Robotics 1

January 24, 2024

Exercise 1

Consider a sequence of three rotations by the angles α, β, and γ around the fixed axes ZXY.

- Provide in symbolic form the rotation matrix $\boldsymbol{R}(\alpha, \beta, \gamma)$ representing the obtained final orientation.
- Solve the inverse problem in closed form for a generic $\boldsymbol{R} \in SO(3)$, including also singular situations.
- Compute all numerical solutions $\{\alpha, \beta, \gamma\}$ of the inverse problem (and check the results!) when

$$\boldsymbol{R} = \begin{pmatrix} \frac{5\sqrt{2}}{8} & \frac{\sqrt{2}}{8} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{6}}{8} & -\frac{3\sqrt{6}}{8} & \frac{1}{4} \end{pmatrix}. \tag{1}$$

- Express the orientation (1) with respect to the frame defined by a sequence of two rotations around the fixed axes ZX obtained using one pair $\{\alpha, \beta\}$ from the solution angles of the above inverse problem.

Exercise 2

The Yaskawa Motoman GP7 shown in Fig. 1 and in the accompanying Extra Sheet is a 6R robot manipulator with a spherical wrist and some offsets. The six joints are also labeled by the manufacturer (in sequence): S, L, U, R, B, T. Define a set of Denavit-Hartenberg (D-H) frames and compute the corresponding table of parameters. The D-H frame RF_0 should be placed on the floor at the robot base, while the origin O_6 of RF_6 is at the center of the end-effector flange and axis $\boldsymbol{z}_6$ is in the approach direction.

Draw the frames on the Extra Sheet, using the side, front, and top views (for better clarity, draw on each view only those DH axes that lie in the associated plane). Provide the numerical values of the constant parameters and the values of the joint variables $\boldsymbol{q}$ in the configuration shown in the sheet. Compute then numerically the position of O_6 in this configuration, as well as in the configuration $\boldsymbol{q} = \boldsymbol{0}$.

Figure 1: Three views of the Yaskawa Motoman GP7 robot.

Exercise 3

For which values of θ_2 in the interval $(-\pi, \pi]$ has this equation real solutions in terms of the angle θ_1?

$$\sin\theta_1 + 2\cos(\theta_1 + \theta_2) = 2 \tag{2}$$

Exercise 4

The kinematics of a 3R spatial robot is defined through the D-H parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	$d_1 > 0$	q_1
2	0	$a_2 > 0$	0	q_2
3	$\pi/2$	$a_3 > 0$	0	q_3

Table 1: D-H parameters of a 3R spatial robot.

- Determine the 6×3 geometric Jacobian matrix $\boldsymbol{J}_g(\boldsymbol{q})$ in symbolic form of this robot.
- Find at least one singularity of its linear part (i.e., of the upper 3×3 matrix $\boldsymbol{J}_L(\boldsymbol{q})$).
- Determine all feasible directions for the angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ (using the lower 3×3 matrix $\boldsymbol{J}_A(\boldsymbol{q})$).
- For a point whose position ${}^3\boldsymbol{p}_D = \begin{pmatrix} 0 & 0 & D \end{pmatrix}^T$ is known and constant in the last D-H frame, compute the expression of its position ${}^0\boldsymbol{p}_D(\boldsymbol{q})$ in the base frame.
- Using the data $a_1 = a_3 = 0.04$, $a_2 = 0.445$, $d_1 = 0.33$, and $D = 0.52$ (all expressed in [m]), compute the velocity $\boldsymbol{v}_D = \dot{\boldsymbol{p}}_D \in \mathbb{R}^3$ in the base frame at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad] for $\dot{\boldsymbol{q}} = (0, \pi/4, \pi/2)$ [rad/s].

Exercise 5

Consider a 3R planar robot with links lengths $l_i > 0$ ($i = 1, 2, 3$) and D-H joint variables q_1 q_2, and q_3. For the task vector $\boldsymbol{r} = (\boldsymbol{p}, \alpha) = (p_x, p_y, \alpha) \in \mathbb{R}^3$, with $\boldsymbol{p}$ being the position of the end-effector and α its orientation angle with respect to the ${}^0\boldsymbol{x}$ axis, the following desired task trajectory $\boldsymbol{r}_d(t)$ is assigned:

$$\begin{aligned} p_{x,d}(t) &= x_0 + R\cos\alpha_d(t) \\ p_{y,d}(t) &= y_0 + R\sin\alpha_d(t) \\ \alpha_d(t) &= \omega\, t, \end{aligned} \tag{3}$$

with $R > 0$, $\omega > 0$, and $t \in [0, \infty)$.

- Determine the analytic expressions of the associated desired joint trajectory $\boldsymbol{q}_d(t)$, and of its velocity $\dot{\boldsymbol{q}}_d(t)$ and acceleration $\ddot{\boldsymbol{q}}_d(t)$.
- Using the data $l_1 = l_2 = l_3 = 1$ [m], $x_0 = y_0 = 1$ [m], $R = 0.5$ m, and $\omega = 2\pi$ rad/s, determine the numerical values of $\boldsymbol{q}_d(\bar{t})$, $\dot{\boldsymbol{q}}_d(\bar{t})$, and $\ddot{\boldsymbol{q}}_d(\bar{t})$ at time $\bar{t} = 0.25$ s.
- Check that the obtained results are consistent with those of the desired task trajectory $\boldsymbol{r}_d(t)$ and of its first and second derivatives at the same time instant.

