Robotics 1

Remote Midterm Test – November 20, 2020

The test has the form of a Questionnaire with 10 questions. Provide as many answers as you can, with short but significant texts and formulas/tables/pictures. Please write clearly. If you wish, you may use the 'Reply Sheet' in the `Exam.net` *environment to type in some answers. Take a picture of each page of your handwritten answers and upload them before submitting. Try to follow the same order of the questions. Number your answers accordingly (don't repeat the text of the questions).*

Question #1

Given three rotations around the sequence of fixed axes ZYX by the angles $\alpha_1 = -\pi/2$, $\alpha_2 = -\pi/4$, and $\alpha_3 = \pi/4$ [rad], provide the rotation matrix $\boldsymbol{R}$ that specifies the final orientation. Compute then a vector $\boldsymbol{r} \in \mathbb{R}^3$, with $\|\boldsymbol{r}\| = 1$, that will not be rotated by $\boldsymbol{R}$.

Question #2

A rigid body rotates from an initial orientation $\boldsymbol{R}_i$ to a final orientation $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find an axis/angle representation $(\boldsymbol{r}, \theta)$ of the rotation. Is the solution unique in this case?

Question #3

The pose of a rigid body $\mathcal{B}$ in 3D space w.r.t. a reference frame is expressed by 6 independent parameters, 3 for its position and 3 for its orientation when using a minimal representation. Why do we need then *only* 4 Denavit-Hartenberg parameters to characterize the pose of a link in a serial manipulator w.r.t. the frame associated to the previous link?

Question #4

For generic $m \geq 1$ and $n > 1$, give the total number of elementary products $N_\times$ and additions N_+ in evaluating, through operations with rotation matrices ${}^{j-1}\boldsymbol{R}_j$, the vectors ${}^0\boldsymbol{v}_i \in \mathbb{R}^3$ by the expression

$${}^0\boldsymbol{v}_i = \left({}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{R}_2 \, \dots \, {}^{n-1}\boldsymbol{R}_n\right) {}^0\boldsymbol{v}_i, \qquad \text{for } i = 1, \dots, m, \tag{1}$$

or by successive matrix-vector products as

$${}^0\boldsymbol{v}_i = {}^0\boldsymbol{R}_1 \left({}^1\boldsymbol{R}_2 \left(\dots \left(\dots \left({}^{n-1}\boldsymbol{R}_n \, {}^0\boldsymbol{v}_i\right)\right) \dots\right)\right), \qquad \text{for } i = 1, \dots, m. \tag{2}$$

Given a value $n > 1$, which is the break-even value of m at which the number of evaluations $N_\times$ using (1) becomes advantageous (or disadvantageous) w.r.t. that using (2)?

Question #5

Robots are multi-body electromechanical systems driven by the torques $\boldsymbol{\tau}$ produced by the motors at the joints. In which sense are we allowed to say that one can move them by commanding just a desired joint velocity $\dot{\boldsymbol{q}}$ (or a joint position $\boldsymbol{q}$)?

Question #6

Consider the 4-dof PRPR robot sketched in Fig. 1, where the base frame RF_0 and the end-effector frame RF_4 are already assigned. The robot has a shoulder offset given by the constant $N > 0$.

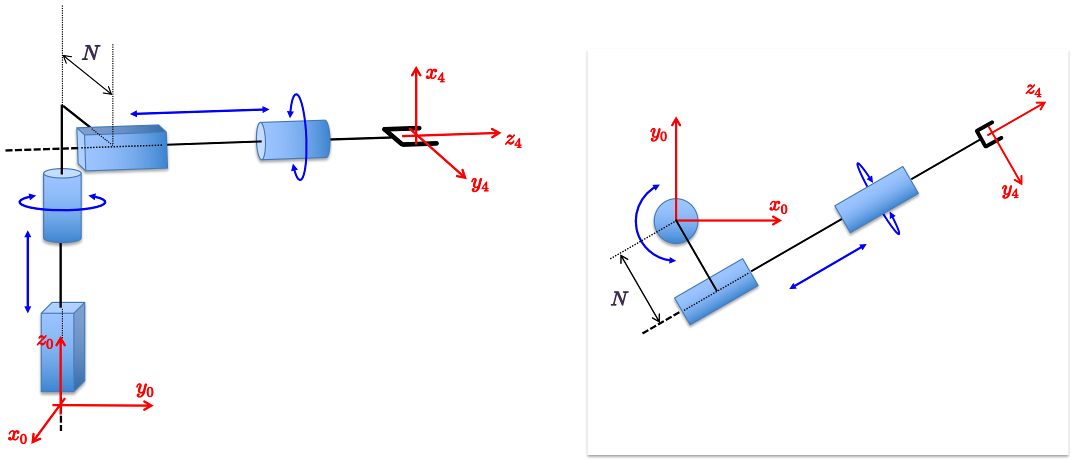

Figure 1: Kinematic skeleton of a PRPR robot. A perspective [left] and a top view (right).

