Robotics 1

July 12, 2021

Exercise #1

Consider the 4-dof spatial RRRP robot in Fig. 1. The robot has a shoulder and an elbow offset.

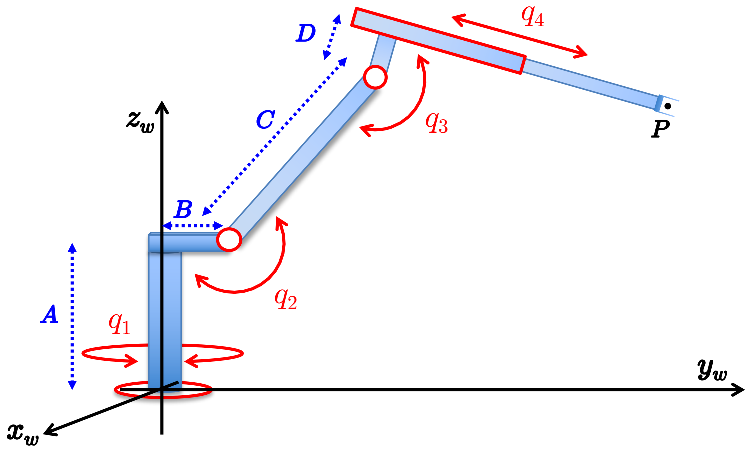

Figure 1: A spatial RRRP robot.

a) Assign a set of Denavit-Hartenberg (DH) frames and derive the associated table of parameters. Place the 0-th DH frame coincident with the world frame RF_w and the last DH frame with the origin in P and the $\boldsymbol{z}$ axis in the approach direction. Draw all the DH frames on the robot. Provide also the (approximate) values of the robot coordinates $\boldsymbol{q}$ in the shown configuration.

b) Compute in symbolic form the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the position of the end-effector. Derive the analytical Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this map.

c) Neglect from now on the shoulder and elbow offsets (i.e., set $B = D = 0$). For the resulting reduced Jacobian $\boldsymbol{J}_{red}(\boldsymbol{q})$, find (at least) a singular configuration $\boldsymbol{q}^*$. In such a configuration, define a *feasible* end-effector velocity $\boldsymbol{v}^* \in \mathbb{R}^3$ and find a joint velocity $\dot{\boldsymbol{q}}^* \in \mathbb{R}^4$ that realizes it.

Exercise #2

A robot link is actuated by a DC motor with rotor inertia $I_m = 1.5 \cdot 10^{-4}$ [kgm^2] via a double gearbox and a transmission shaft. An incremental encoder with $N = 2000$ pulses/turn is mounted on the motor axis (without extra electronics for quadrature count). A first gearbox with reduction ratio $n_{r1} = 10:1$ is placed at the motor output. This drives a long transmission shaft having rotational inertia $I_t = 0.5 \cdot 10^{-2}$ [kgm^2]. A second gearbox is placed at the end of the shaft with a reduction ratio $n_{r2} \geq 1$ which has *to be defined*. The final payload is the robot link, with an inertia around its rotation axis $I_\ell = 0.8 \cdot 10^{-1}$ [kgm^2]. Neglecting all dissipative effects, model this robot joint structure and find the value n_{r2} that minimizes the motor torque τ_m needed to accelerate the link by $\ddot{\theta}_\ell = a > 0$. Accordingly, determine the angular resolution $\Delta\theta_\ell$ of the link position provided by the encoder measurement on the motor side. For a bang-bang, rest-to-rest trajectory rotating the link by $\theta_{\ell,d} = -\pi/4$ in $T = 0.5$ s, find the maximum absolute value $\tau_{m,max}$ (in [Nm]) of the torque that the motor needs to produce.

Exercise #3

Plan a smooth rest-to-rest trajectory using a minimal representation of the orientation by means of ZYX Euler angles (α, β, γ) from the initial orientation

$$\boldsymbol{R}_1 = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

to the final orientation

$$\boldsymbol{R}_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

passing through the intermediate orientation

$$\boldsymbol{R}_{via} = \begin{pmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}.$$

In the inverse problems, use always the solution with the '+' sign when there is such an option. The time durations for the two subintervals are $T_1 = 2.5$ [s] (from $\boldsymbol{R}_1$ to $\boldsymbol{R}_{via}$) and $T_2 = 1$ [s] (from $\boldsymbol{R}_{via}$ to $\boldsymbol{R}_2$), with total motion time $T = T_1 + T_2$. The planned orientation trajectory should be continuous up to the acceleration for all $t \in (0, T)$ (so, everywhere except at the initial and final instants). At the end of the computations, sketch the time evolution of the three Euler angles $(\alpha(t), \beta(t), \gamma(t))$ and check whether or not a representation singularity is encountered during the planned motion.