Exercise 6

- Define a trajectory $q(t)$ for a robot joint that should start at rest from q_i at a given time t_i and arrive at time t_f in q_f with a final velocity $v_f \neq 0$. All the symbolic values are here generic.
- For a motion time $T = t_f - t_i$, find $v_{max} = \max_{t \in [t_i, t_f]} |\dot{q}(t)|$, i.e., the maximum absolute value of the joint velocity, and the instant of time $t^* \in [t_i, t_f]$ at which this value is attained.
- Compute the value of v_{max} (in [rad/s]) and the instant t^* (in [s]) for the following set of data: $t_i = 1.5$ s, $t_f = 2$ s, $q_i = \pi/2$ rad, $q_f = \pi$ rad, $v_f = -4$ rad/s.

[5 hours; open books]

Solution

January 24, 2024

Exercise 1

The orientation obtained with three rotations around the sequence of *fixed* axes ZXY with angles α, β, and γ is given by

$$
\begin{aligned}
\boldsymbol{R}_{ZXY}(\alpha,\beta,\gamma) &= \boldsymbol{R}_{\boldsymbol{y}}(\gamma)\boldsymbol{R}_{\boldsymbol{x}}(\beta)\boldsymbol{R}_{\boldsymbol{z}}(\alpha) \\
&= \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\beta s_\gamma \\ s_\alpha c_\beta & c_\alpha c_\beta & -s_\beta \\ s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma & c_\beta c_\gamma \end{pmatrix}.
\end{aligned}
\tag{4}
$$

Given a matrix $\boldsymbol{R} \in SO(3)$, with elements denoted by R_{ij}, the inverse problem for this RPY-type sequence of angles is solved in closed form as follows. The second angle is obtained comparing the elements of the second row in (4) with those in $\boldsymbol{R}$:

$$\beta^{+,-} = \text{ATAN2}\left\{-R_{23}, \pm\sqrt{R_{21}^2 + R_{22}^2}\right\} \tag{5}$$

If $R_{21}^2 + R_{22}^2 = \cos^2\beta \neq 0$, we are in the regular case (two solution triples). The remaining angles are then found as

$$\alpha^+ = \text{ATAN2}\left\{R_{21}, R_{22}\right\}, \qquad \gamma^+ = \text{ATAN2}\left\{R_{13}, R_{33}\right\}, \tag{6}$$

when the + sign ($\cos\beta > 0$) has been chosen in (5), and as

$$\alpha^- = \text{ATAN2}\left\{-R_{21}, -R_{22}\right\}, \qquad \gamma^- = \text{ATAN2}\left\{-R_{13}, -R_{33}\right\}, \tag{7}$$

when the $-$ sign ($\cos\beta < 0$) has been chosen in (5). From eqs. (5)–(7), the two solution triples are $\{\alpha^+, \beta^+, \gamma^+\}$ and $\{\alpha^-, \beta^-, \gamma^-\}$.

In the singular case, $R_{21} = R_{22} = \cos\beta = 0$, one has the identity[1]

$$\begin{pmatrix} c_\alpha c_\gamma \pm s_\alpha s_\gamma & \pm c_\alpha s_\gamma - s_\alpha c_\gamma & 0 \\ 0 & 0 & -s_\beta \\ \pm s_\alpha c_\gamma - c_\alpha s_\gamma & s_\alpha s_\gamma \pm c_\alpha c_\gamma & 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha \mp \gamma) & -\sin(\alpha \mp \gamma) & 0 \\ 0 & 0 & -s_\beta \\ \pm\sin(\alpha \mp \gamma) & \pm\cos(\alpha \mp \gamma) & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & 0 \\ 0 & 0 & \mp 1 \\ R_{31} & R_{32} & 0 \end{pmatrix}.$$

If $R_{23} = -1$, then it is $\beta = \pi/2$ and one can solve only for the difference $\alpha - \gamma = \text{ATAN2}\{-R_{12}, R_{11}\}$. If instead $R_{23} = 1$, then it is $\beta = -\pi/2$ and one can solve only for the sum $\alpha + \gamma = \text{ATAN2}\{-R_{12}, R_{11}\}$.

Considering the rotation matrix $\boldsymbol{R}$ in (1), it is easy to see that we are in the regular case (with $c_\beta = 0.5$). Thus, the two solution triples are

$$\{\alpha^+, \beta^+, \gamma^+\} = \{0.7854, -1.0472, -1.0472\} = \{\pi/4, -\pi/3, -\pi/3\} \text{ [rad]} \tag{8}$$

and

$$\{\alpha^-, \beta^-, \gamma^-\} = \{-2.3562, -2.0944, 2.0944\} = \{-3\pi/4, -2\pi/3, 2\pi/3\} \text{ [rad]}. \tag{9}$$

When inserted in (4) as a check, both solutions return as expected the given $\boldsymbol{R}$.