Assign the other frames according to the Denavit-Hartenberg convention and build the associated table of parameters so that the position of the origin O_4 of the end-effector frame will be given by

$$^0\boldsymbol{p}_4(\boldsymbol{q}) = \begin{pmatrix} N\cos q_2 - q_3 \sin q_2 \\ N \sin q_2 + q_3 \cos q_2 \\ q_1 \end{pmatrix}. \tag{3}$$

Determine the symbolic expression of $^0\boldsymbol{R}_4(\boldsymbol{q})$ in the direct kinematics. Further, provide a numerical matrix $\boldsymbol{R} \in SO(3)$ representing an orientation that the end-effector of this robot can never assume.

Question #7

Given a desired $\boldsymbol{p} \in \mathbb{R}^3$ for $^0\boldsymbol{p}_4(\boldsymbol{q})$ in (3), find all the analytical solutions $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ to the associated inverse kinematics problem in the regular case. Assuming there are no joint limits, sketch also the primary workspace WS_1 of the 4-dof PRPR robot. Finally, compute the numerical solutions to this inverse kinematics problem for $\boldsymbol{p} = \begin{pmatrix} 0 & 2 & 1.5 \end{pmatrix}^T$ with $N = 0.3$ [m].

Question #8

What are the pros and cons in estimating online or offline the velocity of a joint from position data measured by an encoder using numerical differentiation formulas. Write a simple code that uses the 1-step BDF (Euler) formula to provide online estimates $\dot{y}_k^e = \dot{y}^e(t_k)$, for $k = 1, \ldots, 10$, of the velocity from the following series of ten position data (noisy and with only 4 significant digits), collected with a sampling frequency of 40 Hz from $t_1 = 0$ on:

$$\{y_k\} = \{0.0007 \quad 0.1251 \quad 0.2500 \quad 0.3741 \quad 0.4977 \quad 0.6187 \quad 0.7397 \quad 0.8579 \quad 0.9739 \quad 1.0876\} \text{ [rad]}$$

Compute also the average value $\bar{\dot{y}}^e$ of the obtained samples of velocity estimates (for comparison, the average value of the true velocity samples $\dot{y}_k$, for $k = 1, \ldots, 10$, is $\bar{\dot{y}} = 4.8239$ [rad/s]).

Question #9

With reference to Fig. 2, the second joint of a 2R planar arm having link length $L_1 = 0.45$ and $L_2 = 0.35$ [m] is actuated by a motor M located at the first joint through a toothed transmission belt inside the body of link 1 (this may represent the situation of the first two dof of a SCARA robot). The belt connects a toothed disk of radius $r_1 = 5$ [cm], placed on the output shaft of motor M, with a second one of radius $r_2 = 0.25$ [m], connected to the axis of joint 2. An incremental encoder with 700 pulses/turn and electronic multiplication by a factor 4 is mounted on the back of motor M, for measuring its angular position θ_M.

a) Suppose that the optical disk of the encoder has generated 300 light pulses while rotating in the CCW direction in a time interval $T = 1.2$ [s]. How large is the rotation $\Delta\theta_2$ (in [rad]) performed by the second link? And what is the average angular speed $\bar{\dot{\theta}}_2$ (in [rad/s]) during T?

b) With the robot in the configuration $\boldsymbol{\theta} = \mathbf{0}$ (stretched arm) and keeping joint 1 at rest, what is the minimal lateral displacement (along the $\boldsymbol{y}_0$ direction) of the tip of link 2 that can be sensed by the encoder?

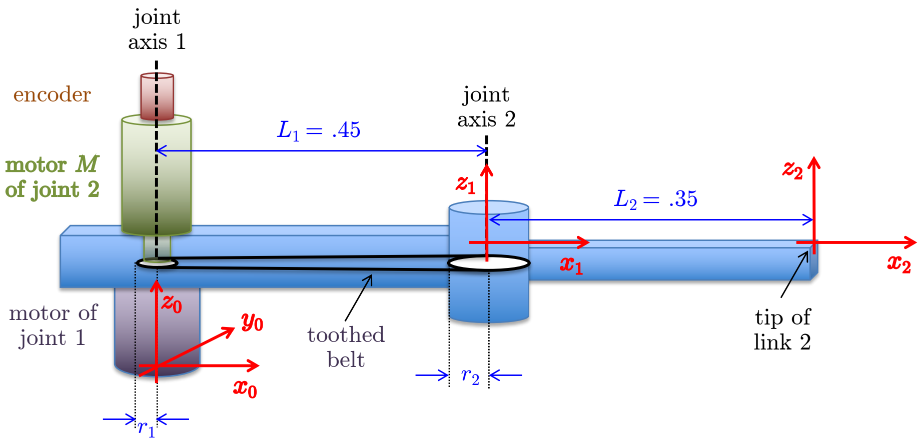

Figure 2: The transmission arrangement for moving joint 2 with a motor M placed at joint 1.

