Robotics 2

Dynamic model of robots:
Lagrangian approach

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Dynamic model

- provides the relation between
  
generalized forces $u(t)$ acting on the robot
  
robot motion, i.e.,
  assumed configurations $q(t)$ over time

\[ \Phi(q, \dot{q}, \ddot{q}) = u \]

a system of 2\textsuperscript{nd} order differential equations

\[ u_{\text{joints}}(t) \quad u_{\text{Cartesian}}(t) \quad q(t) \]
Direct dynamics

- **direct relation**

\[ u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \rightarrow \text{robot arm} \rightarrow \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = q(t) \]

**input** for \( t \in [0, T] \) and **initial state** at \( t = 0 \)

- **experimental solution**
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time \( T_c \))

- **solution by simulation**
  - use dynamic model and **integrate** numerically the differential equations (with simulation step \( T_s \leq T_c \))

\[ \Phi(q, \dot{q}, \ddot{q}) = u \]
Inverse dynamics

- inverse relation

\[ q_d(t), \dot{q}_d(t), \ddot{q}_d(t) \Rightarrow u_d(t) \]

desired motion for \( t \in [0,T] \)
required input for \( t \in [0,T] \)

- experimental solution
  - repeated motion trials of direct dynamics using \( u_k(t) \), with iterative learning of nominal torques updated on trial \( k + 1 \) based on the error in \([0,T]\) measured in trial \( k \): \( u_k(t) \Rightarrow u_d(t) \)

- analytic solution
  - use dynamic model and compute algebraically the values \( u_d(t) \) at every time instant \( t \)

Robotics 2
Approaches to dynamic modeling

Euler-Lagrange method
(energy-based approach)
- dynamic equations in symbolic/closed form
- best for study of dynamic properties and analysis of control schemes

Newton-Euler method
(balance of forces/torques)
- dynamic equations in numeric/recursive form
- best for implementation of control schemes (inverse dynamics in real time)

- many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:
  - principle of d’Alembert, of Hamilton, of virtual works, ...
Euler-Lagrange method (energy-based approach)

basic assumption: the \( N \) links in motion are considered as **rigid bodies** (+ possibly, **concentrated elasticity** at the joints)

\[ q \in \mathbb{R}^N \]  generalized coordinates (e.g., joint variables, but not only!)

\[
L(q, \dot{q}) = T(q, \dot{q}) - U(q)
\]

Lagrangian  kinetic energy – potential energy

- least action principle of Hamilton
- virtual works principle

Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \quad i = 1, \ldots, N
\]

non-conservative (external or dissipative)  generalized forces performing work on \( q_i \)
Dynamics of actuated pendulum
a first example

\[ \dot{\theta}_m = n \dot{\theta} \quad \Rightarrow \quad \theta_m = n\theta + \theta_{m0} \]
\[ \tau = n \tau_m \]
\[ q = \theta \quad \text{(or} \quad q = \theta_m) \]
\[ T = T_m + T_l \]

\[ T_m = \frac{1}{2} I_m \dot{\theta}_m^2 \]
\[ T_l = \frac{1}{2} (I_l + md^2) \dot{\theta}^2 \]

Kinetic energy

\[ T = \frac{1}{2} \left( I_l + md^2 + n^2 I_m \right) \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2 \]
Dynamics of actuated pendulum (cont)

\[ U = U_0 - m g_0 d \cos \theta \]

potential energy

\[ L = T - U = \frac{1}{2} I \dot{\theta}^2 + m g_0 d \cos \theta - U_0 \]

\[ \frac{\partial L}{\partial \ddot{\theta}} = I \dot{\theta} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = I \ddot{\theta} \]

\[ \frac{\partial L}{\partial \theta} = -m g_0 d \sin \theta \]

\[ \hat{\dot{p}}_x = l \cos \theta \cdot \dot{\theta} = J_x \dot{\theta} \]

\[ u = n \tau_m - b_l \dot{\theta} - n b_m \dot{\theta}_m + J_x^T F_x = n \tau_m - (b_l + n^2 b_m) \dot{\theta} + l \cos \theta F_x \]

applied or dissipated torques on motor side are multiplied by \( n \) when moved to the link side

equivalent joint torque due to force \( F_x \) applied to the tip at point \( p_x \)

"sum" of non-conservative torques
Dynamics of actuated pendulum (cont)

Dynamic model in $q = \theta$

\[ I \ddot{\theta} + mg_0 d \sin \theta = n\tau_m - (b_l + n^2 b_m)\dot{\theta} + l \cos \theta \cdot F_x \]