The orientation obtained with two rotations by some angles α and β around the sequence of *fixed* axes ZX is given by $\boldsymbol{R}_{ZX} = \boldsymbol{R}_{\boldsymbol{x}}(\beta)\boldsymbol{R}_{\boldsymbol{z}}(\alpha)$. The expression of a generic orientation $\boldsymbol{R}$ with respect to such rotated

[1] Use of the $\mp$ signs: take always either the upper sign or the lower sign in *all* terms.

frame[2] is given then by ${}^{ZX}\boldsymbol{R} = \boldsymbol{R}_{ZX}^T\,\boldsymbol{R}$. For the matrix in (1), using the elementary rotation matrices in (4) and the values (α^+, β^+) from (8) and, respectively, (α^-, β^-) from (9) gives

$$
{}^{ZX}\boldsymbol{R}^+ = \begin{pmatrix} 0.5625 & 0.8125 & -0.1531 \\ -0.6875 & 0.5625 & 0.4593 \\ 0.4593 & -0.1531 & 0.8750 \end{pmatrix} \quad \text{and} \quad {}^{ZX}\boldsymbol{R}^- = \begin{pmatrix} -0.3125 & -0.5625 & 0.7655 \\ 0.9375 & -0.3125 & 0.1531 \\ 0.1531 & 0.7655 & 0.6250 \end{pmatrix}.
$$

Exercise 2

Views of a possible assignment of D-H frames for the 6R Yaskawa robot of Fig. 1 are shown in Fig. 2.

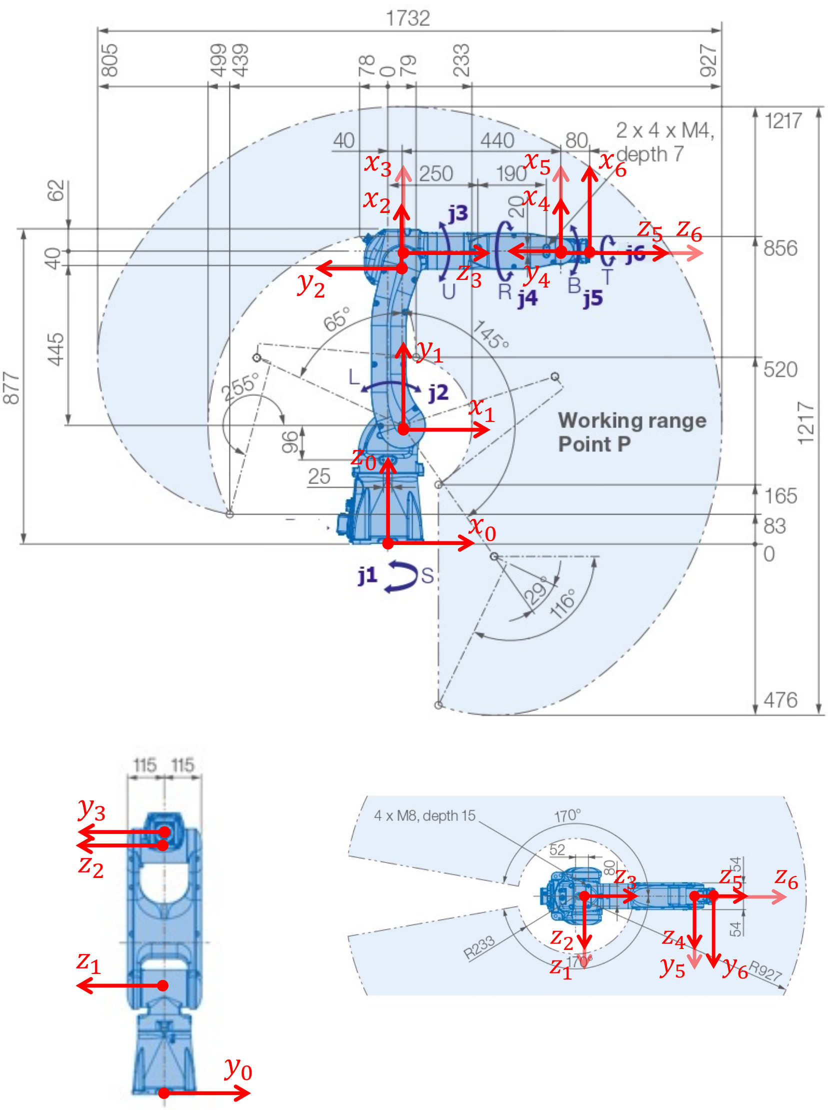

Figure 2: Side, front, and top views of the D-H frame assignment for the Yaskawa robot of Fig. 1.

[2]As usual, when no left superscript is present in a vector or matrix, the default is that this quantity is expressed in an absolute reference frame, say frame 0.

The corresponding D-H parameters are reported in Tab. 2. Note that the parameter d_1 is computed from the data sheet as follows $d_1 = 877 - (445 + 40 + 62) = 330$ mm.

i	α_i	a_i	d_i	θ_i
1 (S)	$\pi/2$	$a_1 = 40$	$d_1 = 330$	$q_1 = 0$
2 (L)	0	$a_2 = 445$	0	$q_2 = \pi/2$
3 (U)	$\pi/2$	$a_3 = 40$	0	$q_3 = 0$
4 (R)	$-\pi/2$	0	$d_4 = 440$	$q_4 = 0$
5 (B)	$\pi/2$	0	0	$q_5 = 0$
6 (T)	0	0	$d_6 = 80$	$q_6 = 0$

Table 2: Table of D-H parameters corresponding to the frames of Fig. 2 for the Yaskawa robot (angles are in [rad], lengths in [mm]). The numerical values of $\boldsymbol{q}$ refer to the configuration shown in the Extra Sheet.

