Robotics 1

Remote Exam – September 11, 2020

Exercise #1

Given a smooth time-varying rotation matrix $\boldsymbol{R}(t) \in SO(3)$, provide a formula to determine the associated angular acceleration vector $\dot{\boldsymbol{\omega}}(t) \in \mathbb{R}^3$ as a function of $\boldsymbol{R}(t)$ and of the angular velocity $\boldsymbol{\omega}(t) \in \mathbb{R}^3$. Apply then this formula to compute $\boldsymbol{\omega}(t)$ and $\dot{\boldsymbol{\omega}}(t)$, given the following rotation matrix:

$$\boldsymbol{R}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ \sin^2 t & \cos t & -\sin t \cos t \\ -\sin t \cos t & \sin t & \cos^2 t \end{pmatrix}.$$

Exercise #2

Consider the 6R Universal Robots UR5 manipulator in Fig. 1, where a feasible set of Denavit-Hartenberg (DH) frames has been assigned. Complete the table of DH parameters and enter also the associated numerical values (expressed in [rad] or [mm]), including those of the joint variables $\boldsymbol{q} = \boldsymbol{\theta}$ in the configuration shown. In the figure, all data are already given in mm.

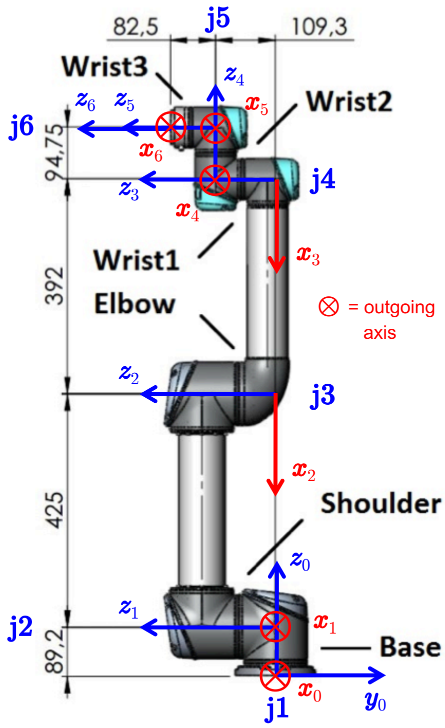

Figure 1: An assignment of DH frames for the UR5 manipulator.

Exercise #3

With reference to Fig. 2, two planar manipulators, a 2R robot (labeled as A) and a 3R robot (labeled as B), both with links of unitary length, should perform a task in cooperation, handing over an object between their end-effector grippers. The base frames of the two robots are positioned with respect to a common world frame by ${}^w\boldsymbol{p}_A = \begin{pmatrix} -2.5 & 1 \end{pmatrix}^T$ and ${}^w\boldsymbol{p}_B = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$. The base of robot B is rotated counterclockwise by an angle $\alpha_B = \pi/6$ [rad] with respect to $\boldsymbol{x}_w$. Robot A holds the object while being in the configuration $\boldsymbol{q}_A = \begin{pmatrix} \pi/3 & -\pi/2 \end{pmatrix}^T$ [rad]. Determine a configuration $\boldsymbol{q}_B$ for robot B such that it can grasp the object held by robot A with the correct orientation.

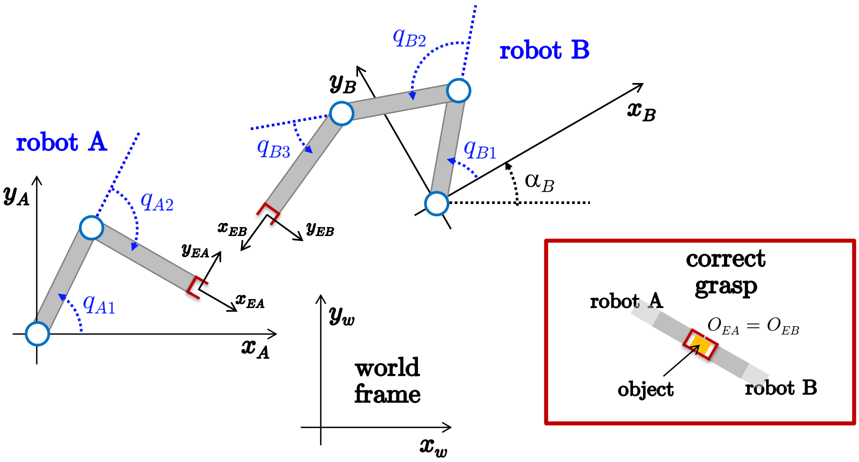

Figure 2: A 2R and a 3R planar manipulators cooperating in a task.

