^2 + z^2) dm & -\int y z dm \\ -\int x z dm & -\int y z dm & \int (x^2 + y^2) dm \end{pmatrix} \]

constant!
Dependence of $T$ from $q$ and $\dot{q}$

$$v_{ci} = J_{Li}(q) \dot{q} = \begin{pmatrix} \cdots & i & \cdots & 0 & \cdots & 0 \end{pmatrix} \dot{q}$$

3 rows

$\omega_i = J_{Ai}(q) \dot{q} = \begin{pmatrix} \cdots & i & \cdots & 0 & \cdots & 0 \end{pmatrix} \dot{q}$

3 rows

(partial) Jacobians typically expressed in $RF_0$
Final expression of $T$

$$T = \frac{1}{2} \sum_{i=1}^{N} \left( m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci} \omega_i \right)$$

$$= \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} m_i J_{Li}(q) J_{Li}(q) + J_{Ai}(q) I_{ci} J_{Ai}(q) \right) \dot{q}$$

NOTE:

in practice, NEVER use this formula (or partial Jacobians) for computing $T$; a better method is available...

Robot (generalized) inertia matrix

- symmetric
- positive definite, $\forall q \Rightarrow$ always invertible
Robot potential energy

assumption: GRAVITY contribution only

\[ U = \sum_{i=1}^{N} U_i \quad \text{N rigid bodies (+ fixed base)} \]

\[ U_i = U_i(q_j, j \leq i) \quad \text{open kinematic chain} \]

\[ U_i = -m_i g^T r_{0,ci} \]

\{ gravity acceleration vector \quad position of the center of mass of link i \} \quad \text{typically expressed in RF}_0

dependence on \( q \)

\[
\begin{pmatrix}
  r_{0,ci} \\
  1
\end{pmatrix} =
\begin{pmatrix}
  {}^0 A_1(q_1) \\
  {}^1 A_2(q_2) \\
  \vdots \\
  {}^{i-1} A_i(q_i)
\end{pmatrix}
\begin{pmatrix}
  r_{i,ci} \\
  1
\end{pmatrix}
\]

NOTE: need to work with homogeneous coordinates
Summarizing ...

**kinetic energy**

\[ T = \frac{1}{2} \dot{q}^T B(q) \dot{q} = \frac{1}{2} \sum_{i,j} b_{ij}(q) \dot{q}_i \dot{q}_j \geq 0 \]

**potential energy**

\[ U = U(q) \]

**Lagrangian**

\[ L(q, \dot{q}) = T(q, \dot{q}) - U(q) \]

**Euler-Lagrange equations**

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k \quad k = 1,...,N \]

non-conservative (active/dissipative) generalized forces performing work on \( q_k \) coordinate

Positive definite quadratic form

\[ T = 0 \iff \dot{q} = 0 \]
Applying Euler-Lagrange equations
(the scalar derivation; see Appendix for vector format)

\[ L(q, \dot{q}) = \frac{1}{2} \sum_{i,j} b_{ij}(q)\ddot{q}_i \dot{q}_j - U(q) \]

\[ \frac{\partial L}{\partial \dot{q}_k} = \sum_j b_{kj} \dot{q}_j \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j \frac{\partial b_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j + \sum_{i,j} \frac{\partial b_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j \]

(dependences on q are not shown)

\[ \frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial b_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial U}{\partial q_k} \]

LINEAR terms in ACCELERATION \( \ddot{q} \)

QUADRATIC terms in VELOCITY \( \dot{q} \)

NONLINEAR terms in CONFIGURATION \( q \)
k-th dynamic equation ...

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k
\]

\[
\sum_j b_{kj}(q)\ddot{q}_j + \sum_{i,j} \left( \frac{\partial b_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial b_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k
\]

\[
\sum_{i,j} \frac{1}{2} \left( \frac{\partial b_{kj}}{\partial q_i} + \frac{\partial b_{ki}}{\partial q_j} - \frac{\partial b_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \cdots
\]

\[
c_{kij} = c_{kji} \quad \text{Christoffel symbols of the first kind}
\]
... and interpretation of dynamic terms

\[
\sum_{j} b_{kj}(q) \ddot{q}_j + \sum_{i,j} c_{kij}(q) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k \quad k = 1,\ldots,N
\]

**INERTIAL terms**

- \( b_{kk}(q) \): inertia at joint k when joint k accelerates (\( b_{kk} > 0 \))!