Dividing by $n$ and substituting $\theta = \theta_m/n$

\[ \frac{I}{n^2} \ddot{\theta}_m + \frac{m}{n} g_0 d \sin \frac{\theta_m}{n} = \tau_m - \left( \frac{b_l}{n^2} + b_m \right) \dot{\theta}_m + \frac{l}{n} \cos \frac{\theta_m}{n} \cdot F_x \]

Dynamic model in $q = \theta_m$
Kinetic energy of a rigid body

When all vectors are referred to a body frame RF\textsubscript{c} attached to the CoM, then
\[ r_c = 0 \quad \Rightarrow \quad \int_B r \, dm = 0 \]

Mass
\[ m = \int_B \rho(x, y, z) \, dx \, dy \, dz = \int_B dm \]

Mass density
\[ \rho(x, y, z) \]

Position of center of mass (CoM)
\[ r_c = \frac{1}{m} \int_B r \, dm \]

Kinetic energy
\[ T = \frac{1}{2} \int_B v^T(x, y, z) \, v(x, y, z) \, dm \]

Kinematic relation for a rigid body
\[ v = v_c + \omega \times r = v_c + S(\omega) \, r \]

Skew-symmetric matrix
Kinetic energy of a rigid body (cont)

\[ T = \frac{1}{2} \int_B (v_c + S(\omega)r)^T (v_c + S(\omega)r) \, dm \]

\[ = \frac{1}{2} \int_B v_c^T v_c \, dm + \int_B v_c^T S(\omega) \, r dm + \frac{1}{2} \int_B r^T S^T(\omega) S(\omega) r \, dm \]

\[ = \frac{1}{2} \begin{bmatrix} m v_c^T v_c \end{bmatrix} \]

translational kinetic energy
(point mass in CoM)

\[ = \frac{1}{2} m v_c^T S(\omega) \int_B r dm = 0 \]

rotational kinetic energy
(of the whole body)

\[ = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \omega^T I_c \omega \end{bmatrix} \]

body inertia matrix
(around the CoM)

König theorem

\[ = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \omega^T I_c \omega \end{bmatrix} \]

Euler matrix

Ex #1: provide the expressions of the elements of Euler matrix \( J_c \)

Ex #2: prove last equality and provide the expressions of the elements of inertia matrix \( I_c \)

\[ a^T b = \text{trace}\{ab^T\} \]

sum of elements on the diagonal of a matrix
Examples of body inertia matrices
homogeneous bodies of mass $m$, with axes of symmetry

**parallelepiped with sides**
a (length/height), $b, c$ (base)

$$I_c = \begin{pmatrix} I_{xx} & I_{yy} & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} m(b^2 + c^2) \\ \frac{1}{12} m(a^2 + c^2) \\ \frac{1}{12} m(a^2 + b^2) \end{pmatrix}$$

**empty cylinder with length $h$, and external/internal radius $a, b$**

$$I_c = \begin{pmatrix} \frac{1}{2} m(a^2 + b^2) \\ \frac{1}{12} m(3(a^2 + b^2)^2 + h^2) \\ I_{zz} \end{pmatrix} \quad I_{zz} = I_{yy}$$

$\quad l_{zz}' = l_{zz} + m \left( \frac{h}{2} \right)^2$ (parallel) axis translation theorem

Steiner theorem

$$I = I_c + m \left( r^T r \cdot E_{3 \times 3} - rr^T \right) = I_c + m S^T(r)S(r)$$

**Ex #3**: prove the last equality

body inertia matrix relative to the CoM

identity matrix
Robot kinetic energy

\[ T = \sum_{i=1}^{N} T_i \quad \text{\( N \) rigid bodies (+ fixed base)} \]

\[ T_i = T_i(q_j, \dot{q}_j; j \leq i) \quad \text{open kinematic chain} \]

König theorem

\[ T_i = \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]

absolute velocity of the center of mass (CoM)

absolute angular velocity of whole body

i-th link (body) of the robot
Kinetic energy of a robot link

\[ T_i = \frac{1}{2} m_i \nu_{ci}^T \nu_{ci} + \frac{1}{2} \omega_i^T I_{ci} \omega_i \]

\( \omega_i, I_{ci} \) should be expressed in the same reference frame, but the product \( \omega_i^T I_{ci} \omega_i \) is invariant w.r.t. any chosen frame