Exercise #4

Consider the 3×3 Jacobian of a 3R spatial robot, with generic link lengths $l_2 > 0$ and $l_3 > 0$:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1(l_2c_2 + l_3c_3) & -l_2c_1s_2 & -l_3c_1s_3 \\ c_1(l_2c_2 + l_3c_3) & -l_2s_1s_2 & -l_3s_1s_3 \\ 0 & l_2c_2 & l_3c_3 \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Find all (singular) configurations $\boldsymbol{q}^\diamond$ where the rank of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is equal to 2 and all configurations $\boldsymbol{q}^*$ where the rank is equal to 1. In a singularity with rank 1, determine a basis for each of the subspaces $\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}^*)\}$, $\mathcal{R}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$, and $\mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\}$.

Exercise #5

A mass $M = 2$ [kg] moves linearly under a bounded force u, with $|u| \le U_{max} = 8$ [N], according to differential equation $M\ddot{x} = u$. The mass starts at $t = 0$ from $x_i = x(0) = 0$ with a negative velocity $\dot{x}_i = \dot{x}(0) = -2$ [m/s], and has to reach the final position $x_f = x(T) = 3$ [m] at rest (i.e., with $\dot{x}_f = \dot{x}(T) = 0$) in minimum time T. Determine the minimum time T and the associated optimal command $u^*(t)$. Sketch the time evolution of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$.

[240 minutes (4 hours); open books]

Solution

September 11, 2020

Exercise #1

We have that

$$\dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}, \quad \text{and thus} \quad \boldsymbol{S}(\boldsymbol{\omega}) = \dot{\boldsymbol{R}}\boldsymbol{R}^T \;\Rightarrow\; \boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{3,2}(\boldsymbol{\omega}) \\ \boldsymbol{S}_{1,3}(\boldsymbol{\omega}) \\ \boldsymbol{S}_{2,1}(\boldsymbol{\omega}) \end{pmatrix}.$$

Differentiating further with respect to time,

$$\ddot{\boldsymbol{R}} = \boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{R} + \boldsymbol{S}(\boldsymbol{\omega})\dot{\boldsymbol{R}} = \boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{R} + \boldsymbol{S}^2(\boldsymbol{\omega})\boldsymbol{R}.$$

Since

$$\begin{aligned}\boldsymbol{S}^2(\boldsymbol{\omega}) &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\left(\omega_y^2+\omega_z^2\right) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & -\left(\omega_x^2+\omega_z^2\right) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & -\left(\omega_x^2+\omega_y^2\right) \end{pmatrix} = \boldsymbol{\omega}\,\boldsymbol{\omega}^T - \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2,\end{aligned}$$

we obtain finally

$$\ddot{\boldsymbol{R}} = \left(\boldsymbol{S}(\dot{\boldsymbol{\omega}}) + \boldsymbol{\omega}\,\boldsymbol{\omega}^T - \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2\right)\boldsymbol{R}, \quad \text{and thus} \qquad \begin{aligned} &\boldsymbol{S}(\dot{\boldsymbol{\omega}}) = \ddot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{I}\,\|\boldsymbol{\omega}\|^2 - \boldsymbol{\omega}\,\boldsymbol{\omega}^T \\ &\Rightarrow\; \dot{\boldsymbol{\omega}} = \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{S}_{3,2}(\dot{\boldsymbol{\omega}}) \\ \boldsymbol{S}_{1,3}(\dot{\boldsymbol{\omega}}) \\ \boldsymbol{S}_{2,1}(\dot{\boldsymbol{\omega}}) \end{pmatrix}. \end{aligned}$$

