Robotics 1

Midterm Test – November 19, 2021

The test has 9 questions. Provide as many answers as you can, with short but significant texts and formulas/tables/pictures. Please write clearly. Take a picture of each of your handwritten answers and upload them to Exam.net *before submitting. Try to follow the same order of the questions. Number your answers accordingly (don't repeat the text of the questions).*

Question #1

A rigid body is rotated first by an angle $\theta = \pi/3$ around the unit vector $\boldsymbol{r} = (1/\sqrt{3}) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and then by an angle $\phi = -\pi/3$ around the fixed $\boldsymbol{y}$-axis. What is the final orientation of the body?

Question #2

An initial orientation $\boldsymbol{R}_i$ and a final orientation $\boldsymbol{R}_f$ are defined by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \boldsymbol{I}.$$

Find the two sequences of ZYZ Euler angles that represent the rotation from $\boldsymbol{R}_i$ to $\boldsymbol{R}_f$.

Question #3

For the 4R robot with a spherical shoulder of Fig. 1, complete the assignment of Denavit-Hartenberg (D-H) frames and fill in the associated table of parameters *[for this, use the extra sheet distributed]*. Keep the quantities that are already defined in the figure unchanged. If needed, provide the transformation between the last D-H frame and the standard frame of an end-effector gripper.

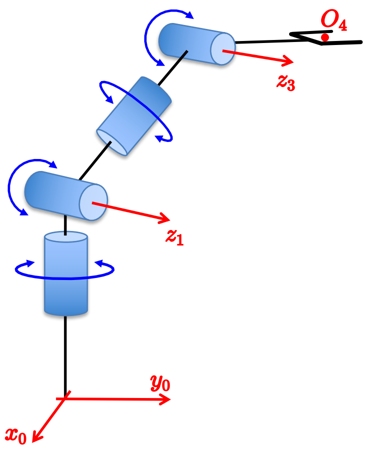

Figure 1: A 4R spatial robot arm with a spherical shoulder.

Question #4

A 2R planar robot with links of equal length L has limited joint ranges as follows: $q_1 \in [-\pi/2, \pi/2]$, $q_2 \in [0, \pi/2]$. Draw the primary workspace $WS_1 \in \mathbb{R}^2$. For $L = 1.4$ [m], is the point $P = (1.6, -0.2)$ reachable by the robot end effector?

Question #5

A branched two-arm planar robot having 5 dofs is sketched in Fig. 2, with generic labels for the link lengths and the actual definition of the joint angles. The sign convention for angles is the usual one (i.e., positive if counterclockwise). Determine the relative pose of the end-effector frame of the left arm with respect to that of the right arm, as expressed by the 4×4 homogeneous matrix ${}^{rE}\boldsymbol{T}_{lE}(\boldsymbol{q})$ with $\boldsymbol{q} = (\theta_0, \theta_{r1}, \theta_{r2}, \theta_{l1}, \theta_{l2})$. Check numerically the obtained symbolic expression when all the links have equal and unitary length and the two-arm robot is in the configuration $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$ —the right arm is horizontal and the left one is vertical and upward.

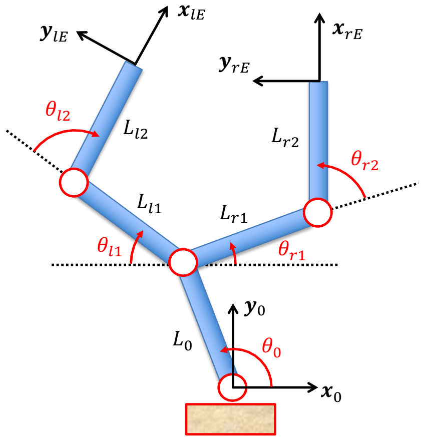

Figure 2: A two-arm robot with 5 dofs.

Question #6

Figure 3 shows a planar RRP robot, with the definition of its joint variables. The task of interest is specified by the position $\boldsymbol{p} = (p_x, p_y)$ of the robot end effector and by the orientation α of the forearm w.r.t. the $\boldsymbol{x}$-axis. The associated direct kinematics is

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}).$$

Determine the analytic solutions to the inverse kinematics problem. Disregard any situation that is unfeasible or singular. Provide at least one solution for the following (feasible) input data: $l_1 = 1$ [m], $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad].

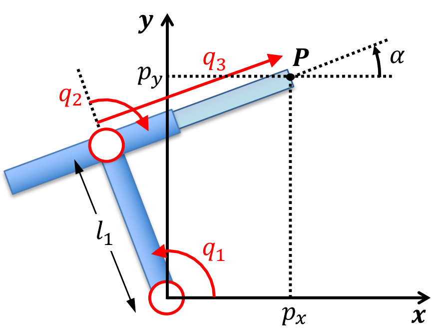

Figure 3: A planar RRP robot.