[180 minutes (3 hours); open books]

Solution

July 12, 2021

Exercise #1

A possible assignment of Denavit-Hartenberg frames for the 4-dof RRRP robot is shown in Fig. 2. The associated parameters are reported in Tab. 1.

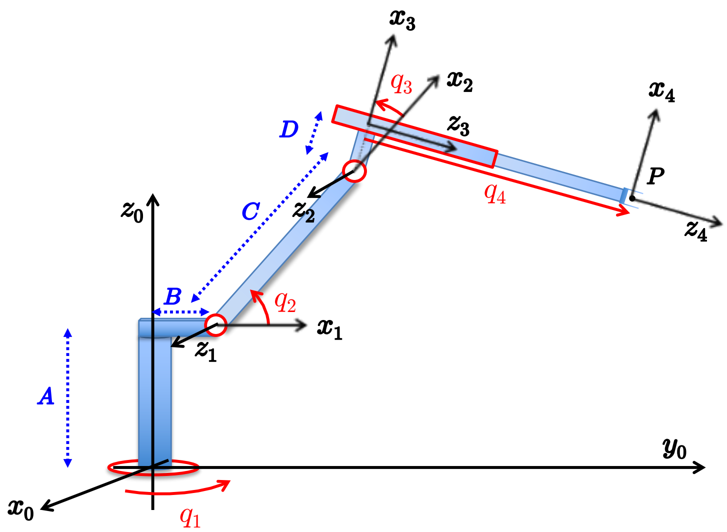

Figure 2: A possible assignment of DH frames for the 4-dof RRRP robot of Fig. 1.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	B	A	$q_1 = \pi/2$
2	0	C	0	$q_2 \simeq \pi/4$
3	$\pi/2$	D	0	$q_3 > 0$
4	0	0	$q_4 > 0$	0

Table 1: DH table of parameters corresponding to Fig. 2. The joint variables q_i (in red) take values associated to the robot configuration shown in the same figure.

With the data in Tab. 1, we construct the homogenous transformation matrices ${}^{i-1}\boldsymbol{A}_i(q_i)$, for $i = 1, \dots, 4$. The position of the robot end-effector in homogeneous coordinates is efficiently computed as

$$\boldsymbol{p}_H = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\, \boldsymbol{A}_2(q_2)\, {}^2\boldsymbol{A}_3(q_3)\, {}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

yielding

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 \left(B + C\cos q_2 + D\cos(q_2+q_3) + q_4 \sin(q_2+q_3)\right) \\ \sin q_1 \left(B + C\cos q_2 + D\cos(q_2+q_3) + q_4 \sin(q_2+q_3)\right) \\ A + C\sin q_2 + D\sin(q_2+q_3) - q_4\cos(q_2+q_3) \end{pmatrix}.$$

Using the usual shorthand notation, the analytic Jacobian is thus

$$
\begin{aligned}
\boldsymbol{J}(\boldsymbol{q}) &= \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \\
&= \begin{pmatrix} -s_1\left(B + Cc_2 + Dc_{23} + q_4 s_{23}\right) & -c_1\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -c_1\left(Ds_{23} - q_4 c_{23}\right) & c_1 s_{23} \\ c_1\left(B + Cc_2 + Dc_{23} + q_4 s_{23}\right) & -s_1\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -s_1\left(Ds_{23} - q_4 c_{23}\right) & s_1 s_{23} \\ 0 & Cc_2 + Dc_{23} + q_4 s_{23} & Dc_{23} + q_4 s_{23} & -c_{23} \end{pmatrix}.
\end{aligned} \tag{1}
$$

