Robotics 1

June 11, 2021

Exercise #1

Consider the 6-dof robot in Fig. 1. The robot has three prismatic joints in a portal arrangement and a spherical wrist. Assign a set of Denavit-Hartenberg (DH) frames and provide the associated table of parameters. Give plausible values for the joint variables $\boldsymbol{q}$ at the configuration shown in the figure. Define the homogeneous transformation matrix ${}^{w}\boldsymbol{T}_0$ relating the DH reference frame RF_0 to the world frame RF_w (for this, introduce geometric quantities as needed).

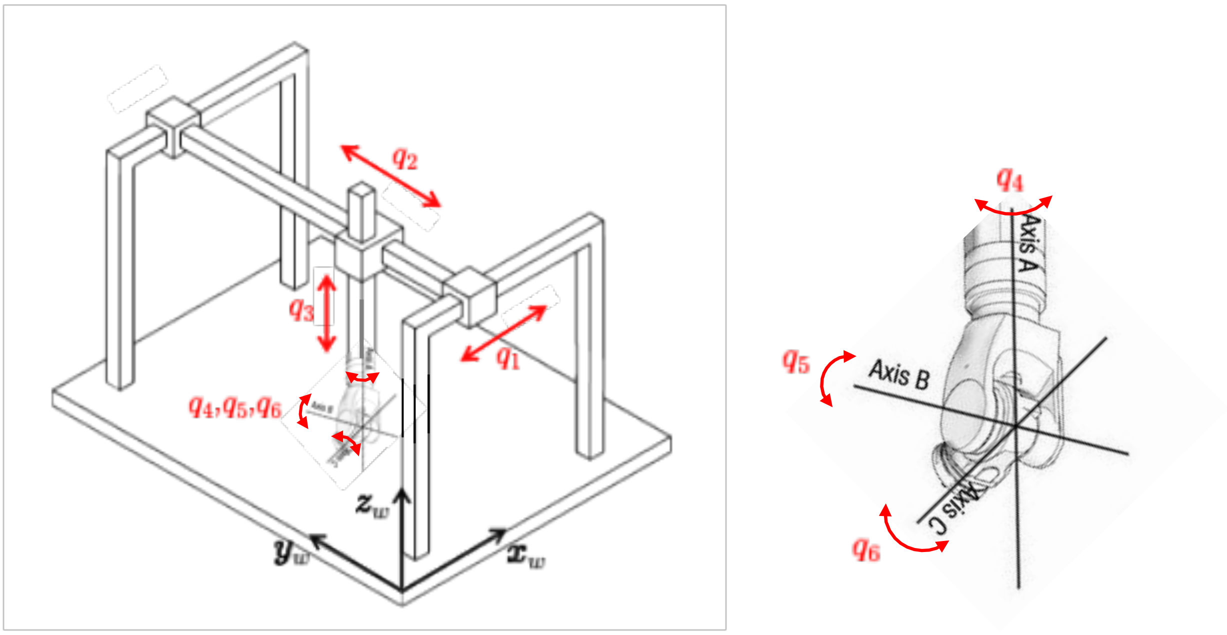

Figure 1: A 6-dof robot with a portal structure (3P) carrying a spherical (3R) wrist. An enlarged view of the wrist is shown on the right.

Exercise #2

A planar RPR robot is shown in Fig. 2, together with the definition of the joint coordinates[1]. The third link has length $L = 0.6$ [m]. The robot has to execute two different tasks, with the end effector placed at the point $P_d = (2, 0.4)$ [m] and pointing downward.

a) In the first task, the robot end effector should start moving inside a tube with a vertical speed $v = -2.5$ [m/s]. Determine the initial joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that realizes this instantaneous motion.

b) In the second task, the robot should keep its initial configuration in the presence of an horizontal force $f = 15$ [N] and a torque $\mu = 6$ [Nm] applied to its end effector. Determine the joint commands $\boldsymbol{\tau} \in \mathbb{R}^3$ (two torques and a force) needed for static balance.

Comments that justify intuitively some of the obtained results are welcome!

[1] Use these coordinates in your developments. Note that q_1 and q_3 are *not* DH variables.