Question #7

With reference to Fig. 4, a motor with inertia J_M drives a link through a gear with toothed wheels (a photo of this is also shown in the figure). The wheel on the motor shaft (aka, the *pinion*) has radius $r_M = 2$ [cm], while the radius of the wheel on the link rotation axis is $r_L = 10$ [cm]. The link has inertia $J_L = 0.3$ [kgm^2] around its rotation axis. Assuming that an optimal inertia matching is realized by the reduction ratio of this transmission, determine the torque τ_M that the motor needs to produce around its $\boldsymbol{z}_M$ axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2]. Neglect dissipative effects as well as the inertia of the transmission components (and of the encoder).

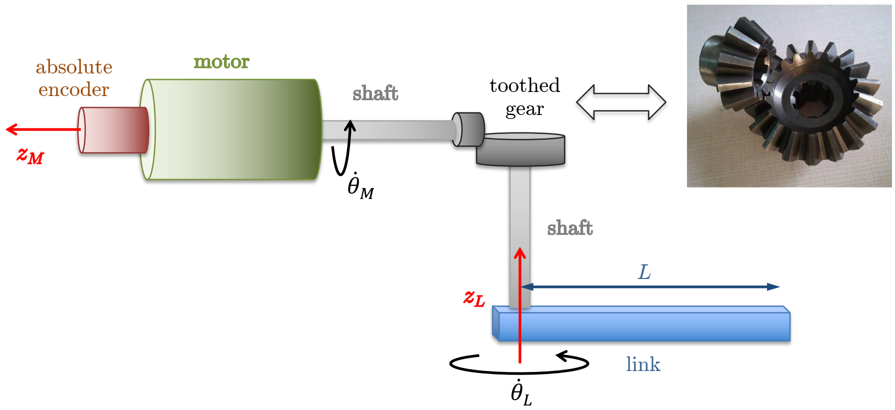

Figure 4: Set up of a motor-transmission-link system using a toothed gear.

Question #8

An absolute encoder is mounted on the motor of the system shown in Fig. 4. If the link length is $L = 0.5$ [m], determine the minimum number of tracks n_t that the encoder needs to have in order to achieve at least a resolution of $\delta = 0.1$ [mm] at the link tip.

Question #9

Explain in exactly three short sentences the specific feature of a SCARA-type robot, its common technical implementation, and the significance in industrial applications.

[150 minutes (2.5 hours); open books]

Solution

November 19, 2021

Question #1

A rigid body is rotated first by an angle $\theta = \pi/3$ around the unit vector $\boldsymbol{r} = (1/\sqrt{3}) \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ and then by an angle $\phi = -\pi/3$ around the fixed $\boldsymbol{y}$-axis. What is the final orientation of the body?

Reply #1

The rotation matrix associated to an axis/angle representation $(\boldsymbol{r}, \theta)$ is

$$\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{r}^T\boldsymbol{r} + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right)\cos\theta + \boldsymbol{S}(\boldsymbol{r})\sin\theta,$$

while the elementary rotation by an angle ϕ around the coordinate axis Y is represented by

$$\boldsymbol{R}_Y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}.$$

Being the sequence of two rotations defined around the fixed axes $\boldsymbol{r}$ and $\boldsymbol{y}$, the final orientation is given by product (in the reverse order)

$$\boldsymbol{R}_{\boldsymbol{r},\boldsymbol{y}} = \boldsymbol{R}_Y\left(-\frac{\pi}{3}\right)\boldsymbol{R}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{\pi}{3}\right) = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0.6220 & -0.7440 & -0.2440 \\ 0.6667 & 0.6667 & -0.3333 \\ 0.4107 & 0.0447 & 0.9107 \end{pmatrix}. \qquad \blacksquare$$

Question #2

An initial orientation $\boldsymbol{R}_i$ and a final orientation $\boldsymbol{R}_f$ are defined by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \boldsymbol{I}.$$

Find the two sequences of ZYZ Euler angles that represent the rotation from $\boldsymbol{R}_i$ to $\boldsymbol{R}_f$.

Reply #2

One has to solve the inverse problem of the ZYZ Euler representation with angles $(\alpha_1, \alpha_2, \alpha_3)$ for the relative rotation matrix

$${}^i\boldsymbol{R}_f = \boldsymbol{R}_i^T\boldsymbol{R}_f = \begin{pmatrix} 0 & 0.5 & -\sqrt{3}/2 \\ -1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0.5 \end{pmatrix}^T \cdot \boldsymbol{I} = \begin{pmatrix} 0 & -1 & 0 \\ 0.5 & 0 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 & 0.5 \end{pmatrix}. \tag{1}$$

The symbolic expression of the ZYZ Euler rotation matrix is

$$
\begin{aligned}
\boldsymbol{R}_{ZYZ}(\alpha_1,\alpha_2,\alpha_3) &= \boldsymbol{R}_Z(\alpha_1)\boldsymbol{R}_Y(\alpha_2)\boldsymbol{R}_Z(\alpha_3)\\
&= \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0\\ \sin\alpha_1 & \cos\alpha_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2\\ 0 & 1 & 0\\ -\sin\alpha_2 & 0 & \cos\alpha_2\end{pmatrix}\begin{pmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0\\ \sin\alpha_3 & \cos\alpha_3 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= \begin{pmatrix} \cos\alpha_1\cos\alpha_2\cos\alpha_3-\sin\alpha_1\sin\alpha_3 & -\cos\alpha_1\cos\alpha_2\sin\alpha_3-\sin\alpha_1\cos\alpha_3 & \cos\alpha_1\sin\alpha_2\\ \sin\alpha_1\cos\alpha_2\cos\alpha_3+\cos\alpha_1\sin\alpha_3 & \cos\alpha_1\cos\alpha_3-\sin\alpha_1\cos\alpha_2\sin\alpha_3 & \sin\alpha_1\sin\alpha_2\\ -\sin\alpha_2\cos\alpha_3 & \sin\alpha_2\sin\alpha_3 & \cos\alpha_2\end{pmatrix}.
\end{aligned}
\tag{2}
$$

