Robotics 2

Dynamic model of robots: Lagrangian approach

Prof. Alessandro De Luca

DIPARTIMENTO DI INGEGNERIA INFORMATICA AUTOMATICA E GESTIONALE ANTONIO RUBERTI

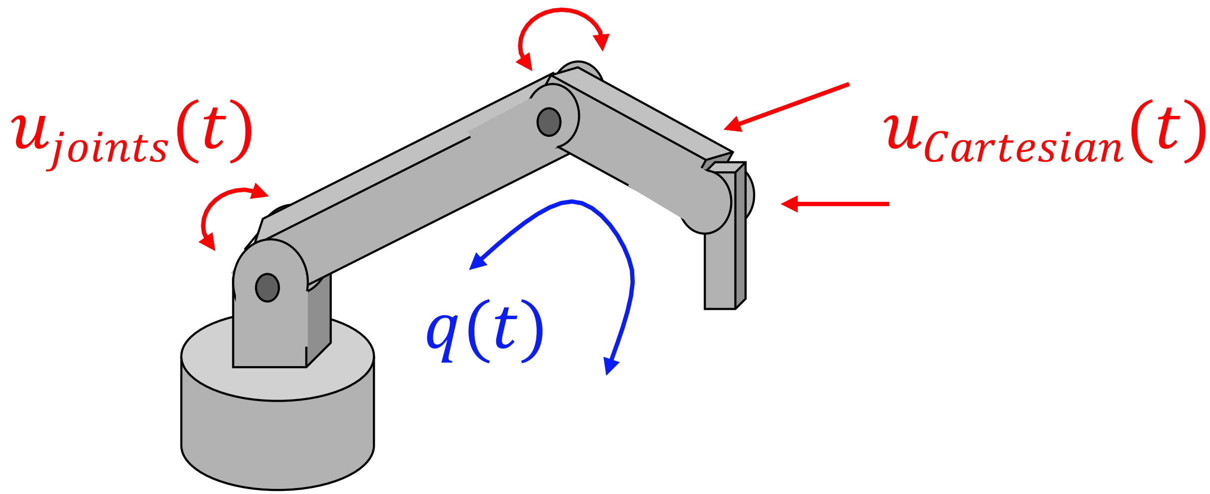
SAPIENZA UNIVERSITÀ DI ROMA

Dynamic model

- provides the relation between

generalized forces $u(t)$ acting on the robot

robot motion, i.e.,
assumed configurations $q(t)$ over time

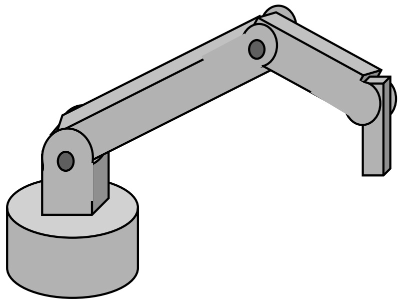


a system of 2nd order differential equations

$$\Phi(q, \dot{q}, \ddot{q}) = u$$

Direct dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \longrightarrow \quad$$  $$\quad \longrightarrow \quad q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

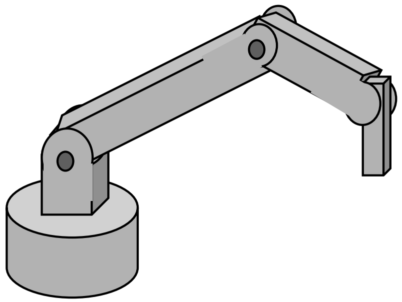
input for $t \in [0, T]$ + $q(0), \dot{q}(0)$ resulting motion

initial state at $t = 0$

- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time T_c)
- solution by simulation $\longleftrightarrow$ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$
  - use dynamic model and integrate numerically the differential equations (with simulation step $T_s \le T_c$)

Inverse dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ ⟶  ⟶ $u_d(t)$

desired motion
for $t \in [0, T]$

required input
for $t \in [0, T]$

- experimental solution
  - e.g., by repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial $k+1$ based on the error in $[0, T]$ measured in trial k: $\lim_{k\to\infty} u_k(t) \Rightarrow u_d(t)$
- analytic solution ⟷ $\boxed{\Phi(q, \dot{q}, \ddot{q}) = u}$
  - use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

Approaches to dynamic modeling

Euler-Lagrange method (energy-based approach) $\rightleftharpoons$ **Newton-Euler method (balance of forces/moments)**

Euler-Lagrange method (energy-based approach):

- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

Newton-Euler method (balance of forces/moments):

- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d'Alembert, of Hamilton, of virtual works, Kane's equations …

Euler-Lagrange method (energy-based approach)

basic assumption: the N links in motion are considered as **rigid bodies**
(+ later on, include also ***concentrated elasticity*** *at the joints)*

$q \in \mathbb{R}^N$ generalized coordinates (e.g., joint variables, but not only!)