For singularity analysis, it is also very convenient to work with the Jacobian matrix expressed in the (rotated) first DH frame:

$$
\begin{aligned}
{}^1\boldsymbol{J}(\boldsymbol{q}) &= {}^0\boldsymbol{R}_1^T(q_1)\,\boldsymbol{J}(\boldsymbol{q}) \\
&= \begin{pmatrix} 0 & -\left(Cs_2 + Ds_{23} - q_4 c_{23}\right) & -\left(Ds_{23} - q_4 c_{23}\right) & s_{23} \\ B + Cc_2 + Dc_{23} + q_4 s_{23} & 0 & 0 & 0 \\ 0 & Cc_2 + Dc_{23} + q_4 s_{23} & Dc_{23} + q_4 s_{23} & -c_{23} \end{pmatrix}.
\end{aligned} \tag{2}
$$

By neglecting now the shoulder offset ($B = 0$) and the elbow offset ($D = 0$), we obtain from (1) and (2)

$$
\boldsymbol{J}_{red}(\boldsymbol{q}) = \begin{pmatrix} -s_1\left(Cc_2 + q_4 s_{23}\right) & -c_1\left(Cs_2 - q_4 c_{23}\right) & q_4 c_1 c_{23} & c_1 s_{23} \\ c_1\left(Cc_2 + q_4 s_{23}\right) & -s_1\left(Cs_2 - q_4 c_{23}\right) & q_4 s_1 c_{23} & s_1 s_{23} \\ 0 & Cc_2 + q_4 s_{23} & q_4 s_{23} & -c_{23} \end{pmatrix},
$$

and, respectively,

$$
{}^1\boldsymbol{J}_{red}(\boldsymbol{q}) = \begin{pmatrix} 0 & -\left(Cs_2 - q_4 c_{23}\right) & q_4 c_{23} & s_{23} \\ Cc_2 + q_4 s_{23} & 0 & 0 & 0 \\ 0 & Cc_2 + q_4 s_{23} & q_4 s_{23} & -c_{23} \end{pmatrix}. \tag{3}
$$

The kinematic singularities of the reduced Jacobian are characterized by

$$
\operatorname{rank}\left\{\boldsymbol{J}_{red}(\boldsymbol{q})\right\} = \operatorname{rank}\left\{{}^1\boldsymbol{J}_{red}(\boldsymbol{q})\right\} < 3.
$$

It is easy to see from (3) that the singularities are of two kinds:

(I)	$Cc_2 + q_4 s_{23} = 0$	$\Longleftrightarrow$	the end-effector point P is on the axis of joint 1;
(II)	$q_4 = 0$ and $s_3 = 0$	$\Longleftrightarrow$	the prismatic joint is fully retracted and link 2 and 4 are orthogonal to each other.

The last question is on generating a joint velocity solution that realizes a desired feasible end-effector velocity in a singular configuration. Two examples are provided next for illustration.

- $\boldsymbol{q}^* = (\pi/2, 0, 0, 0)$ (simple singularity of type II)

  The robot is in the configuration sketched in Fig. 3 [left]. The reduced Jacobian is

$$
\boldsymbol{J}^*_{red} = \boldsymbol{J}_{red}(\boldsymbol{q}^*) = \begin{pmatrix} -C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C & 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \operatorname{rank}\left\{\boldsymbol{J}^*_{red}\right\} = 2.
$$

A feasible end-effector velocity $\boldsymbol{v}^* \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}^* \in \mathbb{R}^4$ that will realize it are

$$\boldsymbol{v}^* = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}^*_{red}\} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}^* = \boldsymbol{J}^{\#}_{red}(\boldsymbol{q}^*)\, \boldsymbol{v}^* = \begin{pmatrix} 0 \\ -\frac{C}{C^2+1} \\ 0 \\ \frac{1}{C^2+1} \end{pmatrix},$$

where the pseudoinverse $\boldsymbol{J}^{\#}_{red}$ of $\boldsymbol{J}_{red}$ has been computed numerically using MATLAB (but it is easy to guess its full expression also by inspection). Another possible solution is given simply by $\dot{\boldsymbol{q}}^{*\prime} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T$.