Denote by R_{hk} the elements of the ${}^i\boldsymbol{R}_f$ matrix in (1). The inverse formulas for the ZYZ Euler representation can be extracted from the simpler elements in the last row and column of the $\boldsymbol{R}_{ZYZ}$ matrix in (2). Since

$$\sin^2\alpha_2 = R_{31}^2 + R_{32}^2 = 0.75 > 0,$$

this is a regular case and there are two solutions. These are computed, e.g., by the Matlab code

```
alfa2=atan2(sqrt(R(3,1)^2+R(3,2)^2),R(3,3))
alfa2bis=-alfa2
alfa1=atan2(R(2,3)/sin(alfa2),R(1,3)/sin(alfa2))
alfa1bis=atan2(R(2,3)/sin(alfa2bis),R(1,3)/sin(alfa2bis))
alfa3=atan2(R(3,2)/sin(alfa2),-R(3,1)/sin(alfa2))
alfa3bis=atan2(R(3,2)/sin(alfa2bis),-R(3,1)/sin(alfa2bis))
```

yielding

$$(\alpha_1,\alpha_2,\alpha_3) = (1.5708, 1.0472, 0) \qquad \text{and} \qquad (\alpha_1',\alpha_2',\alpha_3') = (-1.5708, -1.0472, 3.1416).$$

It is always good to check the result by plugging each of these two triples into (2) and verifying that the obtained rotation matrix is equal to ${}^i\boldsymbol{R}_f$. ■

Question #3

For the 4R robot with a spherical shoulder of Fig. 1, complete the assignment of Denavit-Hartenberg (D-H) frames and fill in the associated table of parameters [for this, use the extra sheet distributed]*. Keep the quantities that are already defined in the figure unchanged. If needed, provide the transformation between the last D-H frame and the standard frame of an end-effector gripper.*

Reply #3

A possible complete assignment of D-H frames for the robot of Fig. 1 is shown in Fig. 5. The associated set of parameters is given in Table 1, where the sign of the constant parameters is also indicated. The last D-H frame does not (and cannot) have its axis $\boldsymbol{z}_4$ along the approach direction of the gripper. Therefore, an extra rotation matrix

$${}^4\boldsymbol{R}_E = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

is needed to align RF_4 with the standard frame RF_E of an end-effector gripper. ■

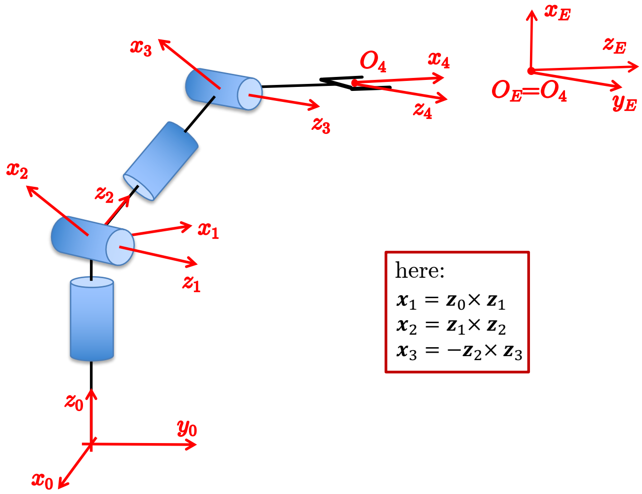

Figure 5: A possible assignment of D-H frames for the 4R robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$-\pi/2$	0	$d_3 > 0$	q_3
4	0	$a_4 > 0$	0	q_4

Table 1: Table of D-H parameters associated to the frame assignment in Fig. 5.

Question #4

A 2R planar robot with links of equal length L has limited joint ranges as follows: $q_1 \in [-\pi/2, \pi/2]$, $q_2 \in [0, \pi/2]$. Draw the primary workspace $WS_1 \in \mathbb{R}^2$. For $L = 1.4$ [m], is the point $P = (1.6, -0.2)$ reachable by the robot end effector?