Lagrangian $\boxed{L(q,\dot{q}) = T(q,\dot{q}) - U(q)}$

kinetic energy – potential energy

- principle of least action of Hamilton
- principle of virtual works

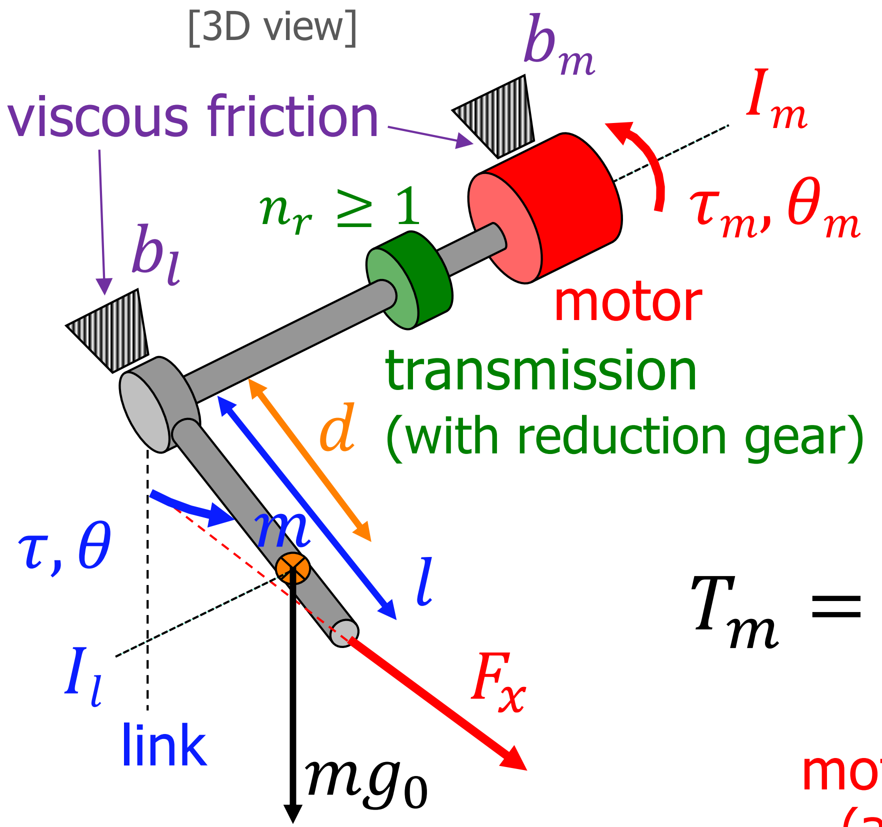
Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \qquad i = 1, \ldots, N$$

non-conservative (external or dissipative) generalized forces performing work on q_i

Dynamics of an actuated pendulum

a first example



$$\dot{\theta}_m = n_r \dot{\theta} \Rightarrow \theta_m = n_r \theta + \underbrace{\cancel{\theta_{m0}}}_{=0}$$

$$\tau = n_r \tau_m$$

$\boxed{q = \theta}$ (or $q = \theta_m$)

$$\boxed{T = T_m + T_l}$$

(… around the ∥ axis through its base)

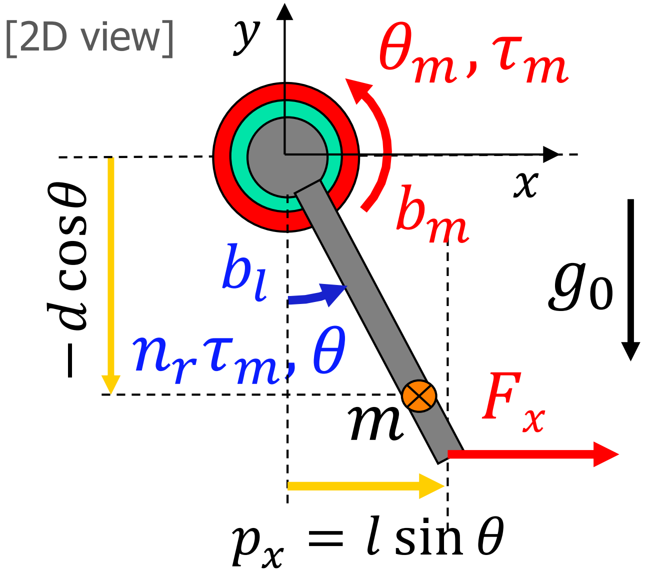
$$T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \qquad T_l = \frac{1}{2}(I_l + md^2)\dot{\theta}^2$$

motor inertia (around its spinning axis)

link inertia (around the z-axis through its center of mass …)

kinetic energy $\boxed{T = \frac{1}{2}(I_l + md^2 + I_m n_r^2)\dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2}$

Dynamics of an actuated pendulum
(continued)



$$\boxed{\dot{p}_x = l\cos\theta \cdot \dot{\theta} = J_x\dot{\theta}}$$

$$U = U_0 - mg_0 d\cos\theta$$ potential energy

$$L = T - U = \frac{1}{2}I\dot{\theta}^2 + mg_0 d\cos\theta - U_0$$

$$\frac{\partial L}{\partial\dot{\theta}} = I\dot{\theta} \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = I\ddot{\theta} \qquad \frac{\partial L}{\partial\theta} = -mg_0 d\sin\theta$$