- $\boldsymbol{q}^{**} = (\pi/2, \pi/2, 0, 0)$ (double singularity: type I and II together)

  The robot is in the configuration sketched in Fig. 3 [right]. The reduced Jacobian is

$$\boldsymbol{J}^{**}_{red} = \boldsymbol{J}_{red}(\boldsymbol{q}^{**}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\,\{\boldsymbol{J}^{**}_{red}\} = 1.$$

  A feasible end-effector velocity $\boldsymbol{v}^{**} \in \mathbb{R}^3$ and a joint velocity $\dot{\boldsymbol{q}}^{**} \in \mathbb{R}^4$ that will realize it are then

$$\boldsymbol{v}^{**} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}^{**}_{red}\} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}}^{**} = \boldsymbol{J}^{\#}_{red}(\boldsymbol{q}^{**})\, \boldsymbol{v}^{**} = \begin{pmatrix} 0 \\ -\frac{C}{C^2+1} \\ 0 \\ \frac{1}{C^2+1} \end{pmatrix} (= \dot{\boldsymbol{q}}^*!),$$

  Another possible solution is given simply by $\dot{\boldsymbol{q}}^{**\prime} = \begin{pmatrix} 0 & -1/C & 0 & 0 \end{pmatrix}^T$.

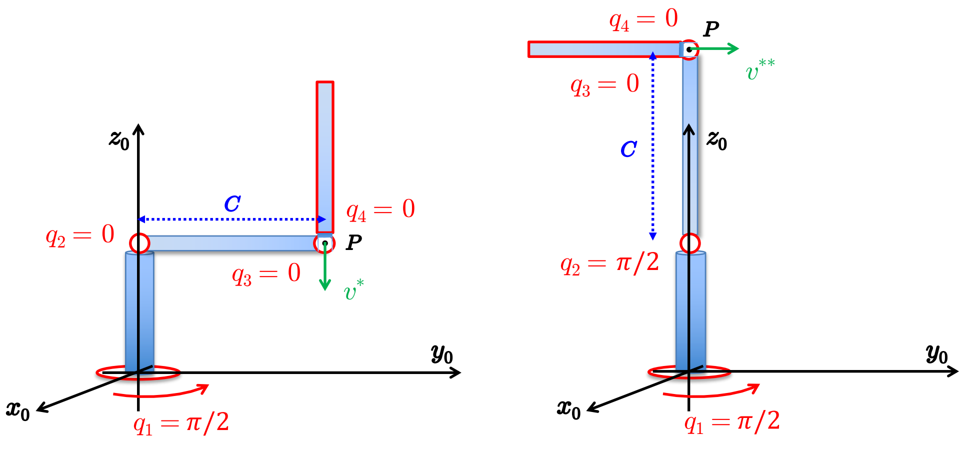

Figure 3: Two singular configurations of the 4-dof RRRP robot with two associated feasible end-effector velocities $\boldsymbol{v}_d$: [left] simple singularity; [right] double singularity.

Exercise #2

Figure 4 shows the motor/sensor/transmission/link arrangement of the considered robot joint.

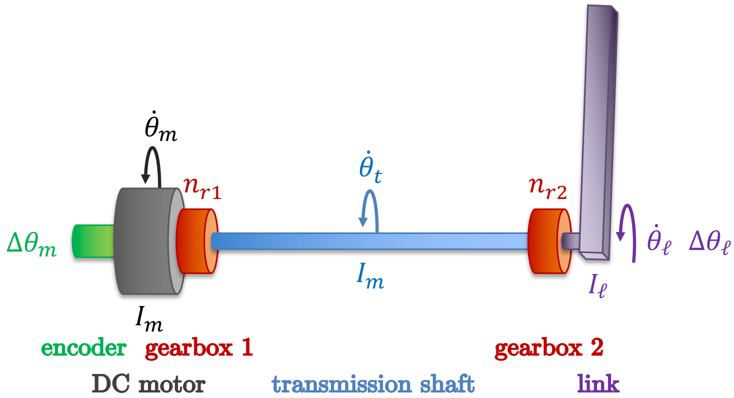

Figure 4: The robot joint with definition of quantities.

The torque balance on the motor axis is given by

$$\tau_m = I_m \ddot{\theta}_m + \frac{1}{n_{r1}} \left(I_t \ddot{\theta}_t \right) + \frac{1}{n_{r1} n_{r2}} \left(I_\ell \ddot{\theta}_\ell \right),$$

whereas the angular velocities are related by

$$\dot{\theta}_t = n_{r2}\, \dot{\theta}_\ell, \qquad \dot{\theta}_m = n_{r1}\, \dot{\theta}_t = n_{r1} n_{r2}\, \dot{\theta}_\ell,$$

and similarly for the angular accelerations. Setting now a generic desired acceleration $\ddot{\theta}_\ell = a > 0$, we obtain

$$\tau_m = I_m \left(n_{r1} n_{r2}\, a \right) + \frac{1}{n_{r1}} I_t \left(n_{r2}\, a \right) + \frac{1}{n_{r1} n_{r2}} I_\ell\, a = \left(\left(I_m\, n_{r1} + \frac{I_t}{n_{r1}} \right) n_{r2} + \frac{I_\ell}{n_{r1}} \frac{1}{n_{r2}} \right) a. \tag{4}$$