Reply #4

For the given joint ranges, the primary workspace of a 2R robot with equal links of length L is shown in Fig. 6. Note that q_1 and q_2 are defined (by default, if not specified otherwise) according to the D-H convention. The boundaries of WS_1 are drawn with red dotted lines. The inner boundary is made by two parts, a half circumference of radius $L\sqrt{2}$ and a quarter circumference of radius L. Also the outer boundary is made by two parts, a quarter circumference of radius L and a half circumference of radius $2L$. If the link length is set to $L = 1.4$, the point P is out of WS_1. This is shown as well (in the proper scale) in Fig. 6. ■

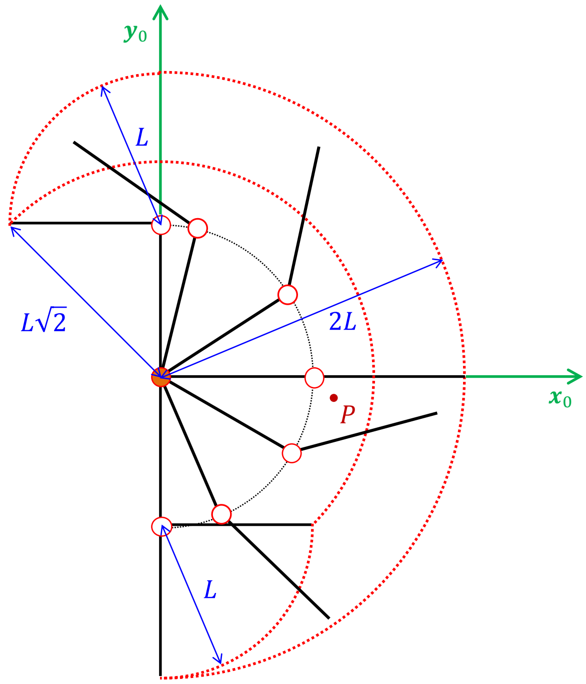

Figure 6: The primary workspace WS_1 of the 2R robot for the given joint limits.

Question #5

A branched two-arm planar robot having 5 dofs is sketched in Fig. 2, with generic labels for the link lengths and the actual definition of the joint angles. The sign convention for angles is the usual one (i.e., positive if counterclockwise). Determine the relative pose of the end-effector frame of the left arm with respect to that of the right arm, as expressed by the 4 × 4 homogeneous matrix ${}^{rE}\boldsymbol{T}_{lE}(\boldsymbol{q})$ with $\boldsymbol{q} = (\theta_0, \theta_{r1}, \theta_{r2}, \theta_{l1}, \theta_{l2})$. Check numerically the obtained symbolic expression when all the links have equal and unitary length and the two-arm robot is in the configuration $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$ —the right arm is horizontal and the left one is vertical and upward.

Reply #5

The result is obtained by computing the direct kinematics of each branch using the 4×4 homogeneous transformation matrices, and then finding the relative pose. The problem is planar (in the plane $z_0 = 0$), and so all rotations will be defined by an angle around the $\boldsymbol{z}_0$-axis normal to the plane. Note also that the joint angles in Fig. 2 are not necessarily part of a D-H convention, but they should be used as defined. For instance, the orientation angles of the right and left forearms are defined w.r.t. the positive and, respectively, negative $\boldsymbol{x}_0$-axis. For the right arm, the position of the end-effector frame RF_{rE} is then given by

$$
{}^{0}\boldsymbol{p}_r = L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} + L_{r1} \begin{pmatrix} \cos\theta_{r1} \\ \sin\theta_{r1} \\ 0 \end{pmatrix} + L_{r2} \begin{pmatrix} \cos(\theta_{r1}+\theta_{r2}) \\ \sin(\theta_{r1}+\theta_{r2}) \\ 0 \end{pmatrix},
$$

while its orientation is parametrized by the angle

$$
\phi_r = \theta_{r1} + \theta_{r2}.
$$

Similarly, for the left arm we have

$$
\begin{aligned}
{}^0\boldsymbol{p}_l &= L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} + L_{l1} \begin{pmatrix} \cos(\pi+\theta_{l1}) \\ \sin(\pi+\theta_{l1}) \\ 0 \end{pmatrix} + L_{l2} \begin{pmatrix} \cos(\pi+\theta_{l1}+\theta_{l2}) \\ \sin(\pi+\theta_{l1}+\theta_{l2}) \\ 0 \end{pmatrix} \\
&= L_0 \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} - L_{l1} \begin{pmatrix} \cos\theta_{l1} \\ \sin\theta_{l1} \\ 0 \end{pmatrix} - L_{l2} \begin{pmatrix} \cos(\theta_{l1}+\theta_{l2}) \\ \sin(\theta_{l1}+\theta_{l2}) \\ 0 \end{pmatrix}
\end{aligned}
$$

and

$$\phi_l = \pi + \theta_{r1} + \theta_{r2}.$$

The addition of π in the angular expressions pertaining to the left arm is needed in order to express the quantities in terms of the common base reference frame RF_0. As a result, from

$$
{}^0\boldsymbol{T}_r = \begin{pmatrix} {}^0\boldsymbol{R}_r(\phi_r) & {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} \cos(\theta_{r1}+\theta_{r2}) & -\sin(\theta_{r1}+\theta_{r2}) & 0 \\ \sin(\theta_{r1}+\theta_{r2}) & \cos(\theta_{r1}+\theta_{r2}) & 0 \\ 0 & 0 & 1 \end{matrix} & {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix}
$$