In frame \( RF_{ci} \) attached to (the center of mass of) link \( i \)

\[
{i} I_{ci} = \begin{pmatrix}
\int (y^2 + z^2) \, dm & - \int xy \, dm & - \int xz \, dm \\
\int (x^2 + z^2) \, dm & \int (x^2 + y^2) \, dm & - \int yz \, dm \\
\text{symm} & \text{symm} & \int (x^2 + y^2) \, dm
\end{pmatrix}
\]

constant!
Dependence of $T$ from $q$ and $\dot{q}$

\[ v_{ci} = J_{Li}(q) \dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \dot{q} \} \text{ 3 rows} \]

\[ \omega_{i} = J_{Ai}(q) \dot{q} = \begin{pmatrix} 1 & \vdots & i & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \dot{q} \} \text{ 3 rows} \]

(partial) Jacobians typically expressed in $RF_0$
Final expression of $T$

$$T = \frac{1}{2} \sum_{i=1}^{N} \left( m_i v_{ci}^T v_{ci} + \omega_i^T I_{ci} \omega_i \right)$$

$$= \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} m_i J_{Li}(q) J_{Li}(q) + J_{Ai}(q) I_{ci} J_{Ai}(q) \right) \dot{q}$$

**NOTE 1:**
in practice, NEVER use this formula (or partial Jacobians) for computing $T$; a better method is available...

**NOTE 2:**
in the past, I have always used the notation $B(q)$ for the robot inertia matrix...

robot (generalized) inertia matrix
- symmetric
- positive definite, $\forall q \Rightarrow$ always invertible

constant if $\omega_i$ is expressed in $RF_{ci}$

else

$$0 I_{ci}(q) = 0 R_i(q) i_{ci} 0 R_i^T(q)$$
Robot potential energy

assumption: GRAVITY contribution only

\[ U = \sum_{i=1}^{N} U_i \quad \text{N rigid bodies (+ fixed base)} \]

\[ U_i = U_i(q_j; j \leq i) \quad \text{open kinematic chain} \]

\[ U_i = -m_i g^T r_{0,ci} \]

\[ \begin{bmatrix} r_{0,ci} \\ 1 \end{bmatrix} = \begin{bmatrix} A_1(q_1) & A_2(q_2) & \cdots & A_{i-1}(q_{i-1}) \end{bmatrix} \begin{bmatrix} r_{i,ci} \\ 1 \end{bmatrix} \]

NOTE: need to work with homogeneous coordinates

typically expressed in RF_0

constant in RF_i
Summarizing ...

kinetic energy

\[ T = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j \]

potential energy

\[ U = U(q) \]

Lagrangian

\[ L = T(q, \dot{q}) - U(q) \]

Euler-Lagrange equations

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k \]

\[ k = 1, \ldots, N \]

positive definite quadratic form

\[ T \geq 0, \quad T = 0 \iff \dot{q} = 0 \]

non-conservative (active/dissipative) generalized forces performing work on \( q_k \) coordinate
Applying Euler-Lagrange equations
(the scalar derivation; see Appendix for vector format)

\[ L(q, \dot{q}) = \frac{1}{2} \sum_{i,j} m_{ij}(q) \dot{q}_i \dot{q}_j - U(q) \]

\[ \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \dot{q}_j \rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j \]

(dependences of elements on \( q \) are not shown)

\[ \frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial m_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial U}{\partial q_k} \]

LINEAR terms in ACCELERATION \( \ddot{q} \)

QUADRATIC terms in VELOCITY \( \dot{q} \)

NONLINEAR terms in CONFIGURATION \( q \)
$k$-th dynamic equation ...

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = u_k
\]

\[
\sum_j m_{kj} \ddot{q}_j + \sum_{i,j} \left( \frac{\partial m_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k
\]

exchanging “mute” indices $i, j$

\[
\cdots + \sum_{i,j} \frac{1}{2} \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i \dot{q}_j + \cdots
\]

$c_{kij} = c_{kji}$ Christoffel symbols of the first kind
... and interpretation of dynamic terms

\[ \sum_j m_{kj}(q) \ddot{q}_j + \sum_{i,j} c_{kij}(q) \dot{q}_i \dot{q}_j + \frac{\partial U}{\partial q_k} = u_k \quad k = 1, \ldots, N \]

**INERTIAL** terms

\( m_{kk}(q) = \text{inertia at joint } k \text{ when joint } k \text{ accelerates} \ (m_{kk} > 0!!) \)