The necessary condition for a minimum of τ_m as a function of the unknown n_{r2} is

$$\frac{\partial\, \tau_m}{\partial\, n_{r2}} = \left(\left(I_m\, n_{r1} + \frac{I_t}{n_{r1}} \right) - \frac{I_\ell}{n_{r1}} \frac{1}{n_{r2}^2} \right) a = 0,$$

or

$$\left(I_m\, n_{r1} + \frac{I_t}{n_{r1}} \right) - \frac{I_\ell}{n_{r1}} \frac{1}{n_{r2}^2} = 0 \qquad \Rightarrow \qquad n_{r2} = \sqrt{\frac{I_\ell}{I_t + I_m\, n_{r1}^2}}.$$

This is indeed also a sufficient condition for a minimum since

$$\frac{\partial^2 \tau_m}{\partial\, {n_{r2}}^2} = \frac{2 I_\ell}{n_{r1}} \frac{1}{n_{r2}^3} > 0.$$

Plugging in the numerical data, we obtain $n_{r2} = 2$. As for the resolution of the angular position of link, we have

$$\Delta\theta_\ell = \frac{1}{n_{r1} n_{r2}} \Delta\theta_m = \frac{1}{10 \cdot 2} \frac{2\pi}{2000} = 1.5708 \cdot 10^{-4}\ [\text{rad}] = 0.009^\circ.$$

Finally, the bang-bang angular acceleration $\pm A_{max}$ for a link that moves from rest to rest by $\theta_{\ell,d}$ in time T is obtained from the triangular velocity profile (with peak absolute speed V_{max} at the halftime $t = T/2$) as

$$V_{max} \cdot \frac{T}{2} = \theta_{\ell,d}, \qquad V_{max} = A_{max} \cdot \frac{T}{2} \qquad \Rightarrow \qquad A_{max} = \frac{4\,|\theta_{\ell,d}|}{T^2}.$$

The associated motor torque will also be bang-bang, $\pm\,\tau_{m,max}$, with the maximum absolute value computed setting $a = A_{max}$ in (4). With the numerical data, one obtains

$$A_{max} = 4\pi = 12.5664 \text{ [rad/s}^2], \qquad \tau_{m,max} = 0.008 \cdot A_{max} = 0.1005 \text{ [Nm]}.$$

Exercise #3

The three given rotation matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_{via}$, and $\boldsymbol{R}_2$ are first converted into their minimal representation of the orientation by means of ZYX Euler angles (α, β, γ). From the direct mapping

$$\begin{aligned} \boldsymbol{R}_{ZY'X''}(\alpha,\beta,\gamma) &= \boldsymbol{R}_Z(\alpha)\boldsymbol{R}_Y(\beta)\boldsymbol{R}_X(\gamma) \\ &= \begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}, \end{aligned}$$

we have the inverse solutions, for a given rotation matrix $\boldsymbol{R} = \{R_{ij}\}$ in the regular case, computed in the order

$$\begin{aligned} \beta &= \text{ATAN2}\left\{-R_{31}, +\sqrt{R_{32}^2 + R_{33}^2}\right\}, \\ \alpha &= \text{ATAN2}\left\{\frac{R_{21}}{\cos\beta}, \frac{R_{11}}{\cos\beta}\right\}, \\ \gamma &= \text{ATAN2}\left\{\frac{R_{32}}{\cos\beta}, \frac{R_{33}}{\cos\beta}\right\}, \end{aligned} \tag{5}$$

where the '+' sign has been used in the expression of β, as requested. Applying (5) to the initial, intermediate, and final rotation matrices yields

$$\begin{array}{lcllll} \boldsymbol{R}_1 & \Rightarrow & \alpha_1 = \dfrac{\pi}{2}, & \beta_1 = 0, & \gamma_1 = \dfrac{\pi}{4}, \\ \boldsymbol{R}_{via} & \Rightarrow & \alpha_v = -\dfrac{\pi}{4}, & \beta_v = \dfrac{\pi}{6}, & \gamma_v = \dfrac{\pi}{2}, \\ \boldsymbol{R}_2 & \Rightarrow & \alpha_2 = 0, & \beta_2 = \dfrac{\pi}{4}, & \gamma_2 = \dfrac{3\pi}{4}. \end{array} \tag{6}$$

No singular case ($\cos\beta = 0$, or $\beta = \pm\pi/2$) was found at these orientations.