Question #10

The base frame RF_0 of a robot has its origin placed in the position ${}^W\boldsymbol{p}_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ and is rotated by an angle $\beta = \pi/2$ [rad] around the $\boldsymbol{z}_w$ axis of the world frame RF_W. In a given configuration, the end-effector pose of the robot is given by

$$
{}^0\boldsymbol{T}_E = \begin{pmatrix} 0 & 0.5 & -\dfrac{\sqrt{3}}{2} & 1 \\ 1 & 0 & 0 & -0.75 \\ 0 & -\dfrac{\sqrt{3}}{2} & -0.5 & 1.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The position of the tip of a tool mounted on the end-effector is ${}^E\boldsymbol{p}_{tool} = \begin{pmatrix} 0 & 0.3 & 0.3 \end{pmatrix}^T$ [m]. Moreover, the tool frame RF_{tool} associated to this point is rotated by an angle $\gamma = -\pi/2$ [rad] around the $\boldsymbol{x}_E$ axis of the end-effector frame RF_E. Compute the position of the tip of the tool in the world frame and the absolute orientation of the tool frame w.r.t. RF_W.

[180 minutes (3 hours); open books]

Solution

November 20, 2020

Question #1

Given three rotations around the sequence of fixed axes ZYX by the angles $\alpha_1 = -\pi/2$, $\alpha_2 = -\pi/4$, and $\alpha_3 = \pi/4$ [rad], provide the rotation matrix $\boldsymbol{R}$ that specifies the final orientation. Compute then a vector $\boldsymbol{r} \in \mathbb{R}^3$, with $\|\boldsymbol{r}\| = 1$, that will not be rotated by $\boldsymbol{R}$.

Reply #1

The assigned sequence is of the Roll-Pitch-Yaw type, with

$$\boldsymbol{R}_Z = \left.\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|_{\alpha_1 = -\pi/2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{R}_Y = \left.\begin{pmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{pmatrix}\right|_{\alpha_2 = -\pi/4} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{R}_X = \left.\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_3 & -\sin\alpha_3 \\ 0 & \sin\alpha_3 & \cos\alpha_3 \end{pmatrix}\right|_{\alpha_3 = \pi/4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

The final orientation is computed by the product of these matrices in the *reverse* order of definition (rotations around fixed axes) as

$$\boldsymbol{R} = \boldsymbol{R}_X \boldsymbol{R}_Y \boldsymbol{R}_Z = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -0.5 & -0.5 \\ -1/\sqrt{2} & 0.5 & 0.5 \end{pmatrix}.$$

The unit vector $\boldsymbol{r} \in \mathbb{R}^3$ that is not rotated (nor scaled) by $\boldsymbol{R}$ is the eigenvector of $\boldsymbol{R}$ associated to its real eigenvalue $\lambda = +1$, i.e., such that

$$\boldsymbol{R}\boldsymbol{r} = \boldsymbol{r} \qquad \Rightarrow \quad \text{normalizing } \boldsymbol{r}\text{, up to the sign} \quad \Rightarrow \qquad \boldsymbol{r} = \pm \begin{pmatrix} -0.5774 \\ 0 \\ 0.8165 \end{pmatrix}.$$

This can be computed, e.g., with the Matlab instruction `[V,D]=eig(R)`, extracting then the (only) real eigenvector from the columns of the matrix `V`. ■

Question #2

A rigid body rotates from an initial orientation $\boldsymbol{R}_i$ to a final orientation $\boldsymbol{R}_f$, as specified by

$$\boldsymbol{R}_i = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & 0 & -0.5 \end{pmatrix}, \qquad \boldsymbol{R}_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find an axis/angle representation $(\boldsymbol{r}, \theta)$ of the rotation. Is the solution unique in this case?

Reply #2

One has to solve the inverse problem for the axis/angle representation $(\boldsymbol{r}, \theta)$ of a rotation matrix

$$\boldsymbol{R}(\boldsymbol{r}, \theta) = {}^{i}\boldsymbol{R}_f = \boldsymbol{R}_i^T \boldsymbol{R}_f = \begin{pmatrix} 0 & -0.5 & -0.8660 \\ 1 & 0 & 0 \\ 0 & -0.8660 & 0.5 \end{pmatrix}, \tag{4}$$

where ${}^{i}\boldsymbol{R}_f$ is the relative rotation from the initial to the final orientation. Denoting by R_{hk} the elements of the ${}^{i}\boldsymbol{R}_f$ matrix, from the inverse formulas we have

$$\sin\theta = \pm\frac{1}{2}\sqrt{(R_{12}-R_{21})^2 + (R_{23}-R_{32})^2 + (R_{13}-R_{31})^2} = 0.9682 \neq 0. \tag{5}$$