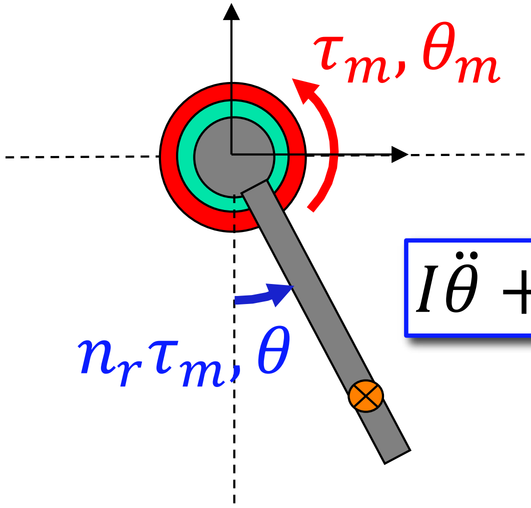
$$u = n_r\tau_m - b_l\dot{\theta} - n_r b_m\dot{\theta}_m + J_x^T F_x = n_r\tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta\, F_x$$

applied or dissipated torques on motor side are multiplied by n_r when moved to the link side

equivalent joint torque due to force F_x applied to the tip at point p_x

"sum" of non-conservative torques

Dynamics of an actuated pendulum (cont)



dynamic model in $q = \theta$

$$\boxed{I\ddot{\theta} + m g_0 d \sin\theta = n_r \tau_m - (b_l + b_m n_r^2)\dot{\theta} + l\cos\theta \cdot F_x}$$

dividing by n_r and substituting $\theta = \theta_m / n_r$

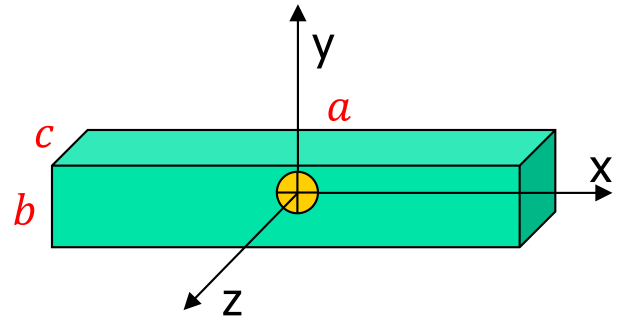
$$\boxed{\frac{I}{n_r^2}\ddot{\theta}_m + \frac{m}{n_r} g_0 d \sin\frac{\theta_m}{n_r} = \tau_m - \left(\frac{b_l}{n_r^2} + b_m\right)\dot{\theta}_m + \frac{l}{n_r}\cos\frac{\theta_m}{n_r} \cdot F_x}$$

dynamic model in $q = \theta_m$

https://en.wikipedia.org/wiki/List_of_moments_of_inertia

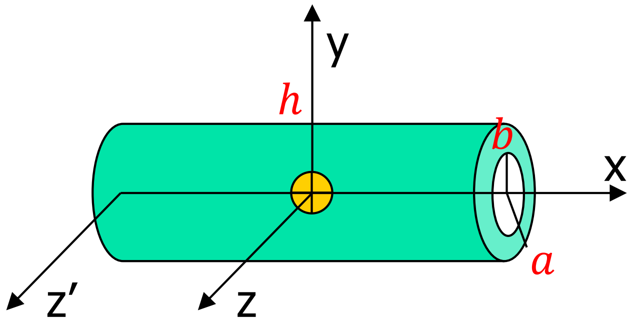
Examples of body inertia matrices

homogeneous bodies of mass m, with axes of symmetry



parallelepiped with sides
a (length/height), b and c (base)

$$I_c = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12}m(b^2+c^2) & & \\ & \frac{1}{12}m(a^2+c^2) & \\ & & \frac{1}{12}m(a^2+b^2) \end{pmatrix}$$



empty cylinder with length h,
and external/internal radius a and b

$$I_c = \begin{pmatrix} \frac{1}{2}m(a^2+b^2) & & \\ & \frac{1}{12}m(3(a^2+b^2)+h^2) & \\ & & I_{zz} \end{pmatrix} \qquad I_{zz} = I_{yy}$$

$$I'_{zz} = I_{zz} + m\left(\frac{h}{2}\right)^2$$

(parallel) axis translation theorem

Steiner theorem

$$\boxed{I = I_c + m(r^T r \cdot E_{3\times 3} - r r^T) = I_c + m\, S^T(r) S(r)}$$

body inertia matrix relative to the CoM

identity matrix

Homework: prove last equality

skew-symmetric

… its generalization:
changes on body inertia matrix due to a pure translation r of the reference frame