and

$$
{}^0\boldsymbol{T}_l = \begin{pmatrix} {}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{p}_r(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \begin{matrix} -\cos(\theta_{l1}+\theta_{l2}) & \sin(\theta_{l1}+\theta_{l2}) & 0 \\ -\sin(\theta_{l1}+\theta_{l2}) & -\cos(\theta_{l1}+\theta_{l2}) & 0 \\ 0 & 0 & 1 \end{matrix} & {}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix},
$$

one obtains

$$
\begin{aligned}
{}^r\boldsymbol{T}_l = {}^0\boldsymbol{T}_r^{-1} \cdot {}^0\boldsymbol{T}_l &= \begin{pmatrix} {}^0\boldsymbol{R}_r^T(\phi_r) & -{}^0\boldsymbol{R}_r^T(\phi_r){}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} {}^0\boldsymbol{R}_r^T(\phi_r)\,{}^0\boldsymbol{R}_l(\phi_l) & {}^0\boldsymbol{R}_r^T(\phi_r)\left({}^0\boldsymbol{p}_l(\theta_0,\theta_{l1},\theta_{l2}) - {}^0\boldsymbol{p}_r(\theta_0,\theta_{r1},\theta_{r2})\right) \\ \boldsymbol{0}^T & 1 \end{pmatrix}
\end{aligned}
$$

or[1]

$$
{}^r\boldsymbol{T}_l(\boldsymbol{q}) =
$$

$$
\begin{pmatrix}
-\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & \sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & 0 & \begin{matrix} -L_{r2} - L_{r1}\cos\theta_{r2} \\ -L_{l1}\cos(\theta_{l1}-\theta_{r1}-\theta_{r2}) \\ -L_{l2}\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) \end{matrix} \\
-\sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & -\cos(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) & 0 & \begin{matrix} L_{r1}\sin\theta_{r2} - L_{l1}\sin(\theta_{l1}-\theta_{r1}-\theta_{r2}) \\ -L_{l2}\sin(\theta_{l1}+\theta_{l2}-\theta_{r1}-\theta_{r2}) \end{matrix} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

[1] A MATLAB program yielding the simplified final output is reported in Appendix 1 at the end of the solution.

When the homogeneous transformation matrix ${}^r\boldsymbol{T}_l(\boldsymbol{q})$ is evaluated at $\boldsymbol{q}^* = (\pi/2, 0, 0, -\pi/2, 0)$, using the numerical data of the problem (all links of equal and unitary length), we obtain

$$
{}^r\boldsymbol{T}_l\left(\boldsymbol{q}^*\right) = \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

It is easy to check that this corresponds to the robot configuration with the right arm horizontal (pointing to the right) and the left arm vertical (pointing upward). The distance between the end effectors is $d = \|\boldsymbol{p}_l - \boldsymbol{p}_r\| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$. Similarly, at $\boldsymbol{q}^{**} = (\pi/2, \pi/4, \pi/4, -\pi/4, -\pi/4)$ —a robot posture that is fully symmetric w.r.t. the $\boldsymbol{y}_0$-axis, we have

$$
{}^r\boldsymbol{T}_l\left(\boldsymbol{q}^{**}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

In this case, there is a displacement $d = \sqrt{2}$ between the two end effectors just along the (now, horizontal) direction $\boldsymbol{y}_{rE}$, and no relative rotation between the two end-effector frames. ■

Question #6

Figure 3 shows a planar RRP robot, with the definition of its joint variables. The task of interest is specified by the position $\boldsymbol{p} = (p_x, p_y)$ of the robot end effector and by the orientation α of the forearm w.r.t. the $\boldsymbol{x}$-axis. The associated direct kinematics is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1 c_1 + q_3 c_{12} \\ l_1 s_1 + q_3 s_{12} \\ q_1 + q_2 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}). \tag{3}
$$

Determine the analytic solutions to the inverse kinematics problem. Disregard any situation that is unfeasible or singular. Provide at least one solution for the following (feasible) input data: $l_1 = 1$ [m], $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad].

Reply #6

The closed-form solution to the inverse kinematics of the RRP robot in Fig. 3 is found as follows. Let $\boldsymbol{r} = \boldsymbol{r}_d = \begin{pmatrix} p_{xd} & p_{yd} & \alpha_d \end{pmatrix}^T$ in (3). Substituting $q_1 + q_2 = \alpha_d$ from the last equation in (3) as argument of the two functions c_{12} and s_{12} in the first two equations, rearranging terms, and then squaring and summing, we obtain

$$
\left(p_{xd} - q_3 \cos\alpha_d\right)^2 + \left(p_{yd} - q_3 \sin\alpha_d\right)^2 = l_1^2 \left(c_1^2 + s_1^2\right) = l_1^2.
$$

Developing the squares, we find a second-order polynomial equation in the single unknown q_3:

$$
q_3^2 - 2\left(p_{xd}\cos\alpha_d + p_{yd}\sin\alpha_d\right) q_3 + \left(p_{xd}^2 + p_{xd}^2 - l_1^2\right) = 0. \tag{4}
$$

Equation (4) has two real roots (regular case) if and only if its discriminant is

$$
\Delta = \left(p_{xd}\cos\alpha_d + p_{yd}\sin\alpha_d\right)^2 - \left(p_{xd}^2 + p_{xd}^2 - l_1^2\right) \geq 0.
$$