Therefore, this is a *regular* case and there will be two opposite solutions $(\boldsymbol{r}, \theta)$ and $(-\boldsymbol{r}, -\theta)$. The solution corresponding to the choice of the + sign in (5) is computed by the four Matlab instructions

```
ctheta=(R(1,1)+R(2,2)+R(3,3)-1)/2
stheta=sqrt((R(1,2)-R(2,1))^2+(R(2,3)-R(3,2))^2+(R(1,3)-R(3,1))^2)/2
ri=1/(2*stheta)*[R(3,2)-R(2,3); R(1,3)-R(3,1); R(2,1)-R(1,2)]
theta=atan2(stheta,ctheta)
```

yielding

$${}^{i}\boldsymbol{r} = \begin{pmatrix} -0.4472 \\ -0.4472 \\ 0.7746 \end{pmatrix}, \qquad \theta = 1.8235 \text{ [rad]} = 104.48°.$$

Note that the unit axis $\boldsymbol{r}$ obtained with this procedure is naturally expressed in the coordinates of the initial frame oriented as $\boldsymbol{R}_i$. From (4), the final orientation is in fact computed by concatenating

$${}^{0}\boldsymbol{R}_i\, \boldsymbol{R}\left({}^{i}\boldsymbol{r}, \theta\right) = {}^{0}\boldsymbol{R}_i\, {}^{i}\boldsymbol{R}_f = {}^{0}\boldsymbol{R}_f,$$

where the coordinate frame of definition of each vector/matrix term has been explicitly indicated by the use of superscripts. Thus, the expression of the invariant axis in the reference frame RF_0 is

$${}^{0}\boldsymbol{r} = \boldsymbol{R}_i\, {}^{i}\boldsymbol{r} = \begin{pmatrix} -0.4472 \\ 0.4472 \\ -0.7746 \end{pmatrix}. \qquad \blacksquare$$

Question #3

The pose of a rigid body $\mathcal{B}$ in 3D space w.r.t. a reference frame is expressed by 6 independent parameters, 3 for its position and 3 for its orientation when using a minimal representation. Why do we need then only *4 Denavit-Hartenberg parameters to characterize the pose of a link in a serial manipulator w.r.t. the frame associated to the previous link?*

Reply #3

This reduction follows from the fact that link i (and so, its associated frame RF_i) is not free to be placed in the 3D space w.r.t. link $i-1$ (and its associated frame RF_{i-1}). The two links are connected at a joint that sets 2 scalar geometric constraints on the 6-dimensional relative pose between link $i-1$ and link i, leaving its characterization specified by only 4 residual parameters. The Denavit-Hartenberg convention is a clever choice of the origins and coordinate axes of the link frames, which shows how to cut down the number of relative pose parameters from 6 to 4. One of these parameters is variable, allowing the motion of frame RF_i around or along the axis, of the 1-dof joint (respectively, revolute or prismatic). ■

Question #4

For generic $m \geq 1$ and $n > 1$, give the total number of elementary products $N_\times$ and additions N_+ in evaluating, through operations with rotation matrices ${}^{j-1}\boldsymbol{R}_j$, the vectors ${}^0\boldsymbol{v}_i \in \mathbb{R}^3$ by the expression

$$ {}^0\boldsymbol{v}_i = \left({}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2\, \ldots\, {}^{n-1}\boldsymbol{R}_n\right) {}^0\boldsymbol{v}_i, \qquad \textit{for } i = 1, \ldots, m, \tag{6} $$

or by successive matrix-vector products as

$$ {}^0\boldsymbol{v}_i = {}^0\boldsymbol{R}_1 \left({}^1\boldsymbol{R}_2 \left(\ldots \left(\ldots \left({}^{n-1}\boldsymbol{R}_n\, {}^0\boldsymbol{v}_i\right)\right) \ldots\right)\right), \qquad \textit{for } i = 1, \ldots, m. \tag{7} $$

Given a value $n > 1$, which is the break-even value of m at which the number of evaluations $N_\times$ using (6) becomes advantageous (or disadvantageous) w.r.t. that using (7)?

Reply #4

The product of a 3×3 matrix by a 3-dimensional vector needs 9 multiplications and 6 additions, whereas the product between two 3×3 matrices needs three times as many, namely 27 multiplications and 18 additions. When doing the $(n-1)$ products of the n rotation matrices first, and then applying it to the m vectors $\boldsymbol{v}_i \in \mathbb{R}^3$ as in the first method (6), one obtains

$$ N_{\times,1} = 27(n-1) + 9\,m \qquad \text{and} \qquad N_{+,1} = 18(n-1) + 6\,m. $$

Using instead the recursive matrix-vector product n times for each of the m vectors v_i as in the second method (7), one has

$$ N_{\times,2} = 9\,mn \qquad \text{and} \qquad N_{+,2} = 6\,mn. $$

Comparing the number of elementary products, gives

$$ N_{\times,1} = 27(n-1) + 9\,m \lesseqgtr 9\,mn = N_{\times,2} \quad \iff \quad 27(n-1) \lesseqgtr 9\,m(n-1) \quad \iff \quad 27 \lesseqgtr 9\,m. $$