The problem is to define a smooth interpolating function in time for each of the three Euler angles, with zero boundary velocities. We need thus a simple spline, namely one constituted by only two cubic polynomials. We rewrite these for a generic angle $\theta(t)$ in the two intervals $t \in [0, T_1]$ and $t \in [T_1, T_1 + T_2] = [T_1, T]$, using conveniently the normalized times $\tau_i = t/T_i$, for $i = 1, 2$:

$$\theta(t) = \begin{cases} \theta_1(\tau_1) = a_1\tau_1^3 + b_1\tau_1^2 + \theta_1, & \tau_1 \in [0,1] \\ \theta_2(\tau_2) = a_2\,(\tau_2 - 1)^3 + b_2\,(\tau_2 - 1)^2 + \theta_2, & \tau_2 \in [0,1]. \end{cases} \tag{7}$$

Its first and second derivatives are

$$\dot{\theta}(t) = \begin{cases} \dot{\theta}_1(\tau_1) = \dfrac{1}{T_1}\left(3a_1\tau_1^2 + 2b_1\tau_1\right), & \tau_1 \in [0,1] \\ \dot{\theta}_2(\tau_2) = \dfrac{1}{T_2}\Big(3a_2\left(\tau_2 - 1\right)^2 + 2b_2\left(\tau_2 - 1\right)\Big), & \tau_2 \in [0,1] \end{cases} \tag{8}$$

and

$$\ddot{\theta}(t) = \begin{cases} \ddot{\theta}_1(\tau_1) = \dfrac{1}{T_1^2}\left(6a_1\tau_1 + 2b_1\right), & \tau_1 \in [0,1] \\ \ddot{\theta}_2(\tau_2) = \dfrac{1}{T_2^2}\left(6a_2\left(\tau_2 - 1\right) + 2b_2\right), & \tau_2 \in [0,1]. \end{cases} \tag{9}$$

The cubics in (7) and the quadratics in (8) automatically satisfy the boundary conditions at $t = 0$ and $t = T$, respectively in position ($\theta_1(0) = \theta_1$ and $\theta_2(1) = \theta_2$) and velocity ($\dot{\theta}_1(0) = \dot{\theta}_2(1) = 0$). Considering also the intermediate passage at θ_v and introducing the common (yet to be defined) velocity v at the via point, we impose four more conditions as

$$\begin{aligned} \theta_1(1) &= a_1 + b_1 + \theta_1 = \theta_v \\ \dot{\theta}_1(1) &= \frac{1}{T_1}\left(3a_1 + 2b_1\right) = v \\ \theta_2(0) &= -a_2 + b_2 + \theta_2 = \theta_v \\ \dot{\theta}_2(0) &= \frac{1}{T_2}\left(3a_2 - 2b_2\right) = v, \end{aligned}$$

and solve the resulting (2×2 decoupled) linear system

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \theta_v - \theta_1 \\ v\,T_1 \\ \theta_v - \theta_2 \\ v\,T_2 \end{pmatrix}$$

in terms of the spline coefficients

$$\begin{aligned} a_1 &= v\,T_1 - 2\left(\theta_v - \theta_1\right) \\ b_1 &= 3\left(\theta_v - \theta_1\right) - v\,T_1 \\ a_2 &= 2\left(\theta_v - \theta_2\right) + v\,T_2 \\ b_2 &= 3\left(\theta_v - \theta_2\right) + v\,T_2. \end{aligned} \tag{10}$$

The remaining unknown v is obtained by imposing continuity of acceleration at the via point, i.e.,

$$\ddot{\theta}_1(1) = \frac{1}{T_1^2}\left(6a_1 + 2b_1\right) = \frac{1}{T_2^2}\left(-6a_2 + 2b_2\right) = \ddot{\theta}_2(0),$$

or

$$4\left(\frac{1}{T_1} + \frac{1}{T_2}\right) v = \frac{6\left(\theta_v - \theta_1\right)}{T_1^2} - \frac{6\left(\theta_v - \theta_2\right)}{T_2^2}$$

yielding

$$v = \frac{3}{2\left(T_1 + T_2\right)}\left(\frac{T_2}{T_1}\left(\theta_v - \theta_1\right) - \frac{T_1}{T_2}\left(\theta_v - \theta_2\right)\right). \tag{11}$$

By replacing (11) in eqs. (10), and these in (7), we get finally the general solution.