- \( b_{kj}(q) \): inertia “seen” at joint k when joint j accelerates

**CENTRIFUGAL (i=j) and CORIOLIS (i\neq j) terms**

- \( c_{kii}(q) \): coefficient of the centrifugal force at joint k when joint i is moving (\( c_{iii} = 0, \forall i \))

- \( c_{kij}(q) \): coefficient of the Coriolis force at joint k when both joint i and joint j are moving

**GRAVITY terms** \( g_k(q) \)
Robot dynamic model
in vector formats

1. \[ B(q)\ddot{q} + c(q,\dot{q}) + g(q) = u \]

\[ c_k(q,\dot{q}) = q^T C_k(q) \dot{q} \]
\[ C_k(q) = \frac{1}{2} \left( \frac{\partial b_k}{\partial q} + \left( \frac{\partial b_k}{\partial q} \right)^T - \frac{\partial B}{\partial q_k} \right) \]

k-th component of vector c
symmetric matrix

2. \[ B(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u \]

\[ s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i \]

factorization of c by S is not unique!

NOTE: these models are in the form \( \Phi(q,\dot{q},\ddot{q}) = u \) as expected

NOT a symmetric matrix

Robotics 2
Dynamic model of a PR robot

\[ T = T_1 + T_2 \]

U = constant
(on horizontal plane)

\[ p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \|v_{c1}\|^2 = p_{c1}^T \dot{p}_{c1} = \dot{q}_1^2 \]

\[ T_1 = \frac{1}{2} m_1 \dot{q}_1^2 \]

\[ T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2 \]

\[ \dot{q}_{c1} = \begin{pmatrix} q_1 + d \cos q_2 \\ d \sin q_2 \\ 0 \end{pmatrix} \quad \Rightarrow \quad v_{c2} = \begin{pmatrix} \dot{q}_1 - d \sin q_2 \dot{q}_2 \\ d \cos q_2 \dot{q}_2 \\ 0 \end{pmatrix} \quad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} \]

\[ T_2 = \frac{1}{2} m_2 \left( \dot{q}_1^2 + d^2 \dot{q}_2^2 - 2d \sin q_2 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2 \]
Dynamic model of a PR robot (cont)

\[
\begin{bmatrix}
  m_1 + m_2 & -m_2 d \sin q_2 \\
  -m_2 d \sin q_2 & I_{c2,zz} + m_2 d^2
\end{bmatrix}
\]

\[
C(q, \dot{q}) = \begin{bmatrix} c_1(q, \dot{q}) \\ c_2(q, \dot{q}) \end{bmatrix}
\]

\[
c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}
\]

where

\[
C_k(q) = \frac{1}{2} \left( \frac{\partial b_k}{\partial q} + \left( \frac{\partial b_k}{\partial q} \right)^T - \frac{\partial B}{\partial q_k} \right)
\]

\[
C_1(q) = \frac{1}{2} \left( \begin{bmatrix} 0 & 0 \\ 0 & -m_2 d \cos q_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -m_2 d \cos q_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)
\]

\[
c_1(q, \dot{q}) = -m_2 d \cos q_2 \dot{q}_2^2
\]

\[
C_2(q) = \frac{1}{2} \left( \begin{bmatrix} 0 & -m_2 d \cos q_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d \cos q_2 & 0 \end{bmatrix} - \begin{bmatrix} -m_2 d \cos q_2 & 0 \\ 0 & 0 \end{bmatrix} \right)
\]

\[
c_2(q, \dot{q}) = 0
\]
Dynamic model of a PR robot (cont)

\[ B(q)\ddot{q} + c(q,\dot{q}) = u \]

\[
\begin{pmatrix}
  m_1 + m_2 & -m_2 d \sin q_2
  
  -m_2 d \sin q_2 & I_{c2,zz} + m_2 d^2
\end{pmatrix}
\begin{pmatrix}
  \ddot{q}_1 \\
  \ddot{q}_2
\end{pmatrix}
+
\begin{pmatrix}
  -m_2 d \cos q_2 \dot{q}_2^2 \\
  0
\end{pmatrix}
=
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}
\]

NOTE: the \( b_{NN} \) element (here, for \( N=2 \)) is always a constant!