Thus, the break-even is obtained exactly at $m = 3$, with the first method (6) becoming more convenient when more than 3 vectors $\boldsymbol{v}_i$ have to be transformed. Note that this result is *independent* of n. Moreover, it can be generalized to matrix/vector computations in any dimension $p \geq 2$ (e.g., with $p = 4$, for 4×4 homogeneous matrices) leading to $m = p$ as break-even value. ■

Question #5

Robots are multi-body electromechanical systems driven by the torques $\boldsymbol{\tau}$ produced by the motors at the joints. In which sense are we allowed to say that one can move them by commanding just a desired joint velocity $\dot{\boldsymbol{q}}$ (or a joint position $\boldsymbol{q}$)?

Reply #5

This statement is correct in so far we assume that a low-level servo system with a feedback loop is present on each actuator —often, an electrical motor— at the robot joints. The velocity command $\dot{\boldsymbol{q}}$ (or the position $\boldsymbol{q}$) will be the reference input for these controllers. For the generic j-th servomotor, with $j = 1, \ldots, n$, a voltage V_j and a current i_j are generated that make the motor produce a torque τ_j on its output shaft. This torque will move the driven link i, possibly through transmission/reduction elements, until the reference velocity $\dot{q}_i$ (or position q_i) will be reached, i.e., when the error between the desired and the measured output variable is zero. ■

Question #6

Consider the 4-dof PRPR robot sketched in Fig. 1, where the base frame RF_0 and the end-effector frame RF_4 are already assigned. The robot has a shoulder offset given by the constant $N > 0$. Assign the other frames according to the Denavit-Hartenberg convention and build the associated table of parameters so that the position of the origin O_4 of the end-effector frame will be given by

$$
{}^0\boldsymbol{p}_4(\boldsymbol{q}) = \begin{pmatrix} N\cos q_2 - q_3 \sin q_2 \\ N \sin q_2 + q_3 \cos q_2 \\ q_1 \end{pmatrix}. \tag{8}
$$

Determine the symbolic expression of ${}^0\boldsymbol{R}_4(\boldsymbol{q})$ *in the direct kinematics. Further, provide a numerical matrix* $\boldsymbol{R} \in SO(3)$ *representing an orientation that the end-effector of this robot can never assume.*

Reply #6

The *unique* assignment of the remaining Denavit-Hartenberg (DH) frames that is consistent with the positional direct kinematics (8) is illustrated by the two views in Fig. 3 and Fig. 4. The associated set of DH parameters is given in Table 1, with the joint variables $\boldsymbol{q}$ taking the values (or just the signs) according to the configuration shown in Fig. 4.

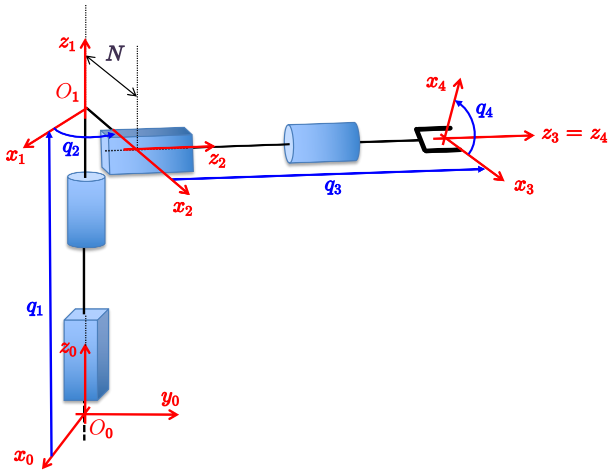

Figure 3: A perspective view of the DH frame assignment for the PRPR robot of Fig. 1.

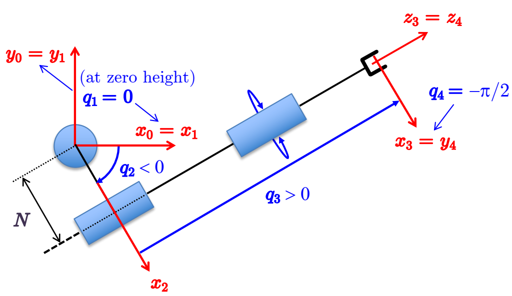

Figure 4: Top view of the frame assignment in Fig. 3, with the robot in a different configuration.

i	α_i	a_i	d_i	θ_i
1	0	0	$q_1 = 0$	0
2	$-\pi/2$	$N > 0$	0	$q_2 < 0$
3	0	0	$q_3 > 0$	0
4	0	0	0	$q_4 = -\pi/2$

Table 1: Table of DH parameters of the PRPR robot: $\boldsymbol{q}$ is associated to the configuration in Fig. 4.