Note that Δ can be rewritten also in the following two equivalent forms

$$
\Delta = l_1^2 - p_{xd}^2 \sin^2\alpha_d - p_{yd}^2 \cos^2\alpha_d + p_{xd} p_{yd} \sin 2\alpha_d = l_1^2 - \left(p_{xd}\sin\alpha_d - p_{yd}\cos\alpha_d\right)^2.
$$

If $\Delta = 0$, the two real roots are indeed coincident (one inverse kinematics solution only, a singular case); if $\Delta < 0$, the two roots of (4) are complex conjugate and there is no solution to the inverse kinematics problem[2]. In the regular case, the two solutions for q_3 are given by

$$q_3^{\pm} = p_{xd} \cos \alpha_d + p_{yd} \sin \alpha_d \pm \sqrt{\Delta}. \tag{5}$$

The associated values for q_1 and q_2 are found from the second and third equation in (3) and, respectively, from the third equation as

$$q_1^{\pm} = \operatorname{atan2} \left\{ p_{yd} - q_3^{\pm} \sin \alpha_d, p_{xd} - q_3^{\pm} \cos \alpha_d \right\}, \qquad q_2^{\pm} = \alpha_d - q_1^{\pm}. \tag{6}$$

With a length $l_1 = 1$ of the first link and for the (regular) input data $\boldsymbol{r}_d = (2, 1, \pi/6)$ [m,m,rad], we obtain from eqs. (5–6)

$$(q_1, q_2, q_3)^+ = (-2.4836, 3.0072, 3.2230) \quad \text{[rad,rad,m]}$$

and

$$(q_1, q_2, q_3)^- = (0.3892, 0.1344, 1.2411) \quad \text{[rad,rad,m]}.$$

The two solutions are shown in Fig. 7 (with the angular values approximated in degrees). ■

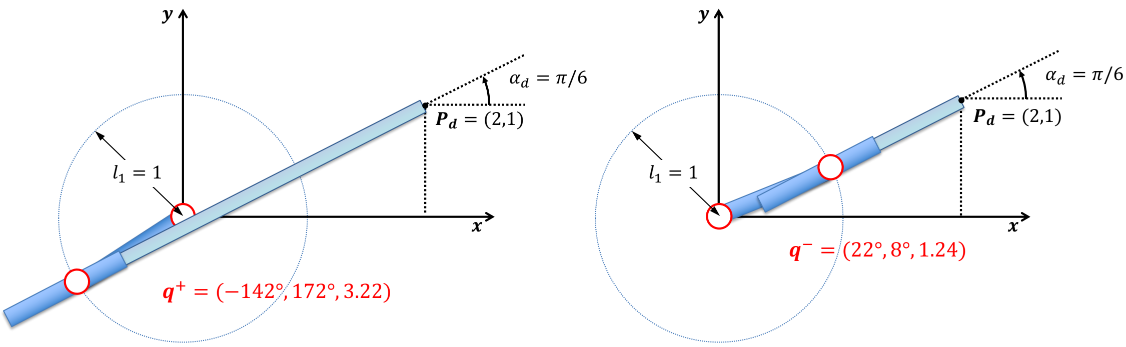

Figure 7: The two inverse kinematics solutions of the RRP robot for the given data.

Question #7

With reference to Fig. 4, a motor with inertia J_M drives a link through a gear with toothed wheels (a photo of this is also shown in the figure). The wheel on the motor shaft (aka, the pinion*) has radius $r_M = 2$ [cm], while the radius of the wheel on the link rotation axis is $r_L = 10$ [cm]. The link has inertia $J_L = 0.3$ [kgm^2] around its rotation axis. Assuming that an optimal inertia matching is realized by the reduction ratio of this transmission, determine the torque τ_M that the motor needs to produce around its $\boldsymbol{z}_M$ axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2]. Neglect dissipative effects as well as the inertia of the transmission components (and of the encoder).*

Reply #7

The reduction ratio of the transmission with toothed wheels is $n_r = r_L/r_M = 10/2 = 5$. On the other hand, the missing information about the motor inertia J_M is recovered from the assumed optimal inertia matching of the transmission:

$$n_r = n_r^* = \sqrt{\frac{J_L}{J_M}} \qquad \Rightarrow \qquad J_M = \frac{J_L}{n_r^2} = 0.012 \quad [\text{kgm}^2].$$

[2]More on these two cases is reported in Appendix 2 at the end of the solution (as additional material not requested in the solution answer).

Therefore, the torque that the motor needs to deliver on its axis in order to accelerate the link at $\ddot{\theta}_L = -5$ [rad/s^2] is given by

$$\tau_M = J_M \left(\ddot{\theta}_L n_r\right) + \frac{1}{n_r} J_L \ddot{\theta}_L = -0.6 \text{ [Nm]}.$$

Note that the motor and the link rotate in the same direction (positive if CCW) as seen from $\boldsymbol{z}_M$ and $\boldsymbol{z}_L$, respectively. So, there is no inversion of the sense of rotation in this toothed gear (as opposed to the planar case). ■

Question #8

An absolute encoder is mounted on the motor of the system shown in Fig. 4. If the link length is $L = 0.5$ [m], determine the minimum number of tracks n_t that the encoder needs to have in order to achieve at least a resolution of $\delta = 0.1$ [mm] at the link tip.