For the given time-varying rotation matrix, we obtain

$$\boldsymbol{R}(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ \sin^2 t & \cos t & -\sin t\cos t \\ -\sin t\cos t & \sin t & \cos^2 t \end{pmatrix} \;\Rightarrow\; \dot{\boldsymbol{R}}(t) = \begin{pmatrix} -\sin t & 0 & \cos t \\ 2\sin t\cos t & -\sin t & \sin^2 t - \cos^2 t \\ \sin^2 t - \cos^2 t & \cos t & -2\sin t\cos t \end{pmatrix},$$

and thus, after simplifications,

$$\boldsymbol{S}(\boldsymbol{\omega}(t)) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) = \begin{pmatrix} 0 & -\sin t & \cos t \\ \sin t & 0 & -1 \\ -\cos t & 1 & 0 \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{\omega}(t) = \begin{pmatrix} 1 \\ \cos t \\ \sin t \end{pmatrix}.$$

Moreover, one can evaluate

$$\ddot{\boldsymbol{R}}(t) = \begin{pmatrix} -\cos t & 0 & -\sin t \\ 2\left(\cos^2 t - \sin^2 t\right) & -\cos t & 4\sin t\cos t \\ 4\sin t\cos t & -\sin t & 2\left(\sin^2 t - \cos^2 t\right) \end{pmatrix}$$

and then compute

$$S(\dot{\omega}(t)) = \ddot{R}(t)R^T(t)+I\|\omega(t)\|^2-\omega(t)\,\omega^T(t) = \begin{pmatrix} 0 & -\cos t & -\sin t \\ \cos t & 0 & 0 \\ \sin t & 0 & 0 \end{pmatrix} \Rightarrow \dot{\omega}(t) = \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix}.$$

However, as one could have expected, we can also obtain $\dot{\omega}(t) = d\omega(t)/dt$ by direct differentiation (or from $S(\dot{\omega}(t)) = dS(\omega(t))/dt$).

Instead, the analytic formula is strictly required in case R, ω, and $\ddot{R}$ are known only numerically at a given instant of time. For example, if we had

$$R = I, \qquad \omega = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \ddot{R} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

we would then compute

$$S(\dot{\omega}) = \ddot{R}R^T + I\|\omega\|^2 - \omega\,\omega^T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\omega} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is nothing else than the considered case for $t = 0$.

Exercise #2

The Denavit-Hartenberg parameters (in mm or rad) of the UR5 manipulator associated to the frames specified in Fig. 1 are given in Tab. 1. Note that both parameters a_2 and a_3 are negative. In fact, to reach O_2 from O_1 we move in the opposite direction of x_2, thus $a_2 < 0$. Similarly, to reach O_3 from O_2 we move in the opposite direction of x_3, thus $a_3 < 0$.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	89.2	$q_1 = 0$
2	0	-425	0	$q_2 = -\pi/2$
3	0	-392	0	$q_3 = 0$
4	$-\pi/2$	0	109.3	$q_4 = \pi/2$
5	$\pi/2$	0	94.75	$q_5 = 0$
6	0	0	82.5	$q_6 = 0$

Table 1: DH parameters of the UR5 manipulator, with values of q in the configuration of Fig. 1.

Exercise #3

To accomplish the cooperative task we need to find the desired position and orientation of the end-effector of robot B, as expressed in its own base reference frame. For this, we will use the mathematics of 4×4 homogeneous transformations, starting from the definition of the position and orientation of the end-effector of robot A, as computed from the direct kinematics of the task in the world frame. Although the entire problem is planar, with positions in $\mathbb{R}^2$ and scalar orientations expressed by an angle around the normal to the plane $(\boldsymbol{x}_w, \boldsymbol{y}_w)$, we will embed objects in 3D. Once the target pose of the end-effector of robot B is available, the configuration $\boldsymbol{q}_B$ of robot B is found by solving a standard inverse kinematics problem.