Constructing the DH homogeneous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, \ldots, 4$, it is immediate to compute the orientation of the end-effector frame RF_4 as

$$
{}^0\boldsymbol{R}_4(\boldsymbol{q}) = {}^0\boldsymbol{R}_1\, {}^1\boldsymbol{R}_2(q_2)\, {}^2\boldsymbol{R}_3\, {}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_2 \cos q_4 & -\cos q_2 \sin q_4 & -\sin q_2 \\ \sin q_2 \cos q_4 & -\sin q_2 \sin q_4 & \cos q_2 \\ -\sin q_4 & -\cos q_4 & 0 \end{pmatrix}.
$$

This matrix is parametrized by the two rotational joint variables q_2 and q_4 only. It is easy to conclude that the end effector has no sufficient mobility to assume an arbitrary orientation in the 3D space. In particular, the end-effector approach axis $\boldsymbol{z}_4$ can never point out of the horizontal plane. This is revealed by the structural 0 in position $(3, 3)$ of matrix ${}^0\boldsymbol{R}_4$. Therefore, unfeasible orientations for the robot end-effector are given, e.g., by the one-dimensional family of rotation matrices

$$
\boldsymbol{R} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \forall \beta \in \mathbb{R}.
$$

Indeed, any matrix $\boldsymbol{R} \in SO(3)$ having $R_{3,3} \neq 0$ will represent an unfeasible end-effector orientation for this 4-dof robot. ■

Question #7

Given a desired $\boldsymbol{p} \in \mathbb{R}^3$ for ${}^0\boldsymbol{p}_4(\boldsymbol{q})$ in (8), find all the analytical solutions $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ to the associated inverse kinematics problem in the regular case. Assuming there are no joint limits, sketch also the primary workspace WS_1 of the 4-dof PRPR robot. Finally, compute the numerical solutions to this inverse kinematics problem for $\boldsymbol{p} = \begin{pmatrix} 0 & 2 & 1.5 \end{pmatrix}^T$ with $N = 0.3$ [m].

Reply #7

The closed-form solution to the inverse kinematics for the end-effector position of the PRP robot (the last rotational joint is irrelevant here) is found as follows. Let ${}^0\boldsymbol{p}_4(\boldsymbol{q}) = \boldsymbol{p} = \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}^T$. Using this in (8), by squaring and summing the first two equations we obtain

$$
(N\cos q_2 - q_3 \sin q_2)^2 + (N \sin q_2 + q_3 \cos q_2)^2 = N^2 + q_3^2 = p_x^2 + p_y^2 \quad \Rightarrow \quad q_3 = \pm\sqrt{p_x^2 + p_y^2 - N^2}.
$$

The argument of the square root should not be negative, which sets in fact the only limitation on the primary workspace $WS_1 = \left\{\boldsymbol{p} \in \mathbb{R}^3 : p_x^2 + p_y^2 \geq N^2\right\}$. In the regular case ($q_3 \neq 0$), for each of the two values $q_3 = q_3^+$ and $q_3 = q_3^-$ we solve the following linear system (whose determinant is always $N^2 + q_3^2 > 0$) in the unknowns $c_2 = \cos q_2$ and $s_2 = \sin q_2$ as

$$
\begin{pmatrix} N & -q_3^{\{+,-\}} \\ q_3^{\{+,-\}} & N \end{pmatrix} = \begin{pmatrix} c_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \quad \Rightarrow \quad q_2 = \operatorname{atan2}\left\{N p_y - q_3^{\{+,-\}} p_x, N p_x + q_3^{\{+,-\}} p_y\right\}.
$$

Finally, we have the unique value for the first prismatic joint $q_1 = p_z$.
Summarizing, in the regular case ($p_x^2 + p_y^2 > N^2$) we have the two solutions:

$$\boldsymbol{q}^I = \begin{pmatrix} p_z \\ \text{atan2}\left\{p_y N - p_x\sqrt{p_x^2+p_y^2-N^2},\, p_x N + p_y\sqrt{p_x^2+p_y^2-N^2}\right\} \\ \sqrt{p_x^2+p_y^2-N^2} \end{pmatrix},$$

$$\boldsymbol{q}^{II} = \begin{pmatrix} p_z \\ \text{atan2}\left\{p_y N + p_x\sqrt{p_x^2+p_y^2-N^2}, p_x N - p_y\sqrt{p_x^2+p_y^2-N^2}\right\} \\ -\sqrt{p_x^2+p_y^2-N^2} \end{pmatrix}.$$

At the (inner) boundary of the primary workspace ($p_x^2 + p_y^2 = N^2$), the two solutions $\boldsymbol{q}^I$ and $\boldsymbol{q}^{II}$ collapse into a single one. Being $N > 0$, we can write this singular solution as

$$\boldsymbol{q}^s = \begin{pmatrix} p_z \\ \text{atan2}\{p_y, p_x\} \\ 0 \end{pmatrix}$$