The numerical values in the last column in the table correspond to the robot configuration $\boldsymbol{q}_s$ shown in Fig. 2. In this configuration, the position of the origin O_6 (as well as the orientation of the D-H frame 6 — which was not requested) are computed through the direct kinematics of the robot as

$$ {}^0\boldsymbol{p}_6(\boldsymbol{q}_s) = \begin{pmatrix} 560 \\ 0 \\ 815 \end{pmatrix} \text{[mm]}, \qquad {}^0\boldsymbol{R}_6(\boldsymbol{q}_s) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. $$

Similarly, in $\boldsymbol{q} = \boldsymbol{0}$ we have

$$ {}^0\boldsymbol{p}_6(\boldsymbol{0}) = \begin{pmatrix} 525 \\ 0 \\ -190 \end{pmatrix} \text{[mm]}, \qquad {}^0\boldsymbol{R}_6(\boldsymbol{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. $$

Exercise 3

Expand the cosine function in (2) to get

$$ \sin\theta_1 + 2\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) = 2, $$

which is of the form

$$ a\sin\theta_1 + b\cos\theta_1 = c, \tag{10} $$

with

$$ a = 1 - 2\sin\theta_2, \qquad b = 2\cos\theta_2, \qquad c = 2. $$

The transcendental eq. (10) has already been studied in the lecture slides (InverseKinematics.pdf, slide #13). From there, we know that this equation has (one or two) real solutions if and only if

$$ a^2 + b^2 \geq c^2 \qquad \Rightarrow \qquad (1 - 2\sin\theta_2)^2 + 4\cos^2\theta_2 \geq 4, \tag{11} $$

or

$$ \sin\theta_2 \leq 0.25 \qquad \Rightarrow \qquad \theta_2 \in (-\pi, 0.2526] \cup [\pi - 0.2526, \pi] \text{ rad}. \tag{12} $$

Under the condition (11), viz. (12), the solutions to (10) are computed as

$$ \theta_1^{+/-} = 2\arctan\frac{a \pm \sqrt{a^2 + b^2 - c^2}}{b + c} = 2\arctan\frac{1 - 2\sin\theta_2 \pm \sqrt{1 - 4\sin\theta_2}}{2(1 + \cos\theta_2)}. \tag{13} $$

For instance, when $\theta_2 = 0$, eq. (2) becomes

$$\sin\theta_1 + 2\cos\theta_1 = 2,$$

which has the two real solutions

$$\begin{aligned}\theta_1^+ &= 2\arctan\left(\frac{1+1}{4}\right) = 2\arctan 0.5 = 0.9273 \text{ rad},\\ \theta_1^- &= 2\arctan\left(\frac{1-1}{4}\right) = 2\arctan 0 = 0.\end{aligned}$$

On the other hand, eq. (2) has a single solution when $\sin\theta_2 = 0.25$. In particular, for $\theta_2 = 0.2526$, the equation becomes

$$0.5\sin\theta_1 + 1.9365\cos\theta_1 = 2,$$

and has the single solution

$$\theta_1 = 0.2499 \text{ rad};$$

similarly, for $\theta_2 = \pi - 0.2526$, the equation becomes

$$0.5\sin\theta_1 - 1.9365\cos\theta_1 = 2,$$

with the single solution

$$\theta_1 = \pi - 0.2499 = 2.8917 \text{ rad}.$$

Exercise 4

From Tab. 1, we compute the D-H homogeneous transformation matrices of this 3R spatial robot:

$${}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 & a_1c_1\\ s_1 & 0 & -c_1 & a_1s_1\\ 0 & 1 & 0 & d_1\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2\\ s_2 & c_2 & 0 & a_2s_2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad {}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} c_3 & 0 & s_3 & a_3c_3\\ s_3 & 0 & -c_3 & a_3s_3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

From these, being the end-effector position coincident with the origin O_3 of frame 3, we obtain all the quantities needed for computing the geometric Jacobian $\boldsymbol{J}_g(\boldsymbol{q})$ as

$$\boldsymbol{J}_g(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q})\\ \boldsymbol{J}_A(\boldsymbol{q})\end{pmatrix} = \begin{pmatrix} \boldsymbol{z}_0\times\boldsymbol{p}_{03} & \boldsymbol{z}_1\times\boldsymbol{p}_{13} & \boldsymbol{z}_2\times\boldsymbol{p}_{23}\\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2\end{pmatrix}, \tag{14}$$

with all quantities being expressed by default in frame 0. For better clarity, we will insert left superscripts in the following. We have

$$\boldsymbol{z}_0 = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}, \quad \boldsymbol{z}_1 = {}^0\boldsymbol{R}_1(q_1)\boldsymbol{z}_0 = \begin{pmatrix} s_1\\ -c_1\\ 0\end{pmatrix}, \quad \boldsymbol{z}_2 = {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\boldsymbol{z}_0 = \begin{pmatrix} s_1\\ -c_1\\ 0\end{pmatrix}, \tag{15}$$

and for the position vectors

$${}^0\boldsymbol{p}_{01,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_{01}(\boldsymbol{q})\\ 1\end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix} \boldsymbol{0}\\ 1\end{pmatrix} = \begin{pmatrix} a_1c_1\\ a_1s_1\\ d_1\\ 1\end{pmatrix},$$

$${}^0\boldsymbol{p}_{02,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_{02}(\boldsymbol{q})\\ 1\end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\begin{pmatrix} \boldsymbol{0}\\ 1\end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_1c_2\\ a_1s_1 + a_2s_1c_2\\ d_1 + a_2s_2\\ 1\end{pmatrix},$$

$$
{}^0\boldsymbol{p}_{03,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(a_1 + a_2 c_2 + a_3 c_{23}) \\ s_1(a_1 + a_2 c_2 + a_3 c_{23}) \\ d_1 + a_2 s_2 + a_3 s_{23} \\ 1 \end{pmatrix}. \tag{16}
$$