With the given data of the problem, the base reference frames of robot A and B are located respectively by

$$
{}^w\boldsymbol{T}_A = \begin{pmatrix} {}^w\boldsymbol{R}_A & {}^w\boldsymbol{p}_A \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} & -2.5 \\ \boldsymbol{I}_{3\times3} & 1 \\ & 0 \\ \boldsymbol{0}^T & 1 \end{pmatrix}
$$

and

$$
{}^w\boldsymbol{T}_B = \begin{pmatrix} {}^w\boldsymbol{R}_B & {}^w\boldsymbol{p}_B \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_B & -\sin\alpha_B & 0 & 1 \\ \sin\alpha_B & \cos\alpha_B & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5 & 0 & 1 \\ 0.5 & 0.8660 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
$$

The direct kinematics of the planar 2R robot A (from its base to the end-effector frame EA), taking into account the unitary length of the links, is computed as

$$
\begin{aligned}
{}^A\boldsymbol{T}_{EA} &= \begin{pmatrix} {}^A\boldsymbol{R}_{EA} & {}^A\boldsymbol{p}_{EA} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(q_{A1}+q_{A2}) & -\sin(q_{A1}+q_{A2}) & 0 & \cos q_{A1} + \cos(q_{A1}+q_{A2}) \\ \sin(q_{A1}+q_{A2}) & \cos(q_{A1}+q_{A2}) & 0 & \sin q_{A1} + \sin(q_{A1}+q_{A2}) \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.8660 & 0.5 & 0 & 1.3660 \\ -0.5 & 0.8660 & 0 & 0.3660 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
\end{aligned}
$$

Finally, the correct grasping condition by robot B requires that the two end-effector frames have the same origin ($O_{EB} = O_{EA}$) and opposite orientations (i.e., with a relative rotation of π around the common $\boldsymbol{z}_w$ axis). Therefore, the associated homogeneous transformation is

$$
{}^{EA}\boldsymbol{T}_{EB} = \begin{pmatrix} {}^{EA}\boldsymbol{R}_{EB} & {}^{EA}\boldsymbol{p}_{EB} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}.
$$

We can write now the kinematic equation of the task using the above homogeneous transformation matrices, equating the end-effector pose ${}^w\boldsymbol{T}_{EB}$ of robot B in the world frame, as evaluated from the side of robot A and from the side of robot B:

$$
{}^w\boldsymbol{T}_A\,{}^A\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB} = {}^w\boldsymbol{T}_B\,{}^B\boldsymbol{T}_{EB}.
$$

Thus, the desired pose of the end-effector of robot B expressed in the reference frame B is:

$$
\begin{aligned}
{}^{B}\boldsymbol{T}_{EB,d} &= \begin{pmatrix} {}^{B}\boldsymbol{R}_{EB,d} & {}^{B}\boldsymbol{p}_{EB,d} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \left({}^{w}\boldsymbol{T}_{B}\right)^{-1}\, {}^{w}\boldsymbol{T}_{A}\, {}^{A}\boldsymbol{T}_{EA}\, {}^{EA}\boldsymbol{T}_{EB} \\
&= \begin{pmatrix} -0.5 & -0.8660 & 0 & -2.1651 \\ 0.8660 & -0.5 & 0 & 0.5179 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi_{B,d} & -\sin\phi_{B,d} & 0 & {}^{B}\boldsymbol{p}_{EB,d_x} \\ -\sin\phi_{B,d} & \cos\phi_{B,d} & 0 & {}^{B}\boldsymbol{p}_{EB,d_y} \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}
\end{aligned}
$$