Finally, there is no solution for $p_x^2 + p_y^2 < N^2$.
The primary workspace[1] WS_1 is obtained by subtracting from the entire Euclidean space $\mathbb{R}^3$ an infinite cylinder of radius N having its axis coincident with the axis $\boldsymbol{z}_0$. For the numerical input data provided, being $\sqrt{p_x^2+p_y^2} = 4 > 0.3 = N$, we will have two regular solutions to the inverse kinematics, namely

$$\boldsymbol{q}^I = \begin{pmatrix} 1.5 \\ 0.1506 \\ 1.9774 \end{pmatrix}, \qquad \boldsymbol{q}^{II} = \begin{pmatrix} 1.5 \\ 2.9910 \\ -1.9774 \end{pmatrix} \quad \text{[m; rad; m]}. \qquad \blacksquare$$

Question #8

What are the pros and cons in estimating online or offline the velocity of a joint from position data measured by an encoder using numerical differentiation formulas. Write a simple code that uses the 1-step BDF (Euler) formula to provide online estimates $\dot{y}_k^e = \dot{y}^e(t_k)$, for $k = 1, \ldots, 10$, of the velocity from the following series of ten position data (noisy and with only 4 significant digits), collected with a sampling frequency of 40 Hz from $t_1 = 0$ on:

$$\{y_k\} = \left\{0.0007 \;\; 0.1251 \;\; 0.2500 \;\; 0.3741 \;\; 0.4977 \;\; 0.6187 \;\; 0.7397 \;\; 0.8579 \;\; 0.9739 \;\; 1.0876\right\} \; \textit{[rad]}$$

Compute also the average value $\bar{\dot{y}}^e$ of the obtained samples of velocity estimates (for comparison, the average value of the true velocity samples $\dot{y}_k$, for $k = 1, \ldots, 10$, is $\bar{\dot{y}} = 4.8239$ [rad/s]).

Reply #8

The results are obtained using, e.g., the following segment of Matlab code (with comments).

[1] Sorry, no figure! It is rather awkward to draw such a workspace when there is no limit to the ranges of the two prismatic joints.

```
\% position data as input
yN=[0.0007 0.1251 0.2500 0.3741 0.4977 0.6187 0.7397 0.8579 0.9739 1.0876];
\%
ns=length(yN);      \% number of samples in yN
Tc=0.025;           \% sampling interval for a 40 Hz frequency
t=[0:Tc:(ns-1)*Tc]; \% sampled instants of time
\% this initialization step is commented further in the text
yprec=0;            \% alternatives: yprec=yN(1); or yprec=2*yN(1)-yN(2);
for i=1:length(t)
     yd1(i)=(yN(i)-yprec)/Tc; \% 1-step (Euler) BDF
     yprec=yN(i);
end
\% output results
disp('1-step BDF (Euler) velocity estimates and average')
yd1
avgyd1=mean(yd1)
```

The output is

$$\{\dot{y}_k^e\} = \left\{0.0280 \quad 4.9760 \quad 4.9960 \quad 4.9640 \quad 4.9440 \quad 4.8400 \quad 4.8400 \quad 4.7280 \quad 4.6400 \quad 4.5480\right\} \text{ [rad/s]}$$

with an average value $\bar{\dot{y}}^e = 4.3504$ [rad/s].

The initialization of the 1-step BDF method (`yprec` in the command line before the `for` loop) is needed for computing the first sample $\dot{y}_1^e$ of the velocity estimate. In fact, there is no 'previous' position sample to be used in the BDF formula $\dot{y}_k^e = (y_k - y_{k-1}))/T_c$ when $k = 1$. Any choice for y_0 (i.e., for initializing `yprec` in the code) is feasible. Since the Euler method is a one-step approximation, this will affect only the first sample of the produced output. Here, we took $y_0 = 0$ as a neutral value. Another reasonable choice is to set $y_0 = y_1 = 0.0007$, i.e., repeating the same first position sample of the data series. This leads to $\dot{y}_1^e = (y_1 - y_0)/T_c = 0$ (as opposed to $\dot{y}_1^e = 0.0280$), with just a slightly larger average $\bar{\dot{y}}^e = 4.3476$ [rad/s]. It is easy to obtain a better approximation of the derivative $\dot{y}_1^e$ at $k = 1$ by using the next position sample y_2. This future knowledge would give for the initialization `yprec`

$$y_0 = y_1 - \left(\frac{y_2 - y_1}{T_c}\right) T_c = 2y_1 - y_2,$$

leading to $\dot{y}_1^e = (y_1 - y_0)/T_c = (y_2 - y_1)/T_c = \dot{y}_2^e = 4.9760$, The average of the output series grows then to $\bar{\dot{y}}^e = 4.8452$, which is much closer to the true value $\bar{\dot{y}} = 4.8239$ [rad/s]. ■