From these, we get

$$
{}^0\boldsymbol{p}_{13}(\boldsymbol{q}) = {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) - {}^0\boldsymbol{p}_{01}(\boldsymbol{q}) = \begin{pmatrix} c_1(a_2 c_2 + a_3 c_{23}) \\ s_1(a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{pmatrix}, \quad {}^0\boldsymbol{p}_{23}(\boldsymbol{q}) = {}^0\boldsymbol{p}_{03}(\boldsymbol{q}) - {}^0\boldsymbol{p}_{02}(\boldsymbol{q}) = \begin{pmatrix} a_3 c_1 c_{23} \\ a_3 s_1 c_{23} \\ a_3 s_{23} \end{pmatrix}.
$$

Performing now the cross products in (14), we obtain

$$
\boldsymbol{J}_L(\boldsymbol{q}) = \begin{pmatrix} -s_1(a_1 + a_2 c_2 + a_3 c_{23}) & -c_1(a_2 s_2 + a_3 s_{23}) & -a_3 c_1 s_{23} \\ c_1(a_1 + a_2 c_2 + a_3 c_{23}) & -s_1(a_2 s_2 + a_3 s_{23}) & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{pmatrix}. \tag{17}
$$

Indeed, this matrix could have been equivalently obtained by analytic differentiation as $\boldsymbol{J}_L(\boldsymbol{q}) = \partial \boldsymbol{p}_{03}/\partial \boldsymbol{q}$. Moreover, from (15)

$$
\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{18}
$$

The determinant of the linear part of the Jacobian is

$$
\det \boldsymbol{J}_L(\boldsymbol{q}) = -a_2 a_3 s_3 \,(a_1 + a_2 c_2 + a_3 c_{23})\,.
$$

Thus, matrix $\boldsymbol{J}_L(\boldsymbol{q})$ in (17) is singular when $s_3 = 0$ ($q_3 = 0$ or π) and/or when $a_1 + a_2 c_2 + a_3 c_{23} = 0$. In view of the expressions in (16), the latter corresponds to $p_{03,x} = p_{03,y} = 0$, namely to a situation in which the origin O_3 of frame 3 is placed on the axis $\boldsymbol{z}_0$ of the first joint.

On the other hand, matrix $\boldsymbol{J}_A(\boldsymbol{q})$ in (18) is always singular, with constant rank $\rho = 2$. Being $\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}$, all feasible directions at $\boldsymbol{q}$ for the angular velocity $\boldsymbol{\omega}$ of the third (last) D-H frame belong to the subspace

$$
\mathcal{R}\left\{\boldsymbol{J}_A(\boldsymbol{q})\right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}.
$$

Finally, the position ${}^3\boldsymbol{p}_D = \begin{pmatrix} 0 & 0 & D \end{pmatrix}^T$ of a fixed point in the third D-H frame is a (constant) vector from the origin O_3 to this point. To compute the vector ${}^0\boldsymbol{p}_D$ starting from the origin O_0 of the base frame and ending at the same point, and expressed in the 0-th frame, we resort to homogeneous transformations:

$$
{}^0\boldsymbol{p}_{D,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}_D(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{A}_3(q_3) \begin{pmatrix} {}^3\boldsymbol{p}_D \\ 1 \end{pmatrix} = \begin{pmatrix} c_1(a_1 + a_2 c_2 + a_3 c_{23} + D\,s_{23}) \\ s_1(a_1 + a_2 c_2 + a_3 c_{23} + D\,s_{23}) \\ d_1 + a_2 s_2 + a_3 s_{23} - D\,c_{23} \\ 1 \end{pmatrix}. \tag{19}
$$

Using the kinematic data of the robot at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad] and $D = 0.52$ m, we obtain

$$
{}^0\boldsymbol{p}_{03} = \begin{pmatrix} 0.04 \\ 0 \\ 0.815 \end{pmatrix} \qquad \text{and} \qquad {}^0\boldsymbol{p}_D = \begin{pmatrix} 0.56 \\ 0 \\ 0.815 \end{pmatrix} \quad \text{[m]}.
$$

