Robotics I

July 11, 2019

Exercise 1

Consider the 3R planar robot in Fig. 1. The robot is placed in a vertical plane and holds firmly a payload P, modeled as a concentrated mass m, which is off-centered with respect to its tip (the relevant kinematic data are defined in the figure). Determine the symbolic expression of the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$ needed to keep the system in static equilibrium at a configuration $\boldsymbol{q}_0$, when the position ${}^e\boldsymbol{p}_{ep}$ of the payload is known in the end-effector frame. Using the DH convention for the joint variables, compute the numerical value of $\boldsymbol{\tau}$ for the following data:

$$L_1 = 1,\ L_2 = 0.5,\ L_3 = 0.25,\ {}^e\boldsymbol{p}_{ep} = \begin{pmatrix} 0.2 & 0.3 & 0 \end{pmatrix}^T \text{ [m]};\ \boldsymbol{q}_0 = \begin{pmatrix} \pi/3 & -\pi/6 & -\pi/6 \end{pmatrix}^T \text{ [rad]};\ m = 7\ \text{[kg]}.$$

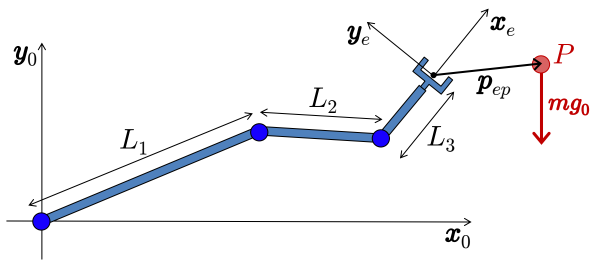

Figure 1: A 3R planar robot holding a payload P of mass m.

Exercise 2

The end effector of the RP planar robot shown in Fig. 2 should trace a linear path between the points $A = \begin{pmatrix} 4.5 & 1.5 \end{pmatrix}^T$ and $B = \begin{pmatrix} 3 & 3 \end{pmatrix}^T$ (units are in [m]), as expressed in a world reference frame ${}^wRF$. The robot has limited joint ranges: $|q_1| \le q_{1,max} = \pi/4$ [rad], $1.5 = q_{2,min} \le q_2 \le q_{2,max} = 3$ [m]. Check if the given task is feasible and, if so, place and orient the robot base frame RF_0 so that the task can be realized. Provide the positions of the two points A and B expressed in the robot base frame and the associated robot configurations.

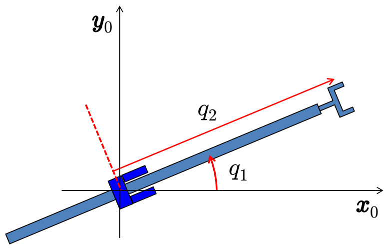

Figure 2: A RP planar robot, with the definition of the joint variables.

Exercise 3

A robot joint should perform a rest-to-rest rotation $\Delta\theta$ in a total time T by using a bang-coast-bang acceleration profile with symmetric acceleration and deceleration phases, each of duration $T_s = T/4$. Given a maximum joint velocity $V_{max} > 0$ and a maximum bound $A_{max} > 0$ for the absolute value of the joint acceleration, find the minimum time T_{min} in this class of trajectories such that the motion is feasible. Provide the general expression of T_{min} in terms of the symbolic parameters of the problem, and then its numerical value for the following data:

$$\Delta\theta = \pi \text{ [rad]}, \qquad V_{max} = 90 \ [°/\text{s}], \qquad A_{max} = 300 \ [°/\text{s}^2].$$

Sketch the resulting angular position, velocity, and acceleration profiles.

[150 minutes, open books]

Solution

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Exercise 1

With reference to Fig. 1 (and embedding the problem in 3D), we need to compute the 3×3 Jacobian $\boldsymbol{J}_P(\boldsymbol{q})$ associated to the linear velocity of the payload point P and then use duality to determine the static torque $\boldsymbol{\tau}_g = -\boldsymbol{J}_P^T(\boldsymbol{q}_0)\boldsymbol{F}_g \in \mathbb{R}^3$ that will balance, at the configuration $\boldsymbol{q}_0$, the effect of the gravity force $\boldsymbol{F}_g$ acting on the payload. This force is in the direction of $-\boldsymbol{y}_0$ and has intensity $mg_0 > 0$. In organizing computations, we take into account that the position ${}^e\boldsymbol{p}_{ep} = \begin{pmatrix} p_{ep,x} & p_{ep,x} & 0 \end{pmatrix}^T$ of the payload mass is given in the end-effector frame RF_e ($= RF_3$ of a DH convention).