Reply #8

The requested angular resolution at the link base and on the motor side of the transmission are

$$\delta_L = \arctan\left(\frac{\delta}{L}\right) \simeq \frac{\delta}{L} = \frac{0.1}{500} = 2 \cdot 10^{-4} \text{ [rad]} \qquad \Rightarrow \qquad \delta_M = \delta_L n_r = 0.001 \text{ [rad]}.$$

Thus, the minimum number of tracks n_t of the absolute encoder in order to get the required resolution is

$$n_t = \left\lceil \log_2\left(\frac{2\pi}{\delta_M}\right)\right\rceil = \lceil 12.6173 \rceil = 13 \text{ tracks}.$$ ■

Question #9

Explain in exactly three short sentences the specific feature of a SCARA-type robot, its common technical implementation, and the significance in industrial applications.

Reply #9

[Sample reply] SCARA stands for Selective Compliance Arm for Robotic Assembly, a robot having 4 joints with vertical axes, the third one being prismatic and the others revolute. The end-effector compliance is present only along horizontal directions, and is usually provided by an harmonic drive on the motor axis of the first joint and by a transmission belt for driving the second joint by another motor mounted also on the first joint. This allows to accommodate in a passive way the lateral forces that may arise in assembly tasks, when the vertical insertion direction is not sufficiently accurate. ■

Appendix 1

A MATLAB code for Question #5

```
clear all; clc;
syms th0 thr1 thr2 thl1 thl2 L0 Lr1 Lr2 Ll1 Ll2 real
disp('right arm')
pr=L0*[cos(th0);sin(th0);0]+Lr1*[cos(thr1);sin(thr1);0]...
   +Lr2*[cos(thr1+thr2);sin(thr1+thr2);0]
phir=thr1+thr2
```

```
Rr=[cos(phir) -sin(phir) 0
    sin(phir)  cos(phir) 0
      0          0       1]
Tr=[  Rr       pr;
    zeros(1,3)  1]
disp('left arm')
pl=L0*[cos(th0);sin(th0);0]+Ll1*[cos(pi+thl1);sin(pi+thl1);0]...
   +Ll2*[cos(pi+thl1+thl2);sin(pi+thl1+thl2);0];
pl=simplify(pl)
phil=pi+thl1+thl2
Rl=[cos(phil) -sin(phil) 0
    sin(phil)  cos(phil) 0
      0          0       1];
Rl=simplify(Rl);
Tl=[  Rl       pl;
    zeros(1,3)  1]
disp('relative homogeneous transformation')
T_rl=simplify(inv(Tr)*Tl)
% data
L0=1;Lr1=1;Lr2=1;Ll1=1;Ll2=1;
disp('numerical evaluation for the given data')
th0=pi/2;thr1=0;thr2=0;thl1=-pi/2;thl2=0;
T_r_l=subs(T_rl)
disp('numerical evaluation in symmetric conditions')
th0=pi/2;thr1=pi/4;thr2=pi/4;thl1=-pi/4;thl2=-pi/4;
T_r_l=subs(T_rl)
% end
```

Appendix 2

Geometric view on the solution of the IK problem for the RRP robot

We present here an extra analysis that pursues more in depth the answer to Question #6.

It can be recognized that, in general, the inverse kinematics (IK) problem for our RRP robot has one, two, or no solution depending on the existence or not of intersections between a line L (with orientation) and a circumference C in the $\mathbb{R}^2$ plane. In fact, this geometric view is equivalent to the algebraic analysis of the roots of the second-order polynomial equation (4). With reference to Fig. 8 (pay attention also to the color codes used), the line L is defined by the point P_d and by the desired direction $\alpha_d \in (-\pi, \pi)$ of the third robot link, as computed from the horizontal $\boldsymbol{x}$-axis (positive if CCW). The circumference C is centered at the origin and has radius equal to the length l_1 of the first link.

The circumference C characterizes a transition from one type of solutions to another. If the point P_d is outside the circumference C (i.e., for $p_{xd}^2 + p_{yd}^2 > l_1^2$), there may be two, one or no intersection between L and C. Figure 8(a) shows a *regular* case with two intersections, and thus two inverse kinematics solutions $\boldsymbol{q}'$ and $\boldsymbol{q}''$, both having q_3' and q_3'' positive. The same figure shows also two *singular* cases, when the line L is tangent to C: there is only one IK solution in each case, but again with a positive q_3. Figure 8(b) shows the same cases considered in (a), but now with desired orientations α_d that differ by $\pm\pi$ from before. The situation is specular w.r.t. case (a), and the values of q_3 are now negative in all (single or double) solutions. Whenever $q_3 < 0$, note that we adopted a dashed line to represent the (retracted) forearm of the RRP robot.