The velocity of the point defined in (19) is easily obtained by time differentiation

$$
\begin{aligned}
\boldsymbol{v}_D = {}^0\dot{\boldsymbol{p}}_D(\boldsymbol{q}) &= \frac{\partial\,{}^0\boldsymbol{p}_D}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}_D(\boldsymbol{q})\,\dot{\boldsymbol{q}} \\
&= \begin{pmatrix} -s_1(a_1 + a_2c_2 + a_3c_{23} + D\,s_{23}) & -c_1(a_2s_2 + a_3s_{23} + D\,c_{23}) & -a_3c_1s_{23} + D\,c_{23} \\ c_1(a_1 + a_2c_2 + a_3c_{23} + D\,s_{23}) & -s_1(a_2s_2 + a_3s_{23} + D\,c_{23}) & -a_3s_1s_{23} + D\,c_{23} \\ 0 & a_2c_2 + a_3c_{23} + D\,s_{23} & a_3c_{23} + D\,s_{23} \end{pmatrix} \dot{\boldsymbol{q}}.
\end{aligned} \tag{20}
$$

Plugging in the robot data, evaluating the Jacobian $\boldsymbol{J}_D(\boldsymbol{q})$ at $\boldsymbol{q} = (0, \pi/2, 0)$ [rad], and commanding the joint velocity $\dot{\boldsymbol{q}} = (0, \pi/4, \pi/2)$ [rad/s], we obtain from eq. (20)

$$
\boldsymbol{v}_D = \begin{pmatrix} 0 & -0.4850 & -0.0400 \\ 0.5600 & 0 & 0 \\ 0 & 0.5200 & 0.5200 \end{pmatrix} \begin{pmatrix} 0 \\ 0.7854 \\ 1.5708 \end{pmatrix} = \begin{pmatrix} -0.4437 \\ 0 \\ 1.2252 \end{pmatrix} \text{ [m/s]}.
$$

Exercise 5

In the first place, we have to solve the inverse kinematics problem for this 3R planar robot, repeatedly and parametrically with respect to the desired trajectory $\boldsymbol{r}_d(t) = (p_{x,d}(t), p_{y,d}(t), \alpha_d(t))$. This is a standard problem at each instant of time t (dropped for compactness in the following).

The direct task kinematics is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{21}
$$

Setting $\boldsymbol{r} = \boldsymbol{r}_d$ in (21) and using the third equation in the first two, one has

$$
\begin{pmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \end{pmatrix} = \begin{pmatrix} p_{x,d} - l_3\,c\alpha_d \\ p_{y,d} - l_3\,s\alpha_d \end{pmatrix}, \tag{22}
$$

with the shorthand notations $c_1 = \cos q_1$, $c_{12} = \cos(q_1 + q_2)$, $c\alpha_d = \cos\alpha_d$ — similarly for the sines. By squaring and summing the two equations in (22), we get

$$
c_{2,d} = \frac{p_{x,d}^2 + p_{y,d}^2 + l_3^2 - 2l_3(p_{x,d}\,c\alpha_d + p_{y,d}\,s\alpha_d) - l_1^2 - l_2^2}{2l_1l_2}, \qquad s_{2,d} = \sqrt{1 - c_{2,d}^2}, \tag{23}
$$

where only the + sign has been considered for $s_{2,d}$ (similar developments hold for the choice $s_{2,d} < 0$). Thus,

$$
q_{2,d} = \text{ATAN2}\left\{s_{2,d}, c_{2,d}\right\}. \tag{24}
$$

This corresponds to an 'elbow down' solution for the first two joints. Using the expressions in (23), eq. (22) is expanded as a linear system in the remaining unknowns c_1 and s_1:

$$
\begin{pmatrix} l_1 + l_2\,c_{2,d} & -l_2\,s_{2,d} \\ l_2\,s_{2,d} & l_1 + l_2\,c_{2,d} \end{pmatrix} \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} p_{x,d} - l_3\,c\alpha_d \\ p_{y,d} - l_3\,s\alpha_d \end{pmatrix}. \tag{25}
$$

Unless the determinant of the coefficient matrix in (25) is zero, namely excluding when $l_1^2 + l_2^2 + 2l_1l_2c_{2,d} = 0$ (which happens if and only if $l_1 = l_2$ and $c_{2,d} = -\pi$, being the determinant always positive otherwise), one can solve for

$$
\begin{aligned}
c_{1,d} &= (l_1 + l_2c_{2,d})(p_{x,d} - l_3\,c\alpha_d) + l_2s_{2,d}(p_{y,d} - l_3\,s\alpha_d) \\
s_{1,d} &= (l_1 + l_2c_{2,d})(p_{y,d} - l_3\,s\alpha_d) - l_2s_{2,d}(p_{x,d} - l_3\,c\alpha_d),
\end{aligned} \tag{26}
$$

and then

$$
q_{1,d} = \text{ATAN2}\left\{s_{1,d}, c_{1,d}\right\}. \tag{27}
$$

Finally,

$$q_{3,d} = \alpha_d - (q_{1,d} + q_{2,d}). \tag{28}$$

The analytic expressions of the components of the desired joint trajectory $\boldsymbol{q}_d(t)$ at any instant of time are obtained from eqs. (23)–(24), (26)–(27) and (28), by plugging the desired task trajectory values from (3).

Moving to the differential level, the task Jacobian associated to (21) is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & -(l_2 s_{12} + l_3 s_{123}) & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix}. \tag{29}$$

Therefore, the joint velocity $\dot{\boldsymbol{q}}_d(t)$ along the desired task trajectory is computed as

$$\dot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{r}}_d(t), \qquad \text{with } \dot{\boldsymbol{r}}_d(t) = \begin{pmatrix} -\omega R \sin \omega t \\ \omega R \cos \omega t \\ \omega \end{pmatrix}, \tag{30}$$

where the Jacobian (29) is first evaluated at $\boldsymbol{q}_d(t)$ and then inverted numerically, provided it is away from its singularities ($\det \boldsymbol{J}(\boldsymbol{q}) = l_1 l_2 \sin q_2 = 0$).