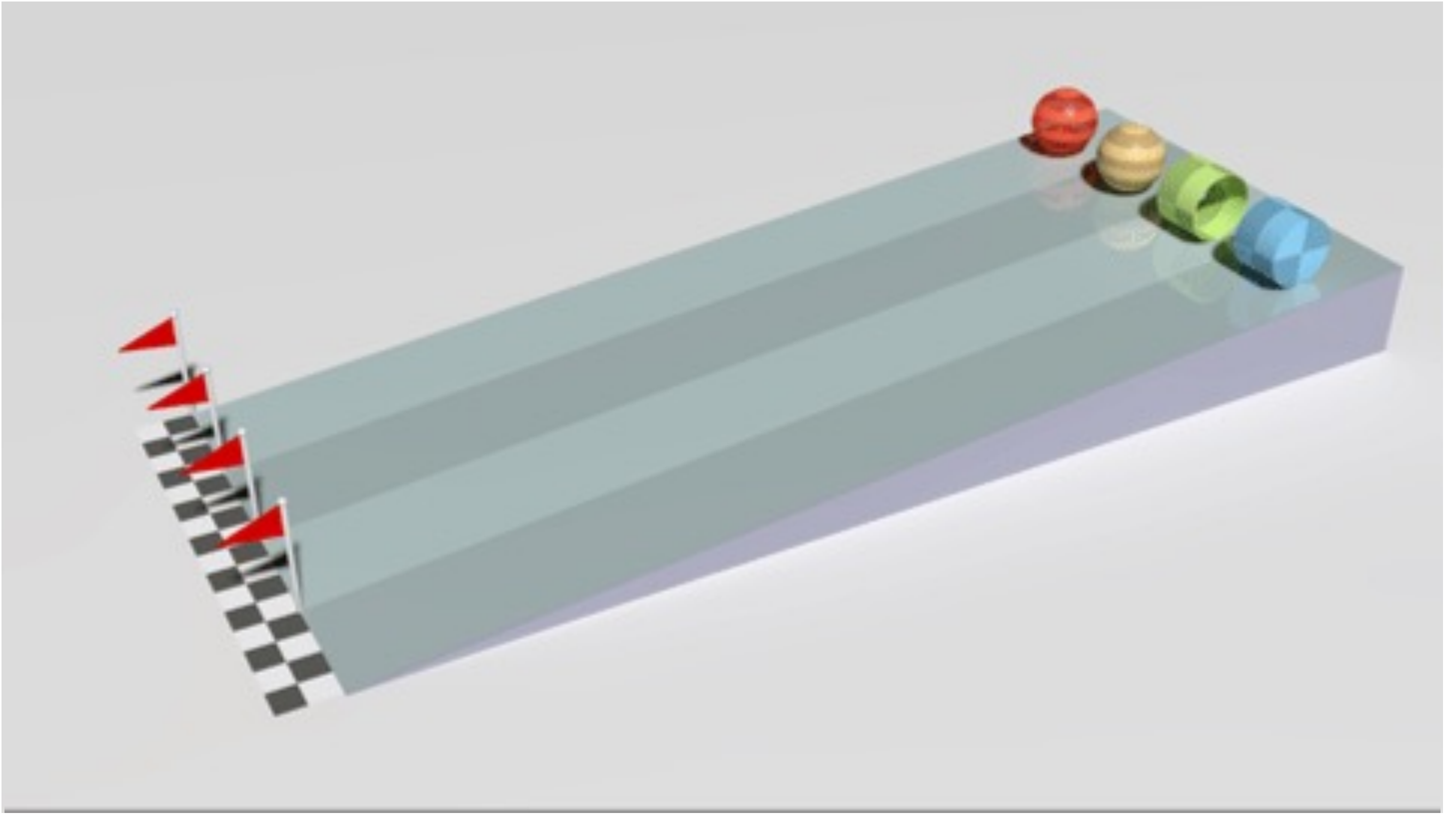
Rolling inertias

https://en.wikipedia.org/wiki/Moment_of_inertia

4 "circular" bodies with the same mass & radius rolling down an inclined plane without slipping

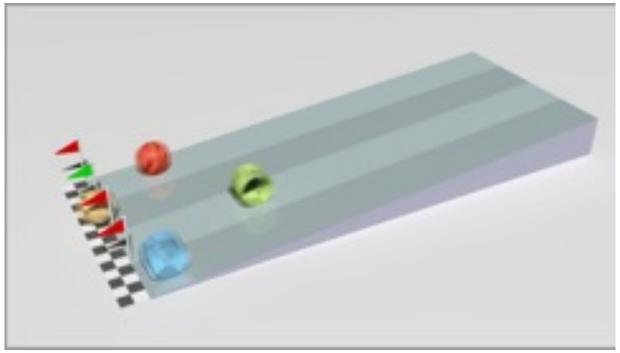
⬇

time to reach the finish line depends on their **moment of inertia** (about rolling axis!)

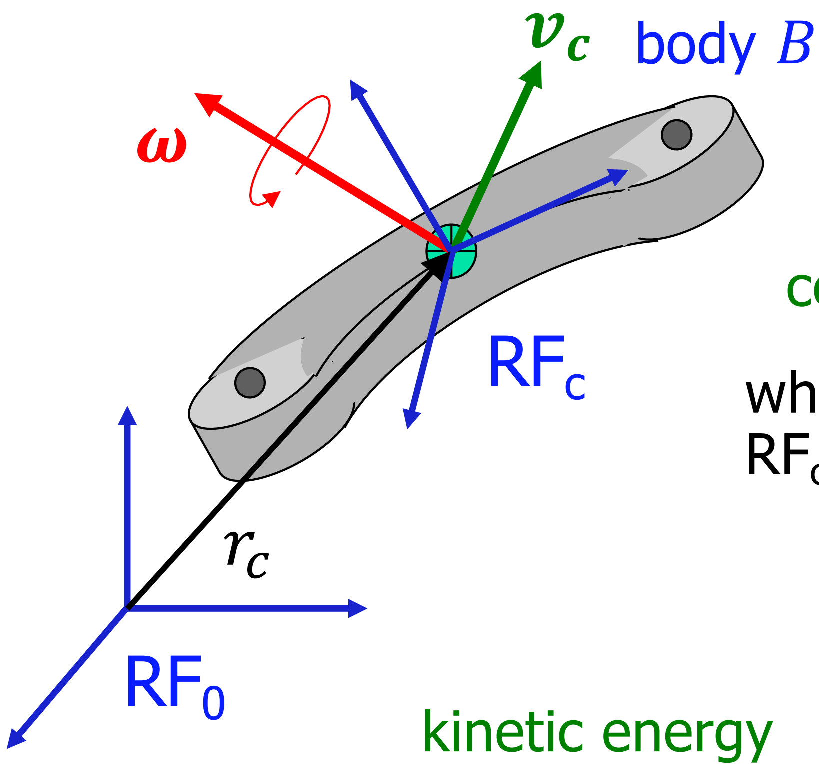


from back to front:

- spherical shell — 3rd
- solid sphere — 1st (smallest)
- cylindrical ring — 4th (largest)
- solid cylinder — 2nd



Kinetic energy of a rigid body



mass density

mass $m = \int_B \rho(x,y,z)\, dxdydz = \int_B dm$

position of center of mass (CoM) $r_c = \frac{1}{m}\int_B r\, dm$

when all vectors are referred to a body frame RF$_c$ attached to the CoM, then

$$r_c = 0 \quad \Rightarrow \quad \int_B r\, dm = 0$$

kinetic energy $T = \frac{1}{2}\int_B v^T(x,y,z)\, v(x,y,z)\, dm$

(fundamental) kinematic relation for a rigid body $v = v_c + \omega \times r = v_c + S(\omega)\, r$

skew-symmetric matrix

Kinetic energy of a rigid body (cont)

$$T = \frac{1}{2}\int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r)\, dm$$

$$= \frac{1}{2}\int_B v_c^T v_c\, dm + \int_B v_c^T S(\omega)\, r\, dm + \frac{1}{2}\int_B r^T S^T(\omega) S(\omega) r\, dm$$

$$\boxed{= \frac{1}{2} m\, v_c^T v_c}$$

$$= v_c^T S(\omega) \int_B r dm = 0$$

$$= \frac{1}{2}\int_B \omega^T S^T(r) S(r) \omega\, dm$$

$$= \frac{1}{2}\omega^T \left(\int_B S^T(r) S(r)\, dm\right) \omega$$

$$\boxed{= \frac{1}{2}\omega^T I_c\, \omega}$$

translational kinetic energy (point mass at CoM)

+

rotational kinetic energy (of the whole body)

Homework
compute "once in your life" the expression of the elements of inertia matrix I_c

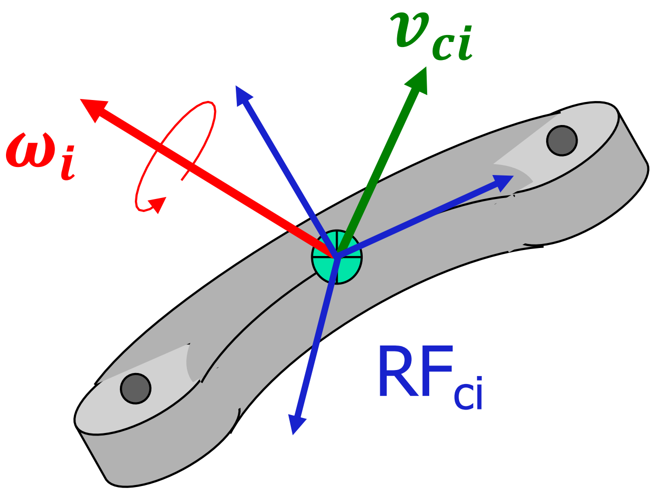
body inertia matrix (around the CoM)

König theorem

Robot kinetic energy

$$T = \sum_{i=1}^{N} T_i$$ ⇐ N rigid bodies (+ fixed base)

$$T_i = T_i(q_j, \dot{q}_j; \underbrace{j \leq i})$$ ⇐ open kinematic chain



i-th link (body)
of the robot

König theorem

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\ \omega_i^T I_{ci}\ \omega_i$$

absolute velocity
of the center of mass
(CoM)

absolute
angular velocity
of whole body

Kinetic energy of a robot link

$$T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2}\, \omega_i^T I_{ci}\, \omega_i$$

ω_i, I_{ci} should be expressed in the **same reference frame**,
but the product $\omega_i^T I_{ci} \omega_i$ is **invariant** w.r.t. any chosen frame

$$^0\omega_i^T\ {}^0I_{ci}(q)\ {}^0\omega_i = \left({}^0R_i(q)\ {}^i\omega_i\right)^T {}^0I_{ci}(q)\left({}^0R_i(q)\ {}^i\omega_i\right) = {}^i\omega_i^T\left({}^0R_i^T(q)\ {}^0I_{ci}(q)\ {}^0R_i(q)\right){}^i\omega_i$$