Question #9

With reference to Fig. 2, the second joint of a 2R planar arm having link length $L_1 = 0.45$ and $L_2 = 0.35$ [m] is actuated by a motor M located at the first joint through a toothed transmission belt inside the body of link 1 (this may represent the situation of the first two dof of a SCARA robot). The belt connects a toothed disk of radius $r_1 = 5$ [cm], placed on the output shaft of motor M, with a second one of radius $r_2 = 0.25$ [m], connected to the axis of joint 2. An incremental encoder with 700 pulses/turn and electronic multiplication by a factor 4 is mounted on the back of motor M, for measuring its angular position θ_M.

a) *Suppose that the optical disk of the encoder has generated 300 light pulses while rotating in the CCW direction in a time interval $T = 1.2$ [s]. How large is the rotation $\Delta\theta_2$ (in [rad]) performed by the second link? And what is the average angular speed $\bar{\dot{\theta}}_2$ (in [rad/s]) during T?*

b) *With the robot in the configuration $\boldsymbol{\theta} = \mathbf{0}$ (stretched arm) and keeping joint 1 at rest, what is the minimal lateral displacement (along the $\boldsymbol{y}_0$ direction) of the tip of link 2 that can be sensed by the encoder?*

Reply #9

a) Since the reduction ratio of the toothed belt transmission is $N_r = r_2/r_1 = 0.25/0.05 = 5$, the answers are

$$\Delta\theta_2 = \frac{\Delta\theta_M}{N_r} = \frac{\#\ \text{pulses}}{\#\ \text{pulses per turn}} \cdot 2\pi \cdot \frac{1}{N_r} = \frac{300}{700} \cdot \frac{2\pi}{5} = 0.5386\ \text{[rad]},$$

$$\bar{\dot{\theta}}_2 = \frac{\Delta\theta_2}{T} = \frac{0.5386}{1.2} = 0.4488\ \text{[rad/s]}.$$

In the first formula, only the fraction of a full turn rotation performed by the motor matters (not the resolution of the position sensor).

b) On the other hand, the minimal displacement of the tip of the second link that can be sensed depends on the actual resolution of the digital encoder mounted on its driving motor. In the given situation (arm stretched along the $\boldsymbol{x}_0$-axis and joint 1 not moving), only the length L_2 of the second link of the 2R robot is involved in this evaluation. We have

$$\Delta \boldsymbol{p}_{tip,y} = L_2 \cdot \Delta\theta_{2,res} = L_2 \cdot \frac{2\pi}{4 \times \#\ \text{pulses per turn}} \cdot \frac{1}{N_r} = \frac{2\pi}{2800} \cdot \frac{0.35}{5} = 0.157\ \text{[mm]}. \quad \blacksquare$$

Question #10

The base frame RF_0 of a robot has its origin placed in the position ${}^W\boldsymbol{p}_0 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$ and is rotated by an angle $\beta = \pi/2$ [rad] around the $\boldsymbol{z}_w$ axis of the world frame RF_W. In a given configuration, the end-effector pose of the robot is given by

$${}^0\boldsymbol{T}_E = \begin{pmatrix} 0 & 0.5 & -\frac{\sqrt{3}}{2} & 1 \\ 1 & 0 & 0 & -0.75 \\ 0 & -\frac{\sqrt{3}}{2} & -0.5 & 1.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The position of the tip of a tool mounted on the end-effector is ${}^E\boldsymbol{p}_{tool} = \begin{pmatrix} 0 & 0.3 & 0.3 \end{pmatrix}^T$ [m]. Moreover, the tool frame RF_{tool} associated to this point is rotated by an angle $\gamma = -\pi/2$ [rad] around the $\boldsymbol{x}_E$ axis of the end-effector frame RF_E. Compute the position of the tip of the tool in the world frame and the absolute orientation of the tool frame w.r.t. RF_W.

Reply #10

The result is obtained by multiplying three homogeneous 4×4 matrices, the given ${}^0\boldsymbol{T}_E$ associated to the robot direct kinematics expressed in its base frame, and the two world-to-base and end effector-to-tool transformations

$${}^W\boldsymbol{T}_0 = \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{z}_w}(\beta = \frac{\pi}{2}) & {}^W\boldsymbol{p}_0 \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and, respectively,

$$ {}^{E}\boldsymbol{T}_{tool} = \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{x}_E}(\gamma = -\dfrac{\pi}{2}) & {}^{E}\boldsymbol{p}_{tool} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.3 \\ 0 & -1 & 0 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Therefore

$$ {}^{W}\boldsymbol{T}_{tool} = {}^{W}\boldsymbol{T}_0\, {}^{0}\boldsymbol{T}_E\, {}^{E}\boldsymbol{T}_{tool} = \begin{pmatrix} -1 & 0 & 0 & 1.75 \\ 0 & 0.8660 & 0.5 & 1.8902 \\ 0 & 0.5 & -0.8660 & 1.0902 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^{W}\boldsymbol{R}_{tool} & {}^{W}\boldsymbol{p}_{tool} \\ \boldsymbol{0}^T & 1 \end{pmatrix}. \quad \blacksquare $$

* * * * *