When P_d is outside C, the line L may also not intersect C (*no solution* to the IK problem). Figure 8(c) shows one such instance. The other two cases in the figure refer to when P_d is inside ($p_{xd}^2 + p_{yd}^2 < l_1^2$) or on ($p_{xd}^2 + p_{yd}^2 = l_1^2$) the circumference C: both situations lead to two *regular* solutions to the IK problem. When the point P_d is *strictly* inside the circumference C, there are *always* two intersections of L with C, and thus two inverse solutions. However, in this case the two solution values of q_3 will have different signs. Finally, Fig. 8(d) suggests that some caution should be used when P_d is on C. In fact, two distinct (regular) inverse solutions will exist unless the orientation α_d is tangent to C, in which case they collapse into a single one with $q_3 = 0$ (again, a singular case). Figure 8 summarizes geometrically all the above possible situations.

Note that all singular solutions (i.e., when the cardinality of the solution set drops to 1 for a given input $\boldsymbol{r}_d$) will occur for $\cos q_2 = 0$. This corresponds to a singularity of the 3×3 analytic Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}_r(\boldsymbol{q})/\partial \boldsymbol{q})$ associated to the direct kinematics (3). The Jacobian matrix arises when considering the differential kinematics of a robot, namely the mapping from joint to task velocities (and vice versa).

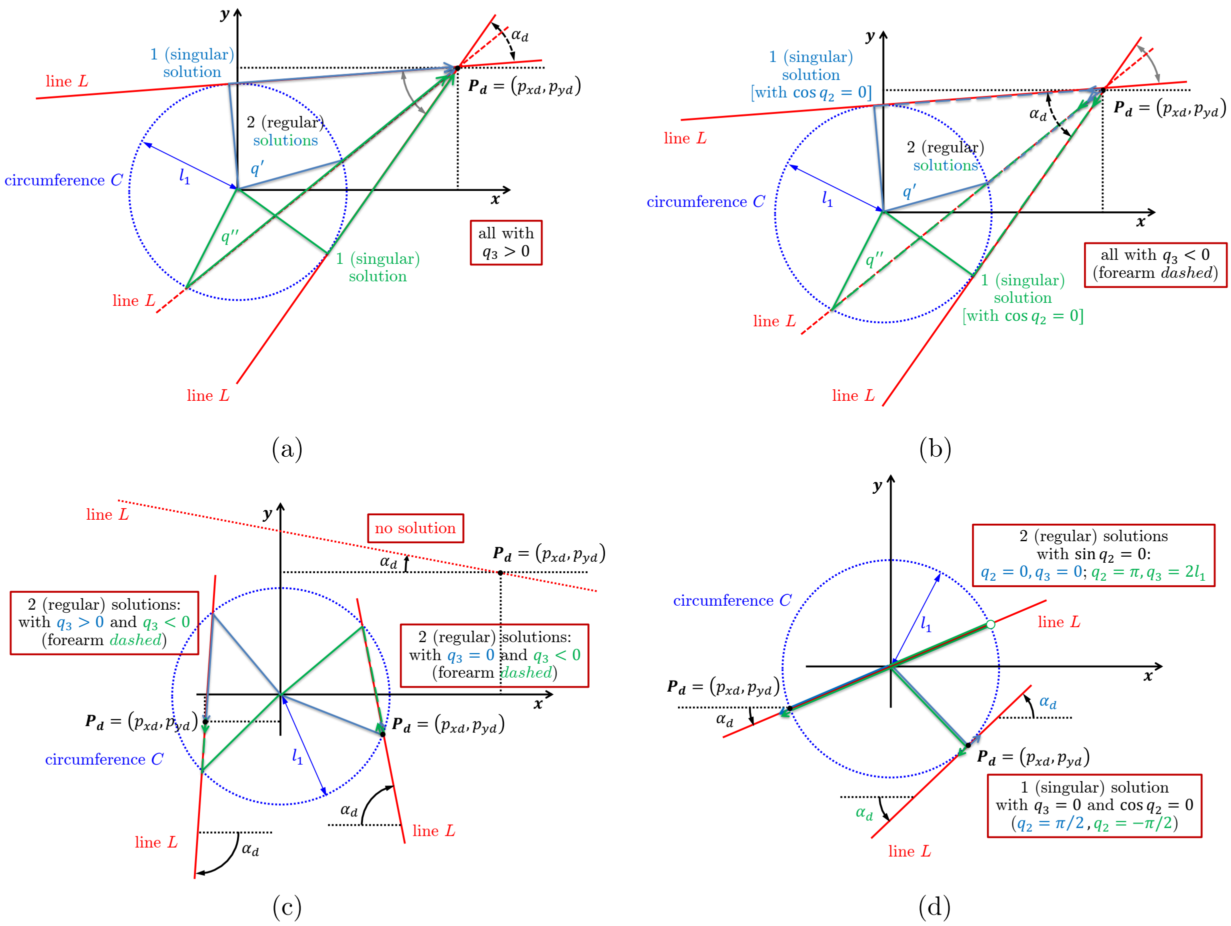

Figure 8: A geometric view of possible situations for the inverse kinematics of the RRP robot. Please refer to the text for a discussion of the cases (a) to (d).

* * * * *