The inverse kinematics problem for the planar 3R robot B requires the solution of

$$
\begin{aligned}
&{}^{B}\boldsymbol{T}_{EB,d} = {}^{B}\boldsymbol{T}_{EB}(\boldsymbol{q}_B) \\
&= \begin{pmatrix} \cos(q_{B1}+q_{B2}+q_{B3}) & -\sin(q_{B1}+q_{B2}+q_{B3}) & 0 & \cos q_{B1}+\cos(q_{B1}+q_{B2})+\cos(q_{B1}+q_{B2}+q_{B3}) \\ \sin(q_{B1}+q_{B2}+q_{B3}) & \cos(q_{B1}+q_{B2}+q_{B3}) & 0 & \sin q_{B1}+\sin(q_{B1}+q_{B2})+\sin(q_{B1}+q_{B2}+q_{B3}) \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix}
\end{aligned}
$$

in terms of the unknown joint variables $\boldsymbol{q}_B = (q_{B1}, q_{B2}, q_{B3})$. The desired angle $\phi_{B,d}$ characterizing the orientation in the plane of the end-effector frame of robot B can be extracted from the elements of the rotation matrix ${}^{B}\boldsymbol{R}_{EB,d}$ as

$$
\begin{aligned}
\phi_{B,d} &= \mathrm{ATAN2}\left\{\sin\phi_{B,d}, \cos\phi_{B,d}\right\} = \mathrm{ATAN2}\left\{{}^{B}\boldsymbol{R}_{EB,d_{21}}, {}^{B}\boldsymbol{R}_{EB,d_{11}}\right\} \\
&= \mathrm{ATAN2}\left\{0.8660, -0.5\right\} = 2.0944 \text{ [rad]} = 120°,
\end{aligned}
$$

the above is equivalent to solving the three nonlinear equations

$$
\begin{pmatrix} \cos q_{B1}+\cos(q_{B1}+q_{B2})+\cos(q_{B1}+q_{B2}+q_{B3}) \\ \sin q_{B1}+\sin(q_{B1}+q_{B2})+\sin(q_{B1}+q_{B2}+q_{B3}) \\ q_{B1}+q_{B2}+q_{B3} \end{pmatrix} = \begin{pmatrix} {}^{B}\boldsymbol{p}_{EB,d_x} \\ {}^{B}\boldsymbol{p}_{EB,d_y} \\ \phi_{B,d} \end{pmatrix} = \begin{pmatrix} -2.1651 \\ 0.5179 \\ 2.0944 \end{pmatrix}.
$$

As usual, this inverse kinematics problem for the planar 3R robot can be decomposed in two parts. First, we solve for the two joint variables q_{B1} and q_{B2} in order to place the tip position $\boldsymbol{p}_{t2}$ of the second link (or, the base of the third link) in the necessary position. Taking again into account the unitary length of the robot links, we have

$$
\boldsymbol{p}_{t2} = \begin{pmatrix} {}^{B}\boldsymbol{p}_{EB,d_x} \\ {}^{B}\boldsymbol{p}_{EB,d_x} \end{pmatrix} - \begin{pmatrix} \cos\phi_{B,d} \\ \sin\phi_{B,d} \end{pmatrix} = \begin{pmatrix} -2.1651 \\ 0.5179 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 0.8660 \end{pmatrix} = \begin{pmatrix} -1.6651 \\ -0.3481 \end{pmatrix} \text{ [m]}.
$$

Thus, a solution for the pair (q_{B1}, q_{B2}) is given by

$$
\begin{aligned}
&c_2 = \frac{\boldsymbol{p}_{t2,x}^2 + \boldsymbol{p}_{t2,y}^2 - 2}{2} = 0.4468, \quad s_2 = \sqrt{1-c_2^2} = 0.8946 \\
&\Rightarrow\ q_{B2} = \mathrm{ATAN2}\left\{s_2, c_2\right\} = 1.1076 \,\text{[rad]} = 63.46°,
\end{aligned}
$$

and[1]

$$
\begin{aligned}
&s_1 = \frac{\boldsymbol{p}_{t2,y}\,(1+c_2) - \boldsymbol{p}_{t2,x} s_2}{2(1+c_2)} = 0.3408, \quad c_1 = \frac{\boldsymbol{p}_{t2,x}\,(1+c_2) + \boldsymbol{p}_{t2,y} s_2}{2(1+c_2)} = -0.9401 \\
&\Rightarrow\ q_{B1} = \mathrm{ATAN2}\left\{s_1, c_1\right\} = 2.7939 \,\text{[rad]} = 160.08°.
\end{aligned}
$$

[1]The common denominator $2(1+c_2) > 0$ in the expressions of s_1 and c_1 can be discarded without affecting the final result in the evaluation of ATAN2.