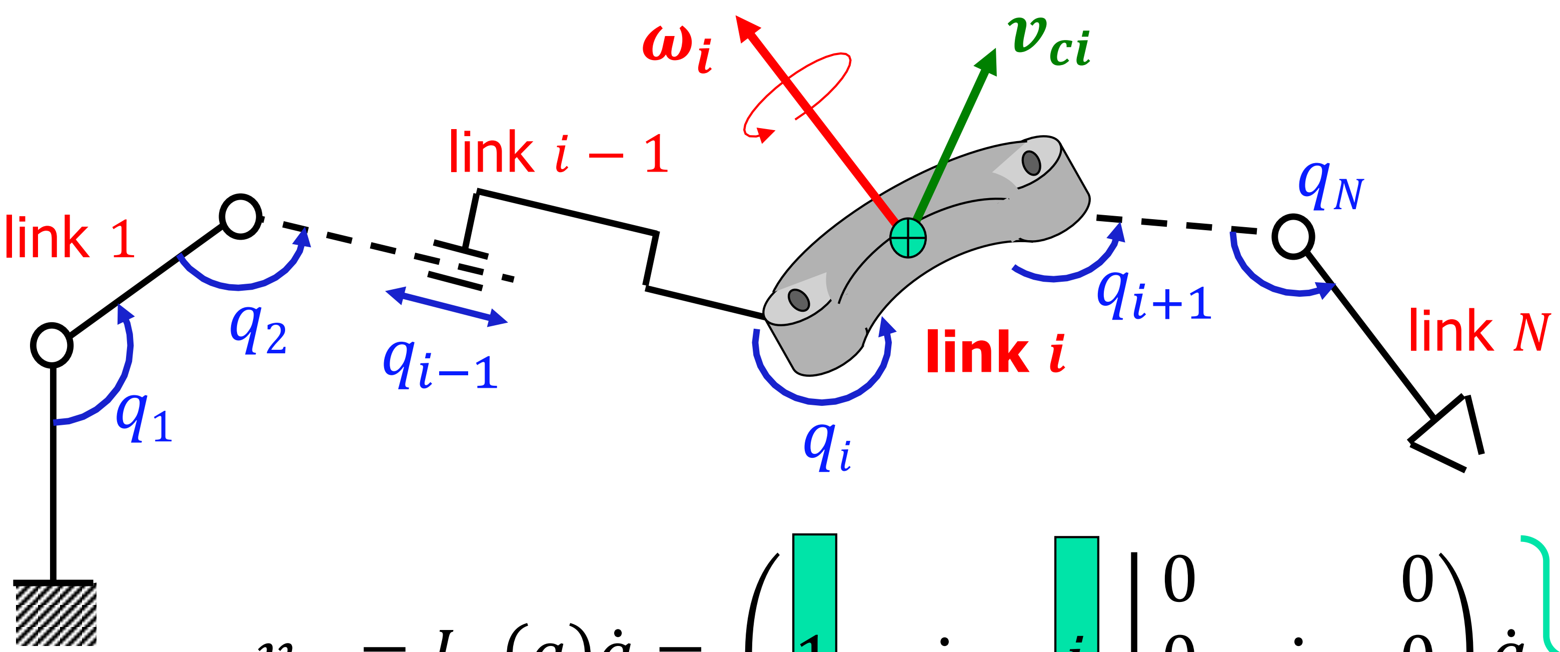
$$= {}^i\omega_i^T\ {}^iI_{ci}\ {}^i\omega_i \qquad \Rightarrow \qquad {}^0I_{ci}(q) = {}^0R_i(q)\ {}^iI_{ci}\ {}^0R_i^T(q)$$

in frame RF_{ci} attached to (the center of mass of) link i

$$^iI_{ci} = \begin{pmatrix} \int (y^2+z^2)dm & -\int xy\, dm & -\int xz\, dm \\ & \int (x^2+z^2)dm & -\int yz\, dm \\ symm & & \int (x^2+y^2)dm \end{pmatrix}$$

constant!

Dependence of T from q and $\dot{q}$



$$v_{ci} = J_{Li}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

(partial) Jacobians

typically expressed in RF_0

$$\omega_i = J_{Ai}(q)\dot{q} = \left(\begin{array}{ccc|ccc} & & & 0 & & 0 \\ 1 & \vdots & i & 0 & \vdots & 0 \\ & & & 0 & & 0 \end{array}\right)\dot{q} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \text{3 rows}$$

Final expression of T

$$\boxed{T = \frac{1}{2}\sum_{i=1}^{N}\left(m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci}\,\omega_i\right)}$$

NOTE 1:
in practice, NEVER
use this formula
(or partial Jacobians)
for computing T
$\Longrightarrow$ a better method
is available...

$$= \frac{1}{2}\,\dot{q}^T\left(\sum_{i=1}^{N} m_i\, J_{Li}^T(q) J_{Li}(q) + J_{Ai}^T(q) I_{ci}(q)\, J_{Ai}(q)\right)\dot{q}$$

constant when ω_i
is expressed in RF_{ci}
else

$$\boxed{{}^0I_{ci}(q) = {}^0R_i(q)\ {}^iI_{ci}\ {}^0R_i^T(q)}$$

$$\boxed{T = \frac{1}{2}\dot{q}^T M(q)\dot{q}}$$

NOTE 2:
I used previously
the notation $B(q)$
for the robot
inertia matrix ...
(see past exams!)

robot (generalized) inertia matrix

- symmetric
- positive definite, $\forall q \Rightarrow$ **always invertible**

Robot potential energy

assumption: GRAVITY contribution only

$$U = \sum_{i=1}^{N} U_i$$ ⇐ N rigid bodies (+ fixed base)

$$U_i = U_i(\underbrace{q_j; j \le i})$$ ⇐ open kinematic chain

$$\boxed{U_i = -m_i g^T r_{0,ci}}$$

{ gravity acceleration vector — position of the center of mass of link i } typically expressed in RF_0

dependence on q

$$\begin{pmatrix} r_{0,ci} \\ 1 \end{pmatrix} = {}^0A_1(q_1)\, {}^1A_2(q_2) \cdots {}^{i-1}A_i(q_i) \begin{pmatrix} r_{i,ci} \\ 1 \end{pmatrix}$$

constant in RF_i

NOTE: need to work with homogeneous coordinates

Summarizing ...

