Robotics I

January 11, 2019

Exercise 1

Consider the spatial 4-dof robot with RRPR sequence of joints shown in Fig. 1.

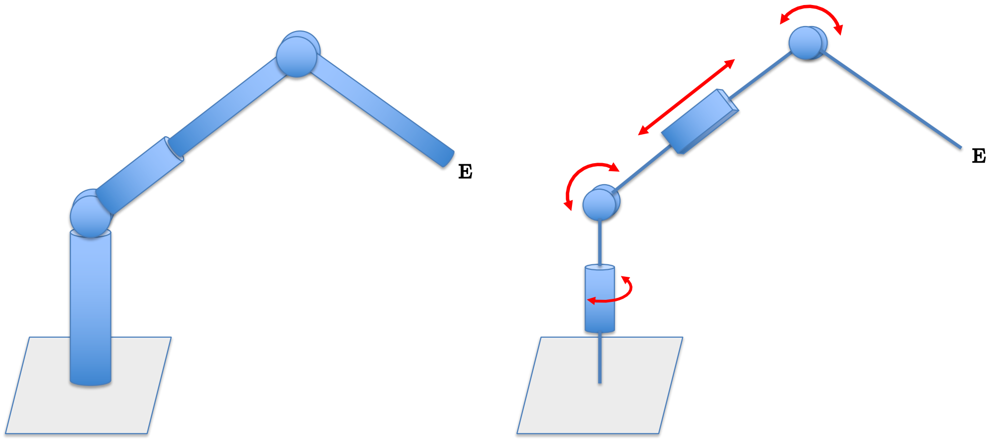

Figure 1: A 4-dof spatial RRPR robot and its kinematic skeleton.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated symbolic table of parameters, choosing only values $\alpha_i \geq 0$ $(i = 1, \ldots, 4)$ for the twist angles, and specifying the signs of all other non-zero constant parameters. The base frame (frame 0) should be placed on the ground and the origin of the (last) frame 4 at the end-effector point E. Draw the frames and fill in the table directly on the extra sheet provided separately.
- Write explicitly the four resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^3\boldsymbol{A}_4(q_4)$ and compute in an efficient way the direct kinematics $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q}) \in \mathbb{R}^3$ for the position of the origin O_4 of the last DH frame.
- Draw the robot in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$ for a generic $L > 0$. Compute the position $\boldsymbol{p}_{4,0} = \boldsymbol{p}_4(\boldsymbol{q}_0)$ as a parametric function of L and of the other constant DH parameters in symbolic form.

Exercise 2

Consider again the robot in Exercise 1.

- Derive the expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of this robot relating the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ to the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ of the end-effector frame. What is the generic rank of the lower 3×4 block $\boldsymbol{J}_A(\boldsymbol{q})$ of this matrix?
- Evaluate $\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0)$, again as a parametric function. At the same previously specified configuration $\boldsymbol{q} = \boldsymbol{q}_0$, provide answers/solutions to the following problems.
  - Find, if possible, a joint velocity $\dot{\boldsymbol{q}}_a \neq \boldsymbol{0}$ that produces no linear velocity ($\boldsymbol{v} = \boldsymbol{0}$) at the end-effector. Would then also $\boldsymbol{\omega} = \boldsymbol{0}$ follow?
  - Determine if the generalized Cartesian velocity $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}^T$ is feasible. If so, provide a joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that instantaneously realizes it.
  - Find, if possible, a non-zero generalized Cartesian force $\boldsymbol{F}_c = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{m}^T \end{pmatrix}^T \in \mathbb{R}^6$ applied at the end-effector that can be statically balanced by zero joint forces/torques ($\boldsymbol{\tau} = \boldsymbol{0}$, with $\boldsymbol{\tau} \in \mathbb{R}^4$). If such a $\boldsymbol{F}_c \neq \boldsymbol{0}$ does not exist, explain why.

Exercise 3

For a planar RP robot with direct kinematics of the end-effector position given by

$$\boldsymbol{p} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \tag{1}$$

consider the planning of a rest-to-rest motion between an initial and a final Cartesian point, respectively, $\boldsymbol{p}_A = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ [m] at $t = 0$ and $\boldsymbol{p}_B = \begin{pmatrix} -3.5355 & 3.5355 \end{pmatrix}^T$ [m] at $t = T$. Optimization of the motion time T is being sought, in two different operative conditions as follows.

a. Define a joint trajectory $\boldsymbol{q}_a^*(t)$ that minimizes the motion time for this task under the bounds on the joint accelerations,

$$|\ddot{q}_1| \leq A_1 = 200\,°/\text{s}^2, \qquad |\ddot{q}_2| \leq A_2 = 5 \text{ m/s}^2. \tag{2}$$

Find the value of the minimum motion time T_a^* and draw the time profiles of the position, velocity and acceleration of the two robot joints.

b. Consider next the additional Cartesian bound on the norm of the end-effector acceleration,

$$\|\ddot{\boldsymbol{p}}\| \leq A_c = 10 \text{ m/s}^2. \tag{3}$$

Verify whether the previous solution $\boldsymbol{q}_a^*(t)$ satisfies the bound (3) or not. If not, propose a modified joint trajectory $\boldsymbol{q}_b^*(t)$ such that both bounds (2) and (3) will be satisfied, while trying to minimize the new motion time. Discuss the rationale of your choice and the supporting equations, provide the resulting motion T_b^*, and sketch the new time profiles of your solution.