Similarly, at the acceleration level one has

$$\ddot{\boldsymbol{q}}_d(t) = \boldsymbol{J}^{-1}(\boldsymbol{q}_d(t)) \left(\ddot{\boldsymbol{r}}_d(t) - \dot{\boldsymbol{J}}(\boldsymbol{q}_d(t))\, \dot{\boldsymbol{q}}_d(t) \right), \qquad \text{with } \ddot{\boldsymbol{r}}_d(t) = - \begin{pmatrix} \omega^2 R \cos \omega t \\ \omega^2 R \sin \omega t \\ 0 \end{pmatrix}, \tag{31}$$

where the term

$$\begin{aligned} \dot{\boldsymbol{J}}(\boldsymbol{q})\, \dot{\boldsymbol{q}} &= - \begin{pmatrix} l_1 c_1 \dot{q}_1 + l_2 c_{12}(\dot{q}_1 + \dot{q}_2) + l_3 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & l_2 c_{12}(\dot{q}_1 + \dot{q}_2) + l_3 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & l_3 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ l_1 s_1 \dot{q}_1 + l_2 s_{12}(\dot{q}_1 + \dot{q}_2) + l_3 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & l_2 s_{12}(\dot{q}_1 + \dot{q}_2) + l_3 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & l_3 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ 0 & 0 & 0 \end{pmatrix} \dot{\boldsymbol{q}} \\ &= - \begin{pmatrix} l_1 c_1 \dot{q}_1^2 + l_2 c_{12}(\dot{q}_1 + \dot{q}_2)^2 + l_3 c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ l_1 s_1 \dot{q}_1^2 + l_2 s_{12}(\dot{q}_1 + \dot{q}_2)^2 + l_3 s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\ 0 \end{pmatrix} \end{aligned}$$

is evaluated using $\boldsymbol{q}_d(t)$, as well as $\dot{\boldsymbol{q}}_d(t)$ from (30).

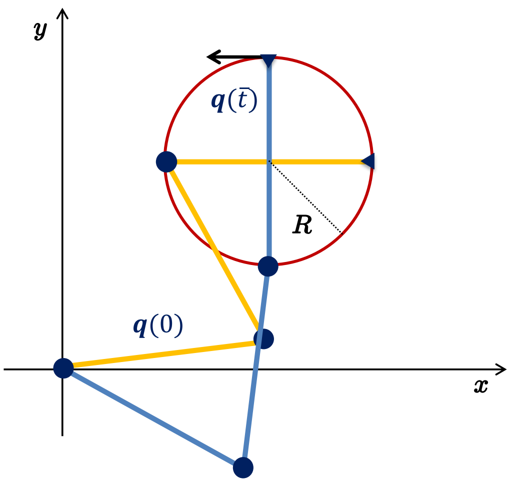

Figure 3: The 3R planar robot executing the desired task trajectory at $t = \bar{t} = 0.25$ s (in blue) and in the initial configuration $\boldsymbol{q}_d(0)$ (in orange).

With the problem data, being for the desired task trajectory at $t = \bar{t} = 0.25$ s

$$\boldsymbol{r}_d(\bar{t}) = \begin{pmatrix} 1 \\ 1.5 \\ \pi/2 \end{pmatrix} \text{[m,m,rad]}, \quad \dot{\boldsymbol{r}}_d(\bar{t}) = \begin{pmatrix} -\pi \\ 0 \\ 2\pi \end{pmatrix} \text{[m/s,m/s,rad/s]}, \quad \ddot{\boldsymbol{r}}_d(\bar{t}) = \begin{pmatrix} 0 \\ -2\pi^2 \\ 0 \end{pmatrix} \text{[m/s}^2\text{,m/s}^2\text{,rad/s}^2\text{]},$$

we obtain the following numerical values

$$\boldsymbol{q}_d(\bar{t}) = \begin{pmatrix} -0.5139 \\ 1.9552 \\ 0.1296 \end{pmatrix} \text{[rad]}, \qquad \dot{\boldsymbol{q}}_d(\bar{t}) = \begin{pmatrix} 0.4378 \\ -3.3889 \\ 9.2343 \end{pmatrix} \text{[rad/s]}, \qquad \ddot{\boldsymbol{q}}_d(\bar{t}) = \begin{pmatrix} 30.4316 \\ -16.6473 \\ -13.7843 \end{pmatrix} \text{[rad/s}^2\text{]}.$$

Figure 3 shows the robot configuration $\boldsymbol{q}_d(\bar{t})$ at the chosen time along the desired circular path, together with the initial (elbow down) configuration $\boldsymbol{q}_d(0) = (0.1296, 1.9552, -2.0847)$ [rad].