kinetic energy
$$T = \frac{1}{2}\dot{q}^T M(q)\dot{q} = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j$$

positive definite quadratic form

$$\boxed{\begin{array}{c} T \geq 0, \\ T = 0 \Leftrightarrow \dot{q} = 0 \end{array}}$$

potential energy
$$U = U(q)$$

Lagrangian
$$L = T(q,\dot{q}) - U(q)$$

Euler-Lagrange equations
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k} \qquad k = 1, \dots, N$$

non-conservative (active/dissipative) generalized forces

performing work on q_k coordinate

Applying Euler-Lagrange equations
(the scalar derivation – see Appendix for vector format)

$$L(q,\dot{q}) = \frac{1}{2}\sum_{i,j} m_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\dot{q}_j \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial m_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

(dependences of elements on q are not shown)

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial m_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial U}{\partial q_k}$$

LINEAR terms in ACCELERATION $\ddot{q}$

QUADRATIC terms in VELOCITY $\dot{q}$

NONLINEAR terms in CONFIGURATION q

k-th dynamic equation ...

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k$$

$$\sum_j m_{kj}\ddot{q}_j + \sum_{i,j}\left(\frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i\dot{q}_j + \frac{\partial U}{\partial q_k} = u_k$$

exchanging
"mute" indices i, j

$$\cdots + \sum_{i,j} \underbrace{\frac{1}{2}\left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)}_{c_{kij} = c_{kji}}\dot{q}_i\dot{q}_j + \cdots$$

Christoffel symbols
of the first kind

… and interpretation of dynamic terms

$$\boxed{\sum_j m_{kj}(q)\ddot{q}_j} + \boxed{\sum_{i,j} c_{kij}(q)\,\dot{q}_i\dot{q}_j} + \boxed{\frac{\partial U}{\partial q_k}} = u_k \qquad k = 1,\dots,N$$

INERTIAL terms

CENTRIFUGAL ($i = j$) and **CORIOLIS** ($i \neq j$) terms

GRAVITY terms $g_k(q)$

$m_{kk}(q)$ = inertia at joint k when joint k accelerates ($m_{kk} > 0$!!)

$m_{kj}(q)$ = inertia "seen" at joint k when joint j accelerates ($= m_{jk}(q)$)

$c_{kii}(q)$ = coefficient of the centrifugal force at joint k when joint i is moving ($c_{iii} = 0, \forall i$)

$c_{kij}(q)$ = coefficient of the Coriolis force at joint k when joint i and joint j are both moving ($= c_{kji}\,(q)$)

Robot dynamic model
in vector formats

1. $$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

extracted from T

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T$$

k-th component of vector c

$$c_k(q,\dot{q}) = \dot{q}^T C_k(q)\dot{q}$$

k-th column of matrix $M(q)$

$$C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$$

symmetric matrix!

2. $$M(q)\ddot{q} + S(q,\dot{q})\dot{q} + g(q) = u$$

NOTE:
the model is in the form

$\Phi(q,\dot{q},\ddot{q}) = u$

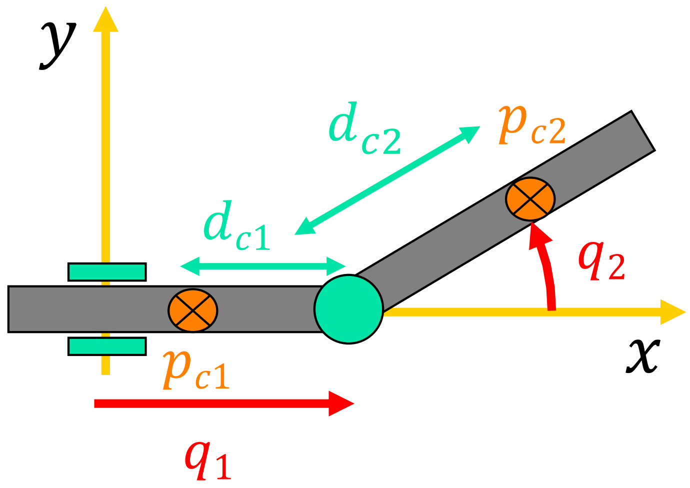
as expected

NOT a symmetric matrix in general

$$s_{kj}(q,\dot{q}) = \sum_i c_{kij}(q)\dot{q}_i$$

factorization of c by S is **not unique!**

Dynamic model of a PR robot



$$T = T_1 + T_2 \qquad U = \text{constant} \Rightarrow g(q) \equiv 0$$

(on horizontal plane)

$$p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \|v_{c1}\|^2 = \dot{p}_{c1}^T \dot{p}_{c1} = \dot{q}_1^2$$

$$\boxed{T_1 = \frac{1}{2} m_1 \dot{q}_1^2}$$

$$T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2$$

$$p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \Rightarrow v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \, \dot{q}_2 \\ d_{c2} \cos q_2 \, \dot{q}_2 \\ 0 \end{pmatrix} \qquad \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix}$$

$$\boxed{T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2 d_{c2} \sin q_2 \, \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2}$$