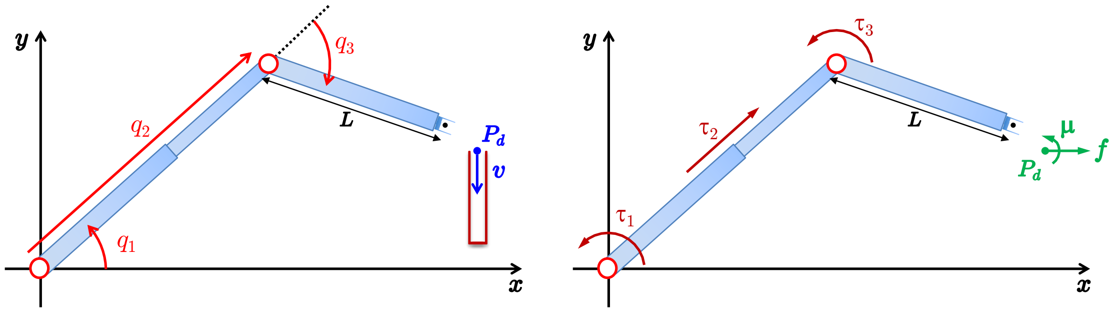

Figure 2: A 3-dof RPR robot, with the definition of a Cartesian motion task [left] and of a static balancing task in the presence of an external force/torque [right]. Both tasks should be executed at point P_d, with the robot end effector pointing downward.

Exercise #3

Plan a smooth rest-to-rest trajectory along a linear path from point $A = (1, 1, 1)$ [m] to point $B = (-1, 5, 0)$ [m], with simultaneous and coordinated change of orientation from

$$\boldsymbol{R}_A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

to

$$\boldsymbol{R}_B = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The total motion time is $T = 2.5$ [s]. The trajectory should be continuous up to the acceleration for all $t \in [0, T]$. Determine the velocity $\boldsymbol{v}_M \in \mathbb{R}^3$, acceleration $\boldsymbol{a}_M \in \mathbb{R}^3$, angular velocity $\boldsymbol{\omega}_M \in \mathbb{R}^3$, and angular acceleration $\dot{\boldsymbol{\omega}}_M \in \mathbb{R}^3$ attained at the time instant(s) when these four vectors assume, respectively, their maximum value in norm. Compute also the absolute orientation $\boldsymbol{R}_{mid} \in SO(3)$ at the midpoint of the planned trajectory.

[180 minutes (3 hours); open books]

Solution

June 11, 2021

Exercise #1

One of the (many) possible assignments of Denavit-Hartenberg frames for the 6-dof portal robot with spherical wrist is shown in Fig. 3. The associated parameters are reported in Tab. 1.