Substituting for θ_1, θ_v, and θ_2 the specific numerical values assigned respectively to α, β, and γ, and using the time intervals $T_1 = 2.5$ and $T_2 = 1$ [s], the following three splines are obtained[1] is for the Euler angles $\alpha(t)$, $\beta(t)$, and $\gamma(t)$:

$$\alpha(t) = \begin{cases} \alpha_1(\tau_1) = \dfrac{207\pi}{112}\,\tau_1^3 - \dfrac{291\pi}{112}\,\tau_1^2 + \dfrac{\pi}{2}, & \tau_1 \in [0,1] \\[2ex] \alpha_2(\tau_2) = -\dfrac{101\pi}{280}\,(\tau_2 - 1)^3 - \dfrac{171\pi}{280}\,(\tau_2 - 1)^2\,, & \tau_2 \in [0,1], \end{cases}$$

$$\beta(t) = \begin{cases} \beta_1(\tau_1) = -\dfrac{13\pi}{336}\,\tau_1^3 + \dfrac{23\pi}{112}\,\tau_1^2, & \tau_1 \in [0,1] \\[2ex] \beta_2(\tau_2) = -\dfrac{41\pi}{840}\,(\tau_2 - 1)^3 - \dfrac{37\pi}{280}\,(\tau_2 - 1)^2 + \dfrac{\pi}{4}, & \tau_2 \in [0,1], \end{cases}$$

$$\gamma(t) = \begin{cases} \gamma_1(\tau_1) = \dfrac{31\pi}{112}\,\tau_1^3 - \dfrac{3\pi}{112}\,\tau_1^2 + \dfrac{\pi}{4}, & \tau_1 \in [0,1] \\[2ex] \gamma_2(\tau_2) = -\dfrac{53\pi}{280}\,(\tau_2 - 1)^3 - \dfrac{123\pi}{280}\,(\tau_2 - 1)^2 + \dfrac{3\pi}{4}, & \tau_2 \in [0,1]. \end{cases}$$

The plots of these three splines are reported in Fig. 5. The cubic polynomials in the two intervals are drawn in different colors (blue for the first, red for the second). From the evolution of $\beta(t)$ it is clear that the singularity $\beta = \pm\pi/2 \simeq \pm 1.57$ [rad] of the ZYX Euler representation is never encountered during the planned motion.

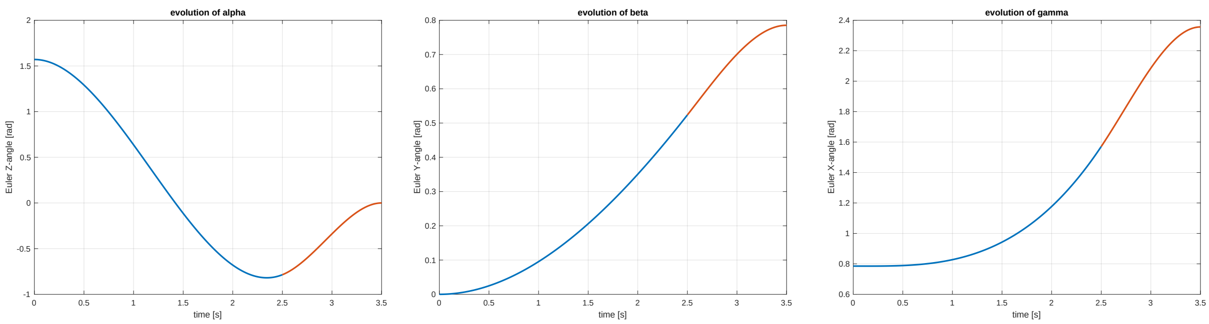

Figure 5: The time evolution of the three interpolating Euler angles $\alpha(t)$, $\beta(t)$, and $\gamma(t)$.

* * * * *

[1] This result has been generated by a symbolic code in MATLAB. Therefore, infinite precision arithmetic is used. Indeed, the same formulas are obtained by a purely numerical code.