The (arbitrary) choice of the + sign for the square root in s_2 results here in an *elbow up* solution for the first two links of the 3R robot. Next, with $(q_{B1}, q_{B2}) = (2.7939, 1.1076)$ [rad], the third joint variable q_{B3} is recovered from the specification $\phi_{B,d} = 2.0944$ [rad] on the end-effector orientation:

$$q_{B3} = \phi_{B,d} - (q_{B1} + q_{B2}) = -1.8071 \text{ [rad]} = -103.54°.$$

The above solution of the inverse kinematics problem is coded in Matlab by the instructions (for unitary lenghts):

```
p_t2=p_Bd-[cos(phi_Bd); sin(phi_Bd)]
px=p_t2(1);
py=p_t2(2);
c2=(px^2+py^2-2)/2
s2=sqrt(1-c2^2)  % sign + on sqrt results in elbow up solution (arbitrary choice)
q_B2=atan2(s2,c2)
s1=py*(1+c2)-px*s2  % denominator (> 0) discarded in s1 and c1
c1=px*(1+c2)+py*s2
q_B1=atan2(s1,c1)
q_B3=phi_Bd-(q_B1+q_B2)
```

Exercise #4

This exercise can be solved with ease either by hand or using the symbolic instructions of Matlab (with caution on simplifications)[2]. To determine the singularities of $\boldsymbol{J}(\boldsymbol{q})$, it is useful to get rid of the dependence of the Jacobian on q_1, by expressing the velocity $\boldsymbol{v}$ in the rotated frame 1 as[3]

$${}^1\boldsymbol{v} = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{v} = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = {}^1\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Thus, we obtain

$${}^1\boldsymbol{J}(\boldsymbol{q}) = \left({}^0\boldsymbol{R}_1\right)^T \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 0 & -l_2 s_2 & -l_3 s_3 \\ l_2 c_2 + l_3 c_3 & 0 & 0 \\ 0 & l_2 c_2 & l_3 c_3 \end{pmatrix}.$$

The determinant is

$$\det \boldsymbol{J}(\boldsymbol{q}) = \det {}^1\boldsymbol{J}(\boldsymbol{q}) = l_2 l_3 s_{2-3} \left(l_2 c_2 + l_3 c_3\right).$$

Therefore, the singularities occur when

$$\sin(q_2 - q_3) = 0 \iff q_3 = \{q_2, q_2 \pm \pi\} \quad \text{(third link stretched or folded w.r.t. the second link)}^4,$$

or when

$$l_2 c_2 + l_3 c_3 = 0 \qquad \text{(end-effector located along the axis of the first joint)},$$

[2]The robot considered in this exercise is similar to the one in Ex. #3 of June 5, 2020. However, absolute angles w.r.t. the horizontal are used here for joints 2 and 3, and the lengths of links 2 and 3 are generic rather than unitary.

[3]Because of the arbitrary definition of frame 0, we know that the variable q_1 will never enter in the definition of singularities of a serial robot manipulator —in this case in the expression of $\det \boldsymbol{J}(\boldsymbol{q})$.

[4]This comment and the next one follow from the fact that the given Jacobian is associated to a 3R spatial robot of the elbow type, with q_2 and q_3 defined as absolute link angles w.r.t. the horizontal plane.

or when both situations occur. In the first two cases, the rank of $\boldsymbol{J}$ drops by one unit. We have[5]

$$\boldsymbol{J}(\boldsymbol{q}^{\diamond}) = \boldsymbol{J}(\boldsymbol{q})|_{\sin(q_2-q_3)=0} = \begin{pmatrix} -(l_2 \pm l_3)s_1c_2 & -l_2c_1s_2 & \mp l_3c_1s_2 \\ (l_2 \pm l_3)c_1c_2 & -l_2s_1s_2 & \mp l_3s_1s_2 \\ 0 & l_2c_2 & \pm l_3c_2 \end{pmatrix}, \qquad \text{rank}\, \boldsymbol{J}(\boldsymbol{q}^{\diamond}) = 2,$$

where $c_2 \neq 0$, otherwise also $l_2c_2 + l_3c_3 = 0$ would follow. Similarly, we have

$$\boldsymbol{J}(\boldsymbol{q}^{\diamond}) = \boldsymbol{J}(\boldsymbol{q})|_{l_2c_2+l_3c_3=0} = \begin{pmatrix} 0 & -l_2c_1s_2 & -l_3c_1s_3 \\ 0 & -l_2s_1s_2 & -l_3s_1s_3 \\ 0 & l_2c_2 & l_3c_3 \end{pmatrix}, \qquad \text{rank}\, \boldsymbol{J}(\boldsymbol{q}^{\diamond}) = 2.$$

On the other hand, when both situations occur simultaneously

$$\boldsymbol{J}(\boldsymbol{q}^{*}) = \boldsymbol{J}(\boldsymbol{q})|_{\sin(q_2-q_3)=0, l_2c_2+l_3c_3=0} = \begin{pmatrix} 0 & -l_2c_1s_2 & \mp l_3c_1s_2 \\ 0 & -l_2s_1s_2 & \mp l_3s_1s_2 \\ 0 & l_2c_2 & \pm l_3c_2 \end{pmatrix}, \qquad \text{rank}\, \boldsymbol{J}(\boldsymbol{q}^{*}) = 1.$$

Choosing for instance the rank 1 singular configuration $\boldsymbol{q}^*$ with $q_2 = q_3 = \pi/2$ (and with an arbitrary q_1)[6], we have

$$\boldsymbol{J}(\boldsymbol{q}^{*}) = \boldsymbol{J}(\boldsymbol{q})|_{q_2=q_3=\pi/2} = \begin{pmatrix} 0 & -l_2c_1 & -l_3c_1 \\ 0 & -l_2s_1 & -l_3s_1 \\ 0 & 0 & 0 \end{pmatrix},$$

We obtain the following subspaces:

$$\mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix}\right\}, \qquad \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ -l_3 \\ l_2 \end{pmatrix}\right\},$$

$$\mathcal{R}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} 0 \\ l_2 \\ l_3 \end{pmatrix}\right\}, \qquad \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q}^*)\} = \text{span}\left\{\begin{pmatrix} -s_1 \\ c_1 \\ * \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Exercise #5

The structure of the optimal command $u^*(t)$ for this state-to-rest minimum time motion problem is found rather intuitively, observing that the net desired displacement is $x_f - x_i = x_f - x(0) = x_f > 0$ and that the mass has an initial velocity in the opposite direction, $\dot{x}_i = \dot{x}(0) < 0$. Thus, we have to apply first the maximum positive feasible force $U_{max} > 0$ in order to stop as soon as possible the motion in the negative direction. This will happen in a finite time T_d. Then, from the reached position $x_d = x(T_d) < 0$, with $\dot{x}(T_d) = 0$, we have a standard rest-to-rest minimum time motion problem for a displacement $x_f - x_d > x_f > 0$. Since there is no velocity limitation in the problem formulation, this second problem is solved by a symmetric bang-bang force (and acceleration) profile in a time T_{bb}. In particular, we will continue to apply the maximum positive force U_{max} for half of the residual motion, switching then to $-U_{max} < 0$ so as to decelerate and stop at the final instant $t = T = T_d + T_{bb}$.

[5] The upper signs in the expression of $\boldsymbol{J}(\boldsymbol{q}^{\diamond})$ apply when $q_3 = q_2$, the lower when $q_3 = q_2 + \pi$. The same situation happens later also in the expression of $\boldsymbol{J}(\boldsymbol{q}^*)$.

[6] The spatial 3R robot will then be fully stretched along the axis of joint 1. Similar computations can be done for $q_2 = q_3 = -\pi/2$, for $q_2 = \pi/2$ and $q_3 = -\pi/2$, or for $q_2 = -\pi/2$ and $q_3 = \pi/2$.