Exercise 6

The desired trajectory is found by solving a 'rest-to-move' interpolation problem between two configurations assigned at two given time instants. A cubic polynomial of the form $q(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ has enough free coefficients to satisfy all boundary conditions at the initial and final time. As customary, however, it is more convenient to use a normalized time

$$\tau = \frac{t - t_i}{t_f - t_i} = \frac{t - t_i}{T}, \qquad \text{with } \tau \in [0, 1] \text{ when } t \in [t_i, t_f],$$

and define the cubic as

$$q(\tau) = q_i + \Delta q \left(a\tau^2 + b\tau^3\right), \qquad \Delta q = q_f - q_i, \tag{32}$$

which already satisfies the boundary conditions at the initial time $\tau = 0$ ($t = t_i$) on position and (zero) velocity. Imposing the other two boundary conditions on the normalized cubic polynomial (32)

$$\begin{matrix} q(t_f) = q_f \\ \dot{q}(t_f) = v_f \end{matrix} \qquad \Rightarrow \qquad \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ v_f T/\Delta q \end{pmatrix},$$

and solving for (a, b) leads to the planned trajectory

$$q(\tau) = q_i + \Delta q \left((3 - \frac{v_f T}{\Delta q})\,\tau^2 + (-2 + \frac{v_f T}{\Delta q})\,\tau^3 \right), \tag{33}$$

with velocity and acceleration given respectively by

$$\dot{q}(\tau) = \frac{\Delta q}{T} \left(2\,(3 - \frac{v_f T}{\Delta q})\,\tau + 3\,(-2 + \frac{v_f T}{\Delta q})\,\tau^2 \right) \tag{34}$$

and

$$\ddot{q}(\tau) = \frac{\Delta q}{T^2} \left(2\,(3 - \frac{v_f T}{\Delta q}) + 6\,(-2 + \frac{v_f T}{\Delta q})\,\tau \right). \tag{35}$$

The maximum velocity (in absolute value) is reached either at the final instant $\tau^* = 1$, being $v_{max} = |v_f|$, or when the acceleration is zero, i.e.,

$$\ddot{q}(\tau^*) = 0 \qquad \Rightarrow \qquad \tau^* = \frac{3 - \dfrac{v_f T}{\Delta q}}{6 - 3\,\dfrac{v_f T}{\Delta q}} \tag{36}$$

as long $\tau^* \in (0, 1)$. Since $T = t_f - t_i > 0$, this condition is *always* satisfied if $v_f/\Delta q \leq 0$, namely when the displacement Δq and the final velocity v_f have opposite signs[3] (in particular, when $v_f = 0$, it is $\tau^* = 0.5$

[3] A solution to $\ddot{q}(\tau) = 0$ inside the interval of definition for τ exists also when $v_f/\Delta q > 0$, provided v_f is not too large in modulus. The details are left to the reader.

— at the trajectory midpoint). The velocity associated to the zero acceleration condition is found by substituting τ^* from (36) in (34), obtaining

$$\dot{q}(\tau^*) = \frac{\Delta q}{T}\,\tau^* \left((6 - 2\frac{v_f T}{\Delta q}) - (6 - 3\frac{v_f T}{\Delta q})\,\tau^* \right) = \frac{\Delta q}{T}\,\frac{\left(3 - \dfrac{v_f T}{\Delta q}\right)^2}{6 - 3\,\dfrac{v_f T}{\Delta q}}. \tag{37}$$

As a result, the maximum absolute velocity will be

$$v_{max} = \max\left\{|\dot{q}(\tau^*)|, |v_f|\right\}.$$

Note that the actual time instant of maximum velocity will be $t^* = t_i + \tau^* T \in [t_i, t_f]$.

With the data of the problem, one has $T = t_f - t_i = 2 - 1.5 = 0.5$ s, $\Delta q = q_f - q_i = \pi - \pi/2 = \pi/2$ rad, and $v_f = -4$ rad/s. Being $v_f/\Delta q < 0$, the above analysis applies and we obtain

$$\tau^* = 0.4352, \qquad t^* = 1.7176\,\text{s}, \qquad v_{max} = 5.8421\,\text{rad/s}.$$

The resulting cubic trajectory is shown in Fig. 4, together with the associated velocity and acceleration. Note the asymmetric profiles with respect to the middle instant of the motion trajectory, as well as the slight overshoot in position close to the final instant.

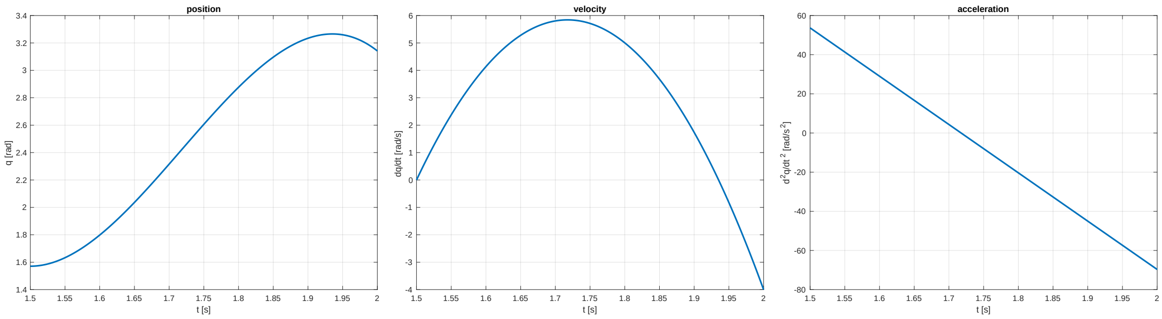

Figure 4: The planned cubic trajectory for the given data (position, velocity, and acceleration).

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