\( m_{kj}(q) = \text{inertia “seen” at joint } k \text{ when joint } j \text{ accelerates} \)

**CENTRIFUGAL** \((i = j)\) terms

\( c_{kii}(q) = \text{coefficient of the centrifugal force at joint } k \text{ when joint } i \text{ is moving} \ (c_{iii} = 0, \forall i) \)

\( c_{kij}(q) = \text{coefficient of the Coriolis force at joint } k \text{ when joint } i \text{ and joint } j \text{ are both moving} \)

**CORIOLIS** \((i \neq j)\) terms

\( g_k(q) = \text{gravity terms} \)

Robotics 2
Robot dynamic model
in vector formats

1. \[ M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u \]

\[ c_k(q, \dot{q}) = \dot{q}^T C_k(q) \dot{q} \]
\[ C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + \left( \frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right) \]

\[ k \text{-th component of vector } c \]

\[ k \text{-th column of matrix } M(q) \]

2. \[ M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) = u \]
\[ s_{kj}(q, \dot{q}) = \sum_i c_{kij}(q)q_i \]

\[ \text{NOT a symmetric matrix in general} \]

\[ \text{factorization of } c \text{ by } S \text{ is not unique!} \]

NOTE: the model is in the form \( \phi(q, \dot{q}, \ddot{q}) = u \) as expected
Dynamic model of a PR robot

\[ T = T_1 + T_2 \]
\[ U = \text{constant} \Rightarrow g(q) \equiv 0 \]
(on horizontal plane)

\[ p_{c1} = \begin{pmatrix} q_1 - d_{c1} \\ 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad ||v_{c1}||^2 = p_{c1}^T \dot{p}_{c1} = \dot{q}_1^2 \]

\[ T_1 = \frac{1}{2} m_1 \dot{q}_1^2 \]

\[ T_2 = \frac{1}{2} m_2 v_{c2}^T v_{c2} + \frac{1}{2} \omega_2^T I_{c2} \omega_2 \]

\[ p_{c2} = \begin{pmatrix} q_1 + d_{c2} \cos q_2 \\ d_{c2} \sin q_2 \\ 0 \end{pmatrix} \quad \rightarrow \quad v_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} \sin q_2 \dot{q}_2 \\ d_{c2} \cos q_2 \dot{q}_2 \\ 0 \end{pmatrix} \]

\[ \omega_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_2 \end{pmatrix} \]

\[ T_2 = \frac{1}{2} m_2 (\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2d_{c2} \sin q_2 \dot{q}_1 \dot{q}_2) + \frac{1}{2} I_{c2,zz} \dot{q}_2^2 \]
Dynamic model of a PR robot (cont)

\[ M(q) = \begin{bmatrix}
  m_1 + m_2 & -m_2 d_{c2} \sin q_2 \\
  -m_2 d_{c2} \sin q_2 & I_{c2,zz} + m_2 d_{c2}^2 
\end{bmatrix} \quad c(q, \dot{q}) = \begin{bmatrix}
  c_1(q, \dot{q}) \\
  c_2(q, \dot{q})
\end{bmatrix} \]

where \[ C_k(q) = \frac{1}{2} \left( \frac{\partial M_k}{\partial q} + \left( \frac{\partial M_k}{\partial q} \right)^T - \frac{\partial M}{\partial q_k} \right) \]

\[ C_1(q) = \frac{1}{2} \left( \begin{bmatrix}
  0 & 0 \\
  0 & -m_2 d_{c2} \cos q_2
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  0 & -m_2 d_{c2} \cos q_2
\end{bmatrix} - \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix} \right) \]

\[ C_2(q) = \frac{1}{2} \left( \begin{bmatrix}
  0 & -m_2 d_{c2} \cos q_2 \\
  0 & 0
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  -m_2 d_{c2} \cos q_2 & 0
\end{bmatrix} \right) = 0 \]

\[ c_1(q, \dot{q}) = -m_2 d_{c2} \cos q_2 \dot{q}_2^2 \]

\[ c_2(q, \dot{q}) = 0 \]
Dynamic model of a PR robot (cont)

\[ M(q)\ddot{q} + c(q, \dot{q}) = u \]

\[
\begin{pmatrix}
  m_1 + m_2 & -m_2 d_{c_2} \sin q_2 \\
  -m_2 d_{c_2} \sin q_2 & I_{c_2,zz} + m_2 d_{c_2}^2
\end{pmatrix}
\begin{pmatrix}
  \ddot{q}_1 \\
  \ddot{q}_2
\end{pmatrix}
+
\begin{pmatrix}
  -m_2 d_{c_2} \cos q_2 \dot{q}_2^2 \\
  0
\end{pmatrix}
=
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}
\]