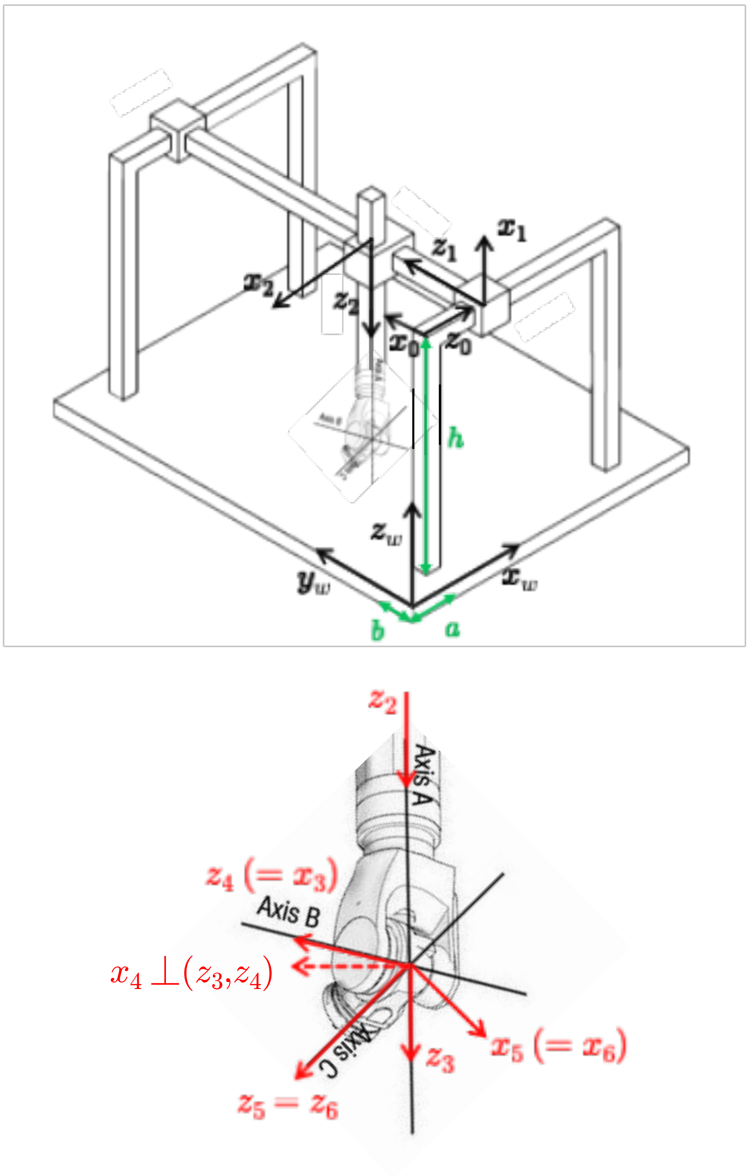

Figure 3: A possible assignment of DH frames for the 6-dof robot of Fig. 1. In the top figure, the world frame RF_w and the first three frames RF_0 to RF_2 of the portal structure are drawn in black, while the extra quantities a, b and h introduced for defining ${}^w\boldsymbol{T}_0$ are shown in green. In the bottom figure, the last four frames RF_3 to RF_6 for the spherical wrist are drawn in red.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$q_1 > 0$	$\pi/2$
2	$\pi/2$	0	$q_2 > 0$	$-\pi/2$
3	0	0	$q_3 > 0$	0
4	$\pi/2$	0	0	$q_4 = 0$
5	$\pi/2$	0	0	$q_5 = 3\pi/4$
6	0	0	0	$q_6 = 0$

Table 1: DH table of parameters corresponding to Fig. 3. The joint variables q_i (in red) take values associated to the configuration shown in the same figure. We have assumed $O_6 \equiv O_5$ ($d_6 = 0$).

With the geometric quantities introduced in Fig. 3, the homogenous matrix that locates the DH base frame RF_0 in the world frame is

$$
{}^{w}\boldsymbol{T}_0 = \begin{pmatrix} 0 & 0 & 1 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Exercise #2

The direct kinematics of interest for the RPR robot in Fig. 2 is

$$
\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \alpha \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 + L\cos(q_1+q_3) \\ q_2 \sin q_1 + L\sin(q_1+q_3) \\ q_1 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{1}
$$

where $\boldsymbol{p}$ is the planar position of the robot tip and α is the absolute angle of its end-effector w.r.t. the $\boldsymbol{x}$ axis. The associated 3×3 analytic Jacobian is

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 - L\sin(q_1+q_3) & \cos q_1 & -L\sin(q_1+q_3) \\ q_2 \cos q_1 + L\cos(q_1+q_3) & \sin q_1 & L\cos(q_1+q_3) \\ 1 & 0 & 1 \end{pmatrix}, \tag{2}
$$

with $\det J(\boldsymbol{q}) = -q_2$.

To find an initial robot configuration $\boldsymbol{q}_d = (q_{1d}, q_{2d}, q_{3d})$ associated to the desired end-effector pose $\boldsymbol{r}_d = (\boldsymbol{p}_d, \alpha_d) = (p_{xd}, p_{yd}, \alpha_d) = (2, 0.4, -\pi/2)$, we solve the inverse kinematics problem in general. From the third equation in (1), we have

$$
q_{1d} + q_{3d} = \alpha_d,
$$

which, substituted in the first two equations, yields

$$
p_{xd} - L\cos\alpha_d = q_{2d}\cos q_{1d}, \qquad p_{yd} - L\sin\alpha_d = q_{2d}\sin q_{1d}. \tag{3}
$$