[210 minutes, open books]

Solution

January 11, 2019

Exercise 1

A DH frame assignment that satisfies the condition on the twist angles, $\alpha_i \geq 0$, $i = 1, \ldots, 4$, is shown in Fig. 2, with the associated parameters given in Tab. 1. The signs of the non-zero symbolic constants are also reported in the table.

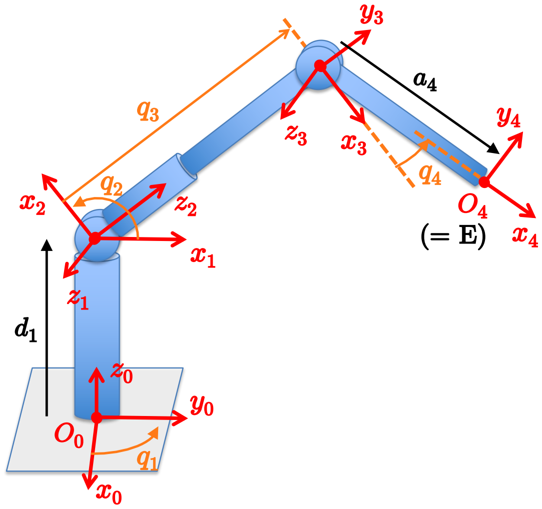

Figure 2: A possible DH frame assignment for the 4-dof spatial RRPR robot.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$\pi/2$	0	0	q_2
3	$\pi/2$	0	q_3	π
4	0	$a_4 > 0$	0	q_4

Table 1: Parameters associated to the DH frames in Fig. 2.

Based on Tab. 1, the four DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} {}^0\boldsymbol{R}_1(q_1) & {}^0\boldsymbol{p}_1 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} {}^1\boldsymbol{R}_2(q_2) & {}^1\boldsymbol{p}_2 \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_2 & 0 & \sin q_2 & 0 \\ \sin q_2 & 0 & -\cos q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} {}^2\boldsymbol{R}_3 & {}^2\boldsymbol{p}_3(q_3) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} {}^3\boldsymbol{R}_4(q_4) & {}^3\boldsymbol{p}_4(q_4) \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 & -\sin q_4 & 0 & a_4 \cos q_4 \\ \sin q_4 & \cos q_4 & 0 & a_4 \sin q_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

An efficient symbolic computation for obtaining the end-effector position $\boldsymbol{p}_4 = \boldsymbol{p}_4(\boldsymbol{q})$ makes use of recursive matrix-vector products in homogeneous coordinates as

$$
\begin{pmatrix} \boldsymbol{p}_4(\boldsymbol{q}) \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] = \begin{pmatrix} \cos q_1 \left(q_3 \sin q_2 - a_4 \cos(q_2 + q_4) \right) \\ \sin q_1 \left(q_3 \sin q_2 - a_4 \cos(q_2 + q_4) \right) \\ d_1 - q_3 \cos q_2 - a_4 \sin(q_2 + q_4) \\ 1 \end{pmatrix}. \quad (4)
$$

Figure 3 shows the robot in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$. The end-effector position is evaluated from (4) as

$$
\boldsymbol{p}_4(\boldsymbol{q}_0) = \begin{pmatrix} L \\ 0 \\ d_1 - a_4 \end{pmatrix}.
$$

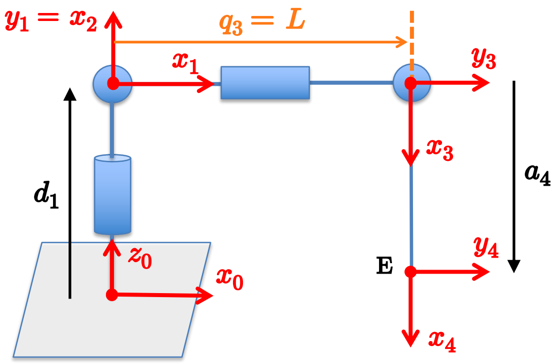

Figure 3: The RRPR robot skeleton in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$.