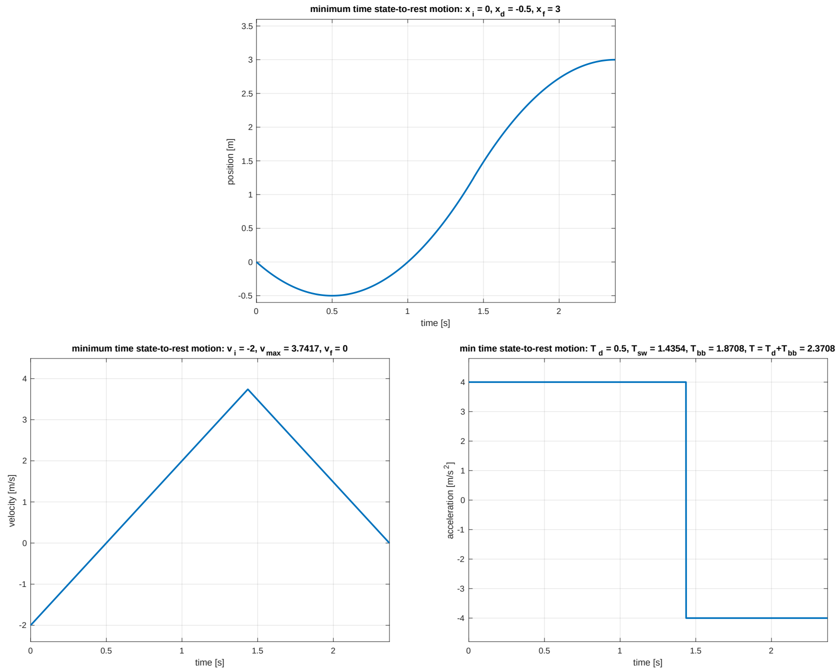

Figure 3: Minimum time state-to-rest motion: mass position, velocity, and acceleration.

Let $A_{max} = U_{max}/M = 8/2 = 4$ [m/s^2] be the maximum feasible acceleration. Applying this from $t = 0$ gives the resulting velocity profile

$$\dot{x}(t) = \dot{x}(0) + A_{max}\, t = -2 + 4\, t \overset{\downarrow}{=} 0 \qquad \Rightarrow \qquad t = T_d = -\frac{\dot{x}(0)}{A_{max}} = 0.5\,[\text{s}].$$

In the interval $t \in [0, T_d]$, the position of the mass evolves as

$$x(t) = x(0) + \dot{x}(0)\, t + A_{max}\frac{t^2}{2} = 0 - 2\, t + 4\,\frac{t^2}{2} = 2\, t\,(t-1) \qquad \Rightarrow \qquad x_d = x(T_d) = -0.5\,[\text{m}].$$

Therefore, the rest-to-rest motion should displace the mass by $L = x_f - x_d = 3 - (-0.5) = 3.5$ [m]. With a symmetric bang-bang acceleration profile, the minimum motion time for this second part of the task is

$$T_{bb} = 2\sqrt{\frac{L}{A_{max}}} = 1.8708\,[\text{s}]$$

and the switching of the command will occur at the middle point $x_d + (L/2) = 1.25$ [m] of this motion, after $T_{bb}/2 = 0.9354$ [s]; in absolute terms, at the instant $t = T_{sw} = T_d + T_{bb}/2 = 1.4354$ [s]. The peak velocity reached at this instant is $V_{max} = A_{max}T_{bb}/2 = 3.7417$ [m/s]. Finally, the minimum motion time is

$$T = T_d + T_{bb} = 2.3708\,[\text{s}].$$

The optimal force command will be

$$u^*(t) = \begin{cases} U_{max} = 8\,[\text{N}], & 0 \le t < T_{sw} = 1.4354\,[\text{s}], \\ -U_{max} = -8\,[\text{N}], & T_{sw} \le t < T = 2.3708\,[\text{s}]. \end{cases}$$

The profiles of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ in the interval $t \in [0, T]$ are shown in Fig. 3. One can clearly appreciate the asymmetry of the bang-bang acceleration profile.

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