Dynamic model of a PR robot (cont)

$$M(q) = \begin{pmatrix} \boxed{\begin{matrix} m_1 + m_2 \\ -m_2 d_{c2} \sin q_2 \end{matrix}} & \boxed{\begin{matrix} -m_2 d_{c2} \sin q_2 \\ I_{c2,zz} + m_2 d_{c2}^2 \end{matrix}} \end{pmatrix}$$

M_1 M_2

$$c(q, \dot{q}) = \begin{pmatrix} c_1(q, \dot{q}) \\ c_2(q, \dot{q}) \end{pmatrix}$$

$$c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q}$$

where $C_k(q) = \frac{1}{2}\left(\frac{\partial M_k}{\partial q} + \left(\frac{\partial M_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right)$

$$C_1(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} \cos q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

$$c_1(q, \dot{q}) = -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2$$

$$C_2(q) = \frac{1}{2}\left(\begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -m_2 d_{c2} \cos q_2 \\ -m_2 d_{c2} \cos q_2 & 0 \end{pmatrix}\right) = 0$$

$$c_2(q, \dot{q}) = 0$$

Dynamic model of a PR robot (cont)

$$\boxed{M(q)\ddot{q} + c(q,\dot{q}) = u}$$

$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\ -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

NOTE: the m_{NN} element (here, for $N = 2$) of $M(q)$ is always constant!

Q1: why is variable q_1 not appearing in $M(q)$? … this is a general property!

Q2: why are Coriolis terms not present?

Q3: when applying a force u_1, does the second joint accelerate? … always?

Q4: what is the expression of a factorization matrix S? … is it unique here?

Q5: what if the PR robot was moving in a vertical plane? … just add $g(q)$!

A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix S)

Proof

$$\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i}\dot{q}_i \qquad 2\boldsymbol{s_{kj}} = \sum_{\boldsymbol{i}} 2\boldsymbol{c_{kij}}\,\boldsymbol{\dot{q}_i} = \sum_i \left(\frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k}\right)\dot{q}_i$$

$$\Rightarrow \quad \dot{m}_{kj} - 2s_{kj} = \sum_i \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j}\right)\dot{q}_i = n_{kj}$$

$$n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left(\frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k}\right)\dot{q}_i = -n_{kj}$$

using the symmetry of M

$$\Rightarrow \quad \boxed{x^T\left(\dot{M} - 2S\right)x = 0, \forall x}$$

Energy conservation

- total robot energy

$$E = T + U = \frac{1}{2}\dot{q}^T M(q)\dot{q} + U(q)$$

- its evolution over time (using the dynamic model)

$$\dot{E} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial U}{\partial q}\dot{q}$$

$$= \dot{q}^T (u - S(q,\dot{q})\dot{q} - g(q)) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{q}^T g(q)$$

$$= \dot{q}^T u + \frac{1}{2}\dot{q}^T \left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q}$$

here, any factorization of vector c by a matrix S can be used

- if $u \equiv 0$, total energy is constant (no dissipation or increase)

$$\dot{E} = 0 \Rightarrow \boxed{\dot{q}^T \left(\dot{M}(q) - 2S(q,\dot{q})\right)\dot{q} = 0, \forall q, \dot{q}} \Rightarrow \boxed{\dot{E} = \dot{q}^T u}$$

it is a weaker property than skew-symmetry, as the external velocity $\dot{q}$ in the quadratic form is the **same** inside the two matrices $\dot{M}$ and S

in general, the variation of the total energy is equal to the work of non-conservative forces

Appendix
dynamic model: alternative vector format derivation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T - \left(\frac{\partial L}{\partial q}\right)^T = u} \qquad L = \frac{1}{2}\dot q^T M(q)\dot q - U(q)$$

$$M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q)\end{pmatrix} = \sum_{i=1}^{N} M_i(q) e_i^T \qquad e_i^T = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0\end{pmatrix}$$

i-th position

dyadic expansion

$$\left(\frac{\partial L}{\partial \dot q}\right)^T = \left(\dot q^T M(q)\right)^T = M(q)\dot q$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)^T = M(q)\ddot q + \dot M(q)\dot q = M(q)\ddot q + \sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q}\right)\dot q_i \dot q$$

$$\left(\frac{\partial L}{\partial q}\right)^T = \left(\frac{1}{2}\dot q^T\left(\sum_{i=1}^{N}\frac{\partial M_i(q)}{\partial q}e_i^T\right)\dot q - \frac{\partial U(q)}{\partial q}\right)^T = \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q}\right)^T \dot q_i\, \dot q - \left(\frac{\partial U}{\partial q}\right)^T$$

special definition of S matrix giving skew-symmetry to $\dot M - 2S$

$$\Rightarrow M(q)\ddot q + \underbrace{\left(\sum_{i=1}^{N}\left(\frac{\partial M_i}{\partial q} - \frac{1}{2}\left(\frac{\partial M_i}{\partial q}\right)^T\right)\dot q_i\right)\dot q}_{S(q,\dot q)\ \text{(generic)}} + \underbrace{\left(\frac{\partial U}{\partial q}\right)^T}_{g(q)} = u$$

k-th row of matrix S

$$\boxed{S_k^T(q,\dot q) = \dot q^T C_k(q)} \longleftrightarrow S(q,\dot q)\ \text{(generic)}$$