**NOTE:** the \( m_{NN} \) element (here, for \( N = 2 \)) of the robot inertia matrix is always **constant**!

Q1: why Coriolis terms are not present?
Q2: when applying a force \( u_1 \), does the second joint accelerate? ... always?
Q3: what is the expression of a factorization matrix \( S \)? ... is it unique?
Q4: which is the configuration with “maximum inertia”?
A structural property

Matrix $\dot{M} - 2S$ is skew-symmetric
(when using Christoffel symbols to define matrix $S$)

Proof

\[
\dot{m}_{kj} = \sum_i \frac{\partial m_{kj}}{\partial q_i} \dot{q}_i \quad 2s_{kj} = \sum_i 2c_{kij} \dot{q}_i = \sum_i \left( \frac{\partial m_{kj}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} - \frac{\partial m_{ij}}{\partial q_k} \right) \dot{q}_i
\]

\[
\dot{m}_{kj} - 2s_{kj} = \sum_i \left( \frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{ki}}{\partial q_j} \right) \dot{q}_i = n_{kj}
\]

\[
n_{jk} = \dot{m}_{jk} - 2s_{jk} = \sum_i \left( \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{ji}}{\partial q_k} \right) \dot{q}_i = -n_{kj}
\]

using the symmetry of $M$

\[
x^T(\dot{M} - 2S)x = 0, \forall x
\]
Energy conservation

- total robot energy
  \[ E = T + U = \frac{1}{2} \dot{q}^T M(q) \dot{q} + U(q) \]

- its evolution over time (using the dynamic model)
  \[
  \dot{E} = \dot{q}^T M(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \frac{\partial U}{\partial q} \dot{q}
  \]
  \[
  = \dot{q}^T (u - S(q, \dot{q}) \dot{q} - g(q)) + \frac{1}{2} \dot{q}^T \dot{M}(q) \dot{q} + \dot{q}^T g(q)
  \]
  \[
  = \dot{q}^T u + \frac{1}{2} \dot{q}^T \left( \dot{M}(q) - 2S(q, \dot{q}) \right) \dot{q}
  \]

- if \( u \equiv 0 \), total energy is constant (no dissipation or increase)
  \[ \dot{E} = 0 \]

  weaker than skew-symmetry, as the external velocity is the same that appears in the internal matrices

  \[ \dot{q}^T \left( \dot{M}(q) - 2S(q, \dot{q}) \right) \dot{q} = 0, \forall q, \dot{q} \]

  in general, the variation of the total energy is equal to the work of non-conservative forces

  \[ \dot{E} = \dot{q}^T u \]
Appendix:
Vector format derivation of dynamic model

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T - \left( \frac{\partial L}{\partial q} \right)^T = u
\]

\[
M(q) = \begin{pmatrix} M_1(q) & \cdots & M_i(q) & \cdots & M_N(q) \end{pmatrix} = \sum_{i=1}^{N} M_i(q)e_i^T
\]

\[
\left( \frac{\partial L}{\partial \dot{q}} \right)^T = (\dot{q}^T M(q))^T = M(q)\dot{q}
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right)^T = M(q)\ddot{q} + \dot{M}(q)\dot{q} = M(q)\ddot{q} + \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} \right) \dot{q} \dot{q}_i
\]

\[
\left( \frac{\partial L}{\partial q} \right)^T = \left( \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} \frac{\partial M_i}{\partial q} e_i^T \right) \dot{q} - \frac{\partial U}{\partial q} \right)^T = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} \right)^T \dot{q}_i \dot{q} - \left( \frac{\partial U}{\partial q} \right)^T
\]

this construction gives to \( \dot{M} - 2S \) skew-symmetry

\[
M(q)\ddot{q} + \left( \sum_{i=1}^{N} \left( \frac{\partial M_i}{\partial q} - \frac{1}{2} \left( \frac{\partial M_i}{\partial q} \right)^T \right) \dot{q}_i \right) \dot{q} + \left( \frac{\partial U}{\partial q} \right)^T = u
\]

k-th row of matrix \( S \)

\[
S_k^T(q, \dot{q}) = \dot{q}^T C_k(q) \rightarrow S(q, \dot{q})
\]

Robotics 2