By squaring and summing the two equations in (3), we find the value q_{2d} as

$$q_{2d} = +\sqrt{(p_{xd} - L\cos\alpha_d)^2 + (p_{yd} - L\sin\alpha_d)^2}, \tag{4}$$

where the positive sign has been chosen for simplicity. Dividing by $q_{2d} > 0$ the two equations in (3), we also obtain

$$q_{1d} = \mathrm{ATAN2}\left\{p_{y,d} - L\sin\alpha_d, p_{x,d} - L\cos\alpha_d\right\}, \tag{5}$$

and finally

$$q_{3d} = \alpha_d - q_{1d}. \tag{6}$$

For the specific data of the problem, note that eqs. (4) and (5) simplify to an intuitive geometric solution. In fact, when the desired point P_d is in the first quadrant and the end effector points vertically and downward ($\alpha_d = -\pi/2$), the base of the third link should be placed at the 'higher' point $P'_d = P_d + (0, L)$, whose position is

$$\boldsymbol{p}'_d = \begin{pmatrix} p_{xd} \\ p_{yd} + L \end{pmatrix}.$$

Therefore, the solution for the first two joints follows immediately as

$$q_{1d} = \arctan\left(\frac{p_{yd} + L}{p_{xd}}\right), \qquad q_{2d} = \sqrt{p_{xd}^2 + (p_{yd} + L)^2}, \tag{7}$$

while q_{3d} is found again by (6).

With the given data, we obtain

$$\boldsymbol{q}_d = \begin{pmatrix} 0.4636 \\ 2.2361 \\ -2.0344 \end{pmatrix} [\mathrm{rad/m/rad}] = \begin{pmatrix} 26.565 \\ 2.2361 \\ -116.565 \end{pmatrix} [°/\mathrm{m}/°].$$

Thus, the Jacobian in this configuration becomes

$$\boldsymbol{J}_d = \boldsymbol{J}(\boldsymbol{q}_d) = \begin{pmatrix} -0.4000 & 0.8944 & 0.6000 \\ 2.0000 & 0.4472 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $\det \boldsymbol{J}_d = -2.2361$, this matrix is invertible. For the motion task, we have that

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} \dot{p}_{xd} \\ \dot{p}_{yd} \\ \dot{\alpha}_d \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix},$$

because the insertion in the tube is feasible only by keeping the vertical, downward orientation of the end effector. For the static balancing task, it is

$$\boldsymbol{F}_d = \begin{pmatrix} f_{xd} \\ f_{yd} \\ \mu_{zd} \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ \mu \end{pmatrix}.$$

Using the numerical data, the solutions for the required tasks are

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_d^{-1}\,\dot{\boldsymbol{r}}_d = \boldsymbol{J}_d^{-1}\begin{pmatrix} 0 \\ -2.5 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1.1180 \\ 1 \end{pmatrix} \text{ [rad/s, m/s, rad/s]} \tag{8}$$

and, respectively,

$$\boldsymbol{\tau} = -\boldsymbol{J}_d^T\boldsymbol{F}_d = -\boldsymbol{J}_d^T\begin{pmatrix} 15 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ -13.4164 \\ -15 \end{pmatrix} \text{ [Nm, N, Nm].} \tag{9}$$

We note that the first and third (revolute) joints compensate their motion in (8), in order not to change the end-effector orientation. Similarly, in (9) the torque at the third joint directly annihilates the presence of the torque μ at the end effector (both acting on the same link), while the force at the second joint is the only one responsible for compensating the horizontal force component f (as projected along the direction of the prismatic joint). The fact that $\tau_1 = 0$ is just an occasional result here, due to the particular combination of input data[2].

Exercise #3

The trajectory is determined in parametric form in terms of a normalized scalar $s \in [0, 1]$ for both the linear and the angular parts, in order to achieve coordinated motion. Then, a sufficiently smooth timing law $s = s(t)$ for $t \in [0, T]$ is assigned, so as to guarantee rest-to-rest motion with continuity up to the acceleration (as a consequence, also the acceleration should be zero at the initial and final instants).

For the linear motion along a straight line from point A to point B, we have

$$\boldsymbol{p}(s) = \boldsymbol{p}_A + (\boldsymbol{p}_B - \boldsymbol{p}_A)\,s = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} s, \qquad s \in [0, 1],$$

with path length $L = \|\boldsymbol{p}_B - \boldsymbol{p}_A\| = 4.5826$ [m]. The velocity and the acceleration are then

$$\dot{\boldsymbol{p}} = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\dot{s}, \qquad \ddot{\boldsymbol{p}} = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\ddot{s},$$

and their maximum values in norm will be attained at the instants where, respectively, $\dot{s}$ or $\ddot{s}$ have a maximum (in absolute value for the latter), with

$$\boldsymbol{v}_M = (\boldsymbol{p}_B - \boldsymbol{p}_A) \cdot \max_{t\in[0,T]} \dot{s}(t) = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\dot{s}_{max} \tag{10}$$

and

$$\boldsymbol{a}_M = (\boldsymbol{p}_B - \boldsymbol{p}_A) \cdot \max_{t\in[0,T]} |\ddot{s}(t)| = (\boldsymbol{p}_B - \boldsymbol{p}_A)\,\ddot{s}_{max}. \tag{11}$$