Q1: why Coriolis terms are not present?
Q2: when applying a force \( u_1 \), does the second joint accelerate? ... always?
Q3: what is the expression of a factorization matrix \( S \)? ... is it unique?
Q4: which is the configuration with “maximum inertia”?
A structural property

matrix $\dot{B} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix $S$)

**Proof**

\[
\dot{b}_{kj} = \sum_i \frac{\partial b_{kj}}{\partial q_i} \dot{q}_i \quad 2s_{kj} = 2 \sum_i c_{kji} \dot{q}_i = 2 \sum_i \frac{1}{2} \left( \frac{\partial b_{kj}}{\partial q_i} + \frac{\partial b_{ki}}{\partial q_j} - \frac{\partial b_{ij}}{\partial q_k} \right) \dot{q}_i
\]

\[
\dot{b}_{kj} - 2s_{kj} = \sum_i \left( \frac{\partial b_{ij}}{\partial q_k} - \frac{\partial b_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}
\]

\[
n_{jk} = \dot{b}_{jk} - 2s_{jk} = \sum_i \left( \frac{\partial b_{ik}}{\partial q_k} - \frac{\partial b_{ji}}{\partial q_i} \right) \dot{q}_i = -n_{kj}
\]

because of the symmetry of $B$

\[
x^T (\dot{B} - 2S)x = 0, \quad \forall x
\]
Energy conservation

- total robot energy

\[ E = T + U = \frac{1}{2} \dot{q}^T B(q) \dot{q} + U(q) \]

- its evolution over time (using the dynamic model)

\[ \dot{E} = \dot{q}^T B(q) \dot{q} + \frac{1}{2} \dot{q}^T \dot{B}(q) \dot{q} + \frac{\partial U}{\partial q} \dot{q} \]

\[ = \dot{q}^T (u - S(q, \dot{q}) \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \dot{B}(q) \dot{q} + \dot{q}^T g(q) \]

\[ = \dot{q}^T u + \frac{1}{2} \dot{q}^T (\dot{B}(q) - 2S(q, \dot{q})) \dot{q} \]

- if \( u \equiv 0 \), total energy is constant (no dissipation or increase)

\[ \dot{E} = 0 \quad \Rightarrow \quad \dot{q}^T (\dot{B} - 2S(q, \dot{q})) \dot{q} = 0, \; \forall q, \dot{q} \]

\[ \dot{E} = \dot{q}^T u \]

weaker than skew-symmetry, as the external velocity is the same that appears in the internal matrices

in general, the variation of the total energy is equal to the work of non-conservative forces
Appendix: 
Vector format derivation of dynamic model

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T - \left( \frac{\partial L}{\partial q} \right)^T = u
\]
\[
L = \frac{1}{2} \dot{q}^T B(q) \ddot{q} - U(q)
\]
\[
B(q) = \begin{bmatrix} b_1(q) & \ldots & b_i(q) & \ldots & b_N(q) \end{bmatrix} = \sum_{i=1}^{N} b_i(q) e_i^T
\]
\[
\begin{bmatrix} \ddot{q}^T \end{bmatrix} = B(q) \ddot{q}
\]
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T = B(q) \ddot{q}
\]
\[
\left( \frac{\partial L}{\partial q} \right)^T = \left( \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} \frac{\partial b_i}{\partial q} e_i^T \right) \dot{q} - \frac{\partial U}{\partial q} \right)^T
\]
\[
= \left( \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} \frac{\partial b_i}{\partial q} \dot{q}_i \right) - \frac{\partial U}{\partial q} \right)^T
\]
\[
= \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial b_i}{\partial q} \right)^T \dot{q}_i \dot{q} - \left( \frac{\partial U}{\partial q} \right)^T
\]
\[
\rightarrow B(q) \ddot{q} + \left[ \sum_{i=1}^{N} \left( \frac{\partial b_i}{\partial q} - \frac{1}{2} \left( \frac{\partial b_i}{\partial q} \right)^T \right) \dot{q}_i \right] \dot{q} + \left( \frac{\partial U}{\partial q} \right)^T = u
\]
\[
\text{s}^T_k (q, \dot{q}) = \dot{q}^T C_k(q) \rightarrow S(q, \dot{q})
\]
\[
g(q)
\]