Using DH angles, the position of the origin of frame RF_e is

$$\boldsymbol{p}_e(\boldsymbol{q}) = \begin{pmatrix} L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} \\ 0 \end{pmatrix},$$

with the usual compact notation for the trigonometric functions (e.g., $c_{12} = \cos(q_1 + q_2)$). The orientation of frame RF_e w.r.t. the robot base frame is expressed by the rotation matrix

$${}^0\boldsymbol{R}_e(\boldsymbol{q}) = \begin{pmatrix} c_{123} & -s_{123} & 0 \\ s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The position of the payload P is then

$$\boldsymbol{p}_P(\boldsymbol{q}) = \boldsymbol{p}_e(\boldsymbol{q}) + {}^0\boldsymbol{R}_e(\boldsymbol{q})\, {}^e\boldsymbol{p}_{ep}.$$

The llnear velocity of the payload is computed as

$$\boldsymbol{v}_P = \dot{\boldsymbol{p}}_P = \frac{\partial \boldsymbol{p}_e(\boldsymbol{q})}{\partial \boldsymbol{q}}\, \dot{\boldsymbol{q}} + {}^0\dot{\boldsymbol{R}}_e(\boldsymbol{q})^e\, \boldsymbol{p}_{ep} = \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{\omega}_e)\, {}^0\boldsymbol{R}_e(\boldsymbol{q})\, {}^e\boldsymbol{p}_{ep},$$

with the skew-symmetric matrix $\boldsymbol{S}(\boldsymbol{\omega}_e)$ given by

$$\boldsymbol{\omega}_e = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{S}(\boldsymbol{\omega}_e) = \begin{pmatrix} 0 & -(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & 0 \\ \dot{q}_1 + \dot{q}_2 + \dot{q}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Performing computations gives

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} -(L_1 s_1 + L_2 s_{12} + L_3 s_{123}) & -(L_2 s_{12} + L_3 s_{123}) & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{S}(\boldsymbol{\omega}_e)\, {}^0\boldsymbol{R}_e(\boldsymbol{q})\, {}^e\boldsymbol{p}_{ep} = \begin{pmatrix} -\left(p_{ep,x} s_{123} + p_{ep,y} c_{123}\right) \\ p_{ep,x} c_{123} - p_{ep,y} s_{123} \\ 0 \end{pmatrix} (\dot{q}_1 + \dot{q}_2 + \dot{q}_3) = \boldsymbol{n}(\boldsymbol{q}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \dot{\boldsymbol{q}}.$$

Thus

$$\boldsymbol{v}_P = \left(\boldsymbol{J}_e(\boldsymbol{q}) + \boldsymbol{n}(\boldsymbol{q}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\right) \dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}},$$

and so

$$\boldsymbol{\tau}_g = -\boldsymbol{J}_P^T(\boldsymbol{q}_0)\boldsymbol{F}_g = \boldsymbol{J}_P^T(\boldsymbol{q}_0) \begin{pmatrix} 0 \\ mg_0 \\ 0 \end{pmatrix} = mg_0 \begin{pmatrix} L_1 c_1 + L_2 c_{12} + \left(L_3 + p_{ep,x}\right) c_{123} - p_{ep,y} s_{123} \\ L_2 c_{12} + \left(L_3 + p_{ep,x}\right) c_{123} - p_{ep,y} s_{123} \\ \left(L_3 + p_{ep,x}\right) c_{123} - p_{ep,y} s_{123} \end{pmatrix}. \qquad (1)$$

Plugging in (1) the numerical values for the link lengths L_i, $i = 1, 2, 3$, the configuration $\boldsymbol{q}_0$, the mass m and the coordinates $p_{ep,x}$ and $p_{ep,y}$ of the payload in RF_e, we obtain finally

$$\boldsymbol{\tau}_g = \begin{pmatrix} 94.9715 \\ 60.6365 \\ 30.9015 \end{pmatrix} \text{ [Nm]}.$$