For the angular motion, it is convenient to choose an axis-angle planning method[3]. In this way, it is immediate to find the resulting angular velocity and acceleration. First, we compute the relative

[2] It is instructive to look at the outcome of the balancing $\boldsymbol{\tau}$ for $f = 15$ only, for $\mu = 6$ only, or for a slight perturbation of one of these two w.r.t. the given input data, e.g., for $f = 15.1$, $\mu = 6$.

[3] Indeed, one may also choose to convert the initial and final rotation matrices $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ into some minimal set of Euler or RPY-type angles, and then plan a trajectory for these angles in a coordinated way. However, the complete procedure would be more lengthy.

orientation between $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$:

$$\boldsymbol{R}_{AB} = \boldsymbol{R}_A^T \boldsymbol{R}_B = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}.$$

From this, we extract the axis-angle solution $(\boldsymbol{r}, \theta_{AB})$ as

$$\begin{aligned} \theta_{AB} &= \text{ATAN2}\left\{\sqrt{(R_{12}-R_{21})^2+(R_{13}-R_{31})^2+(R_{23}-R_{32})^2}, R_{11}+R_{22}+R_{33}-1\right\} \\ &= 2.5936 \text{ [rad]}, \\ \boldsymbol{r} &= \frac{1}{2\sin\theta_{AB}} \begin{pmatrix} R_{32}-R_{23} \\ R_{13}-R_{31} \\ R_{21}-R_{12} \end{pmatrix} = \begin{pmatrix} -0.6786 \\ 0.6786 \\ 0.2811 \end{pmatrix}, \end{aligned}$$

being in a regular condition ($\sin\theta_{aB} \neq 0$). The profile of the rotation angle around the unit vector $\boldsymbol{r}$ is then defined parametrically as

$$\theta(s) = \theta_{AB}\, s, \qquad s \in [0,1].$$

The angular velocity and acceleration vectors are aligned with the unit vector $\boldsymbol{r} \in \mathbb{R}^3$, with profiles

$$\boldsymbol{\omega} = \theta_{AB}\dot{s}\ \boldsymbol{r}, \qquad \dot{\boldsymbol{\omega}} = \theta_{AB}\ddot{s}\ \boldsymbol{r}.$$

As before, their maximum values in norm will be attained at the instants where, respectively, $\dot{s}$ or $\ddot{s}$ have a maximum (in absolute value), with

$$\boldsymbol{\omega}_M = \theta_{AB}\boldsymbol{r} \cdot \max_{t\in[0,T]} \dot{s}(t) = (\theta_{AB}\boldsymbol{r})\, \dot{s}_{max} \tag{12}$$

and

$$\dot{\boldsymbol{\omega}}_M = \theta_{AB}\boldsymbol{r} \cdot \max_{t\in[0,T]} |\ddot{s}(t)| = (\theta_{AB}\boldsymbol{r})\, \ddot{s}_{max}. \tag{13}$$

The simplest timing law that guarantees a rest-to-rest motion in time T with continuous acceleration in the whole interval $[0,T]$ is given by the doubly-normalized quintic polynomial

$$s(t) = 6\left(\frac{t}{T}\right)^5 - 15\left(\frac{t}{T}\right)^4 + 10\left(\frac{t}{T}\right)^3 \qquad \Rightarrow \quad s(0)=0,\ s(T)=1,$$

with first time derivative

$$\dot{s}(t) = \frac{1}{T}\left(30\left(\frac{t}{T}\right)^4 - 60\left(\frac{t}{T}\right)^3 + 30\left(\frac{t}{T}\right)^2\right) \qquad \Rightarrow \quad \dot{s}(0)=\dot{s}(T)=0,$$

and second time derivative

$$\ddot{s}(t) = \frac{1}{T^2}\left(120\left(\frac{t}{T}\right)^3 - 180\left(\frac{t}{T}\right)^2 + 60\left(\frac{t}{T}\right)\right) \qquad \Rightarrow \quad \ddot{s}(0)=\ddot{s}(T)=0.$$