Exercise 2

The simplest way to derive the symbolic expression of the 6×4 geometric Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of the spatial PPRP robot in

$$
\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}}
$$

is to compute the 3×4 upper block $\boldsymbol{J}_L(\boldsymbol{q})$ by partial differentiation of the position vector $\boldsymbol{p}_4(\boldsymbol{q})$, and the

3×4 lower block $\boldsymbol{J}_A(\boldsymbol{q})$ by using the standard formula. From eq. (4), we obtain

$$\boldsymbol{J}_L(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_4(\boldsymbol{q})}{\partial \boldsymbol{q}} =$$

$$\begin{pmatrix} -\sin q_1 (q_3 \sin q_2 - a_4 \cos(q_2+q_4)) & \cos q_1 (q_3 \cos q_2 + a_4 \sin(q_2+q_4)) & \cos q_1 \sin q_2 & a_4 \cos q_1 \sin(q_2+q_4) \\ \cos q_1 (q_3 \sin q_2 - a_4 \cos(q_2+q_4)) & \sin q_1 (q_3 \cos q_2 + a_4 \sin(q_2+q_4)) & \sin q_1 \sin q_2 & a_4 \sin q_1 \sin(q_2+q_4) \\ 0 & q_3 \sin q_2 - a_4 \cos(q_2+q_4) & -\cos q_2 & -a_4 \cos(q_2+q_4) \end{pmatrix}. \tag{5}$$

Further, being ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for all i, we have

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{z}_1 & \boldsymbol{0} & {}^0\boldsymbol{R}_1(q_1){}^1\boldsymbol{R}_2(q_2){}^2\boldsymbol{R}_3{}^3\boldsymbol{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & \sin q_1 & 0 & \sin q_1 \\ 0 & -\cos q_1 & 0 & -\cos q_1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

It follows from (6) that the generic rank of matrix $\boldsymbol{J}_A(\boldsymbol{q})$ is equal to 2. Moreover, at the previously specified configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & L & 0 \end{pmatrix}^T$, we evaluate the geometric Jacobian from (5) and (6) as

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & a_4 & 1 & a_4 \\ L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{L0} \\ \boldsymbol{J}_{A0} \end{pmatrix}. \tag{7}$$

It is easy to see that

$$\rho_0 = \text{rank}\{\boldsymbol{J}_0\} = 4, \qquad \rho_{L0} = \text{rank}\,\{\boldsymbol{J}_{L0}\} = 3, \qquad \rho_{A0} = \text{rank}\,\{\boldsymbol{J}_{A0}\} = 2 \;\; \text{(as expected in general)}. \tag{8}$$

Using (7) and (8), we provide the following answers/solutions when the robot is at the configuration $\boldsymbol{q}_0$.

- Joint velocities $\dot{\boldsymbol{q}}_a \in \mathbb{R}^4$ that produce zero linear velocity at the end-effector, i.e., $\boldsymbol{v} = \boldsymbol{J}_{L0}\,\dot{\boldsymbol{q}}_a = \boldsymbol{0}$, belong to the null space of $\boldsymbol{J}_{L0}$. Since the robot has $n = 4$ joints and $\rho_{L0} = 3$, the null space of $\boldsymbol{J}_{L0}$ has dimension $n - \rho_{L0} = 1$. Thus, there are ∞^1 joint velocities $\dot{\boldsymbol{q}}_a$, all having the form

$$\dot{\boldsymbol{q}}_a = \alpha \begin{pmatrix} 0 \\ 0 \\ -a_4 \\ 1 \end{pmatrix}, \qquad \alpha \lesseqgtr 0,$$

  that produce zero linear velocity at the end-effector.

- Since

$$\boldsymbol{J}_{A0}\,\dot{\boldsymbol{q}}_a = \alpha \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{\omega}_a,$$

  any non-vanishing ($\alpha \neq 0$) joint velocity $\dot{\boldsymbol{q}}_a$ in the null space of $\boldsymbol{J}_{L0}$ will be associated with a non-zero angular velocity, i.e., $\boldsymbol{\omega}_a \neq \boldsymbol{0}$.

- A generalized Cartesian velocity $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{v}^T & \boldsymbol{\omega}^T \end{pmatrix}^T \in \mathbb{R}^6$ will be feasible if and only if it belongs to the range space (or image) of $\boldsymbol{J}_0$. Since $\rho_0 = 4$, the range space of this matrix is given by the linear combinations of all its four columns. A simple test to verify whether or not the Cartesian velocity

$\boldsymbol{V} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}^T$ belongs to the image of $\boldsymbol{J}_0$ is to border the matrix $\boldsymbol{J}_0$ with the column vector $\boldsymbol{V}$ and check the rank of the resulting matrix. Since

$$\text{rank}\{\boldsymbol{J}_0\} = \text{rank}\left\{\begin{pmatrix} 0 & a_4 & 1 & a_4 \\ L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\right\} = 4 < 5 = \text