Exercise 2

We should check first whether the linear path from A to B fits in the bounded workspace of the RP planar robot. The workspace is represented in Fig. 3, together with a segment of length $L = \|B - A\| = \sqrt{4.5} = 2.2113$ [m] that joins ${}^0A$ with ${}^0B$, i.e., the two given points A and B as expressed in frame RF_0 (rather than in RF_w). It is immediate to see that the linear path is fully contained in the robot workspace, once the robot base is suitably placed and rotated. In fact, there are infinite ways for doing so. We shall work with the choice made in Fig. 3, which makes derivations easier: the segment $\overline{AB}$ is placed symmetrically w.r.t. the axis $\boldsymbol{x}_0$ and at a distance $D = q_{2,max}/\sqrt{2} = 2.1213$ [m] from the origin of RF_0. As a result, the coordinates of the two points in frame RF_0 are

$${}^0A = \begin{pmatrix} D \\ -L/2 \end{pmatrix} = \begin{pmatrix} 2.1213 \\ -1.0607 \end{pmatrix}, \qquad {}^0B = \begin{pmatrix} D \\ L/2 \end{pmatrix} = \begin{pmatrix} 2.1213 \\ 1.0607 \end{pmatrix} \text{ [m]}.$$

Moreover, the robot configurations corresponding to the initial and final points of the linear path are

$$\boldsymbol{q}_A = \begin{pmatrix} \text{ATAN2}\{-L/2, D\} \\ \|{}^0A\| \end{pmatrix} = \begin{pmatrix} -0.4636 \\ 2.3717 \end{pmatrix}, \quad \boldsymbol{q}_B = \begin{pmatrix} \text{ATAN2}\{L/2, D\} \\ \|{}^0B\| \end{pmatrix} = \begin{pmatrix} 0.4636 \\ 2.3717 \end{pmatrix} \text{ [rad]}.$$

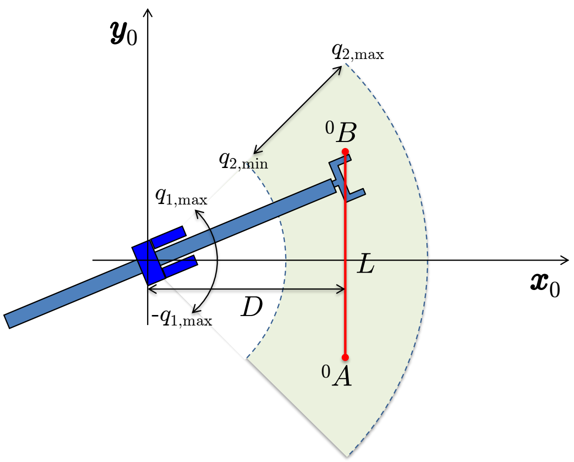

Figure 3: The workspace of the RP robot, as specified by the limited joint ranges.

The position and orientation of the base of the RP planar robot with respect to RF_w can be expressed by a (planar/2D) homogeneous transformation matrix of the form

$${}^w\boldsymbol{A}_0 = \begin{pmatrix} {}^w\boldsymbol{R}_0 & {}^w\boldsymbol{p}_0 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & p_x \\ \sin\theta_0 & \cos\theta_0 & p_y \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, from the kinematic identity for the representations of vectors in homogeneous coordinates we have

$$
{}^{w}A_{hom} = \begin{pmatrix} 4.5 \\ 1.5 \\ 1 \end{pmatrix} = {}^{w}\boldsymbol{A}_0 \, {}^{0}A_{hom} = {}^{w}\boldsymbol{A}_0 \begin{pmatrix} D \\ -L/2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} D\cos\theta_0 + \dfrac{L}{2}\sin\theta_0 + p_x & = & 4.5 \\ D\sin\theta_0 - \dfrac{L}{2}\cos\theta_0 + p_y & = & 1.5 \end{array} \right. \tag{2}
$$

and

$$
{}^{w}B_{hom} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = {}^{w}\boldsymbol{A}_0 \, {}^{0}B_{hom} = {}^{w}\boldsymbol{A}_0 \begin{pmatrix} D \\ L/2 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} D\cos\theta_0 - \dfrac{L}{2}\sin\theta_0 + p_x & = & 3 \\ D\sin\theta_0 + \dfrac{L}{2}\cos\theta_0 + p_y & = & 3. \end{array} \right. \tag{3}
$$