It is easy to see that, apart from the two boundary instants $t=0$ and $t=T$, $\ddot{s}(t)=0$ has a root also at $t = T_{mid} = T/2$, where the pseudo-velocity $\dot{s}$ has a maximum. We obtain

$$\dot{s}_{max} = \dot{s}\left(T_{mid}\right) = \frac{1}{T}\left(30\left(\frac{1}{2}\right)^4 - 60\left(\frac{1}{2}\right)^3 + 30\left(\frac{1}{2}\right)^2\right) = \frac{7.5}{4T} = 0.75, \tag{14}$$

where $T = 2.5$ [s] has been used. On the other hand, the maximum value (in module) for the pseudo-acceleration $\ddot{s}$ is found by solving for the roots of $\dddot{s}(t) = 0$, or

$$6\left(\frac{t}{T}\right)^2 - 6\left(\frac{t}{T}\right) + 1 = 0$$

$$\Rightarrow \qquad t = T_{a1} = \left(0.5 - \frac{\sqrt{3}}{6}\right) T = 0.2113\,T, \qquad t = T_{a2} = \left(0.5 + \frac{\sqrt{3}}{6}\right) T = 0.7887\,T.$$

We obtain

$$\ddot{s}_{max} = |\ddot{s}\,(T_{a1})| = |\ddot{s}\,(T_{a2})| = \frac{60}{T^2}\left|2\left(0.5 \pm \frac{\sqrt{3}}{6}\right)^3 - 3\left(0.5 \pm \frac{\sqrt{3}}{6}\right)^2 + \left(0.5 \pm \frac{\sqrt{3}}{6}\right)\right| = 0.9238, \tag{15}$$

where $T = 2.5$ [s] was used again. The time behaviors of $s(t)$, $\dot{s}(t)$ and $\ddot{s}(t)$ are shown in Fig. 4.

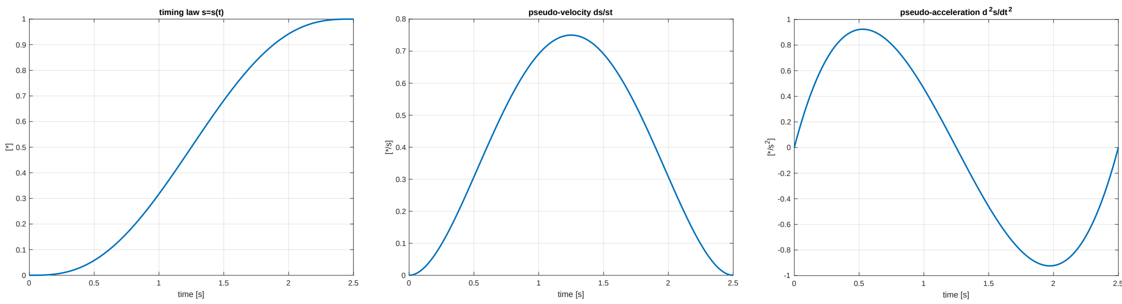

Figure 4: Profile of the timing law $s = s(t)$, with first and second time derivatives. The maximum values of the latter two are $\dot{s}_{max} = 0.75$ and $\ddot{s}_{max} = 0.9238$.

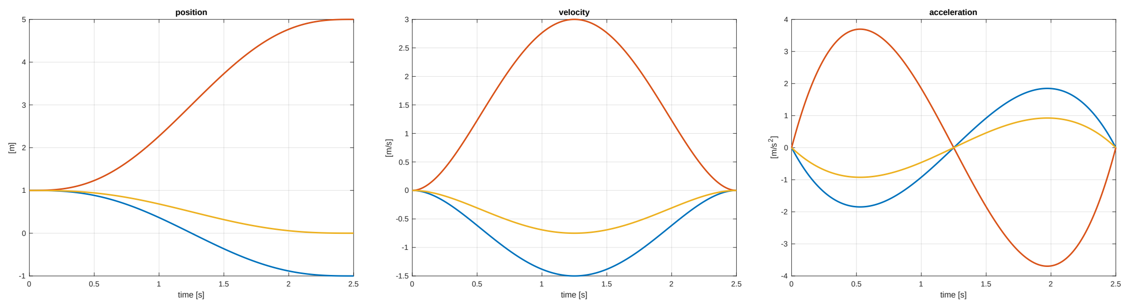

Figure 5: Planned position $\boldsymbol{p}(t)$, velocity $\boldsymbol{v}(t)$, and acceleration $\boldsymbol{a}(t)$ (x-components in blue, y in yellow, z in red).

According to (10–11) and (12–13) and by using (14) and (15), the values of the velocity and acceleration vectors attained at the time instant(s) when they assume their maximum value in norm are

$$\boldsymbol{v}_M = \dot{\boldsymbol{p}}(T_{mid}) = \begin{pmatrix} -1.50 \\ 3.00 \\ -0.75 \end{pmatrix} \text{ [m/s]}, \qquad \boldsymbol{a}_M = \ddot{\boldsymbol{p}}(T_{a1}) = -\ddot{\boldsymbol{p}}(T_{a2}) = \begin{pmatrix} -1.8475 \\ 3.6950 \\ -0.9238 \end{pmatrix} \text{ [m/s}^2\text{]},$$

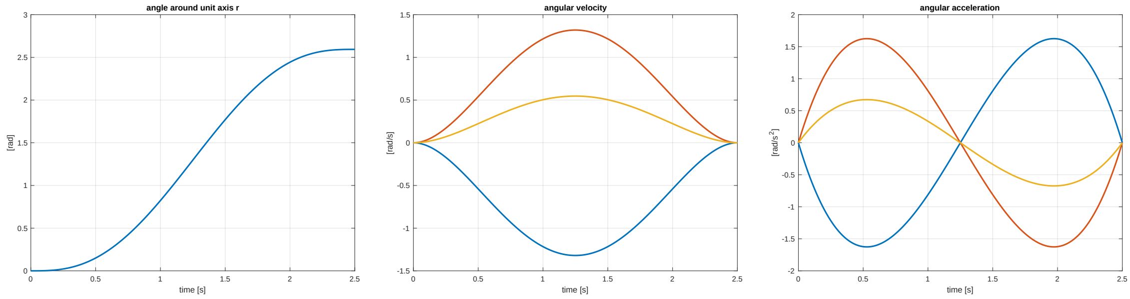

Figure 6: Planned angle $\theta(t) = \theta_{AB} s(t)$, angular velocity $\boldsymbol{\omega}(t) = \theta_{AB} \dot{s}(t) \boldsymbol{r}$, and angular acceleration $\dot{\boldsymbol{\omega}}(t) = \theta_{AB} \ddot{s}(t) \boldsymbol{r}$ (for vectors: x-components in blue, y in yellow, z in red).

see also Fig. 5 for the time behaviors of the single components. Similarly, for the angular velocity and acceleration vectors,

$$\boldsymbol{\omega}_M = \boldsymbol{\omega}(T_{mid}) = \begin{pmatrix} -1.3200 \\ 1.3200 \\ -0.5468 \end{pmatrix} \text{ [rad/s]}, \qquad \dot{\boldsymbol{\omega}}_M = \dot{\boldsymbol{\omega}}(T_{a1}) = -\dot{\boldsymbol{\omega}}(T_{a2}) = \begin{pmatrix} -1.6258 \\ 1.6258 \\ 0.6734 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

see also Fig. 6.

Finally, the absolute orientation at the midpoint of the planned motion (namely, at $t = T_{mid}$, where $\theta_{mid} = \theta(T_{mid}) = \theta_{AB}/2$) is expressed using the rotation matrix of the axis-angle method as

$$\boldsymbol{R}_{mid} = \boldsymbol{R}_A \left(\boldsymbol{r}\boldsymbol{r}^T + \left(\boldsymbol{I} - \boldsymbol{r}\boldsymbol{r}^T\right) \cos(\theta_{AB}/2) + \boldsymbol{S}(\boldsymbol{r}) \sin(\theta_{AB}/2) \right) = \begin{pmatrix} -0.0653 & 0.6065 & 0.7924 \\ 0.6065 & -0.6065 & 0.5142 \\ 0.7924 & 0.5142 & -0.3282 \end{pmatrix},$$

where $\boldsymbol{S}(\boldsymbol{r})$ is the skew-symmetric matrix built with $\boldsymbol{r}$.

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