The linear system (2–3) of four equations in the four unknowns $\cos\theta_0$, $\sin\theta_0$, p_x and p_y is nonsingular, and thus we have a unique solution to the problem (up to a reflection of the robot placement w.r.t. the line containing the path). The solution is obtained numerically from the given data as

$$
\theta_0 = \frac{\pi}{4} \text{ [rad]} \quad (\sin\theta_0 = \cos\theta_0 = \frac{\sqrt{2}}{2}), \qquad p_x = 2.25, \quad p_y = 0.75 \text{ [m]}. \tag{4}
$$

It is interesting to note that one can obtain this solution also in closed symbolic form. The linear system to be solved is in fact

$$
\begin{pmatrix} D & L/2 & 1 & 0 \\ -L/2 & D & 0 & 1 \\ D & -L/2 & 1 & 0 \\ L/2 & D & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ B_x \\ B_y \end{pmatrix} \qquad \Longleftrightarrow \qquad \boldsymbol{M}\boldsymbol{x} = \boldsymbol{b} = \begin{pmatrix} 4.5 \\ 1.5 \\ 3 \\ 3 \end{pmatrix}.
$$

The determinant of the coefficient matrix is $\det \boldsymbol{M} = L^2 \neq 0$. The unique solution is found then as

$$
\boldsymbol{x} = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ p_x \\ p_y \end{pmatrix} = \boldsymbol{M}^{-1}\boldsymbol{b} = \begin{pmatrix} 0 & -1/L & 0 & 1/L \\ 1/L & 0 & -1/L & 0 \\ 0.5 & D/L & 0.5 & -D/L \\ -D/L & 0.5 & D/L & 0.5 \end{pmatrix} \begin{pmatrix} 4.5 \\ 1.5 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/(2L) \\ 3/(2L) \\ 3.75 - 3D/(2L) \\ 2.25 - 3D/(2L) \end{pmatrix}.
$$

Substituting now $D = 3/\sqrt{2} = 2.1213$ and $L = 1.5\sqrt{2} = 2.1213$ returns the numerical values in (4).

Exercise 3

The acceleration profile for the rest-to-rest motion trajectory $\theta(t)$ is *assigned* to be of the bang-coast-bang type, having symmetric initial and final acceleration/deceleration phases, each of duration $T_s = T/4$ and with $\ddot{\theta} = \pm A$ (to be determined), and a cruising phase that lasts for half of the motion time, i.e., $T/2$, with constant velocity $\dot{\theta} = V$. From this motion structure, choosing as arbitrary initial angle $\theta(0) = 0$, it is easy to compute the following quantities:

$$
V = \dot{\theta}\left(\frac{T}{4}\right) = A\,\frac{T}{4}, \qquad \Delta\theta_s = \theta\left(\frac{T}{4}\right) = \frac{1}{2}A\left(\frac{T}{4}\right)^2 = \frac{AT^2}{32}, \qquad \Delta\theta = \theta(T) = 2\Delta\theta_s + V\,\frac{T}{2} = \frac{3AT^2}{16}.
$$

Thus, for a desired total displacement $\Delta\theta > 0$ and a given motion time T, we have for the acceleration A and cruise velocity V

$$
A = \frac{16\Delta\theta}{3T^2} > 0 \qquad \Rightarrow \qquad V = \frac{4\Delta\theta}{3T} > 0. \tag{5}
$$

Note that, when the acceleration phase ends at time $t = T_s = T/4$, the performed angular displacement is $\Delta\theta_s = \Delta\theta/6$. By symmetry, when the deceleration phase begins at time $t = T - T_s = 3T/4$, the displacement done so far will be $\Delta\theta - \Delta\theta_s = 5\Delta\theta/6$.

Imposing now on (5) the two constraints

$$V \leq V_{max}, \qquad A \leq A_{max},$$

yields the minimum feasible motion time

$$T_{min} = \max\left\{4\sqrt{\frac{\Delta\theta}{3A_{max}}}, \frac{4\Delta\theta}{3V_{max}}\right\}.$$

With the data $\Delta\theta = \pi$ [rad], $V_{max} = 90$ [°/s] $\cdot$ $(\pi/180°) = \pi/2 = 1.5708$ [rad/s], and $A_{max} = 300$ [°/s^2] $\cdot$ $(\pi/180°) = 5\pi/3 = 5.2360$ [rad/s^2], we find the value of the optimal motion time

$$T_{min} = 2.6667 \text{ [s]}.$$

From (5), it follows $A = 2.3562$ [rad/s^2] and $V = V_{max} = 1.5708$ [rad/s]. Saturation of the constraint has occurred in the cruise phase. Figure 4 shows the resulting profiles of the angular position, velocity, and acceleration.

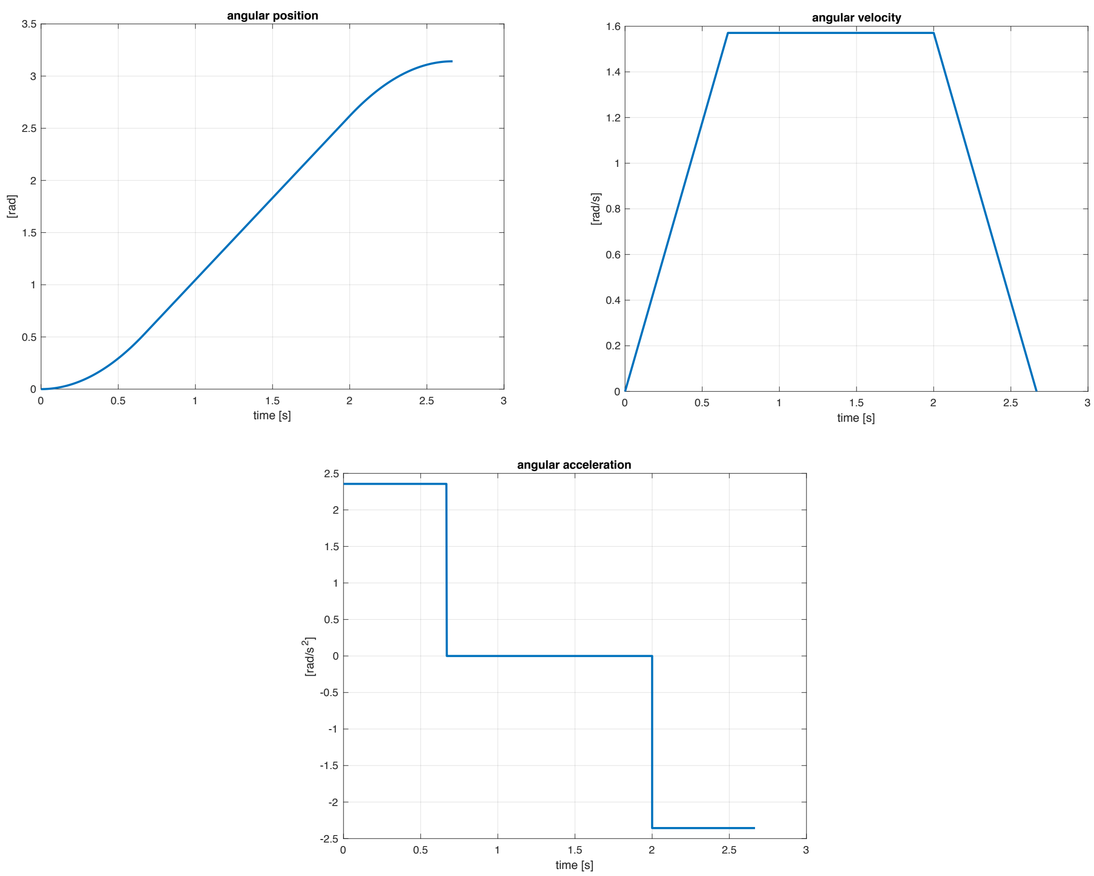

Figure 4: Position, velocity, and acceleration profiles for the considered minimum time rest-to-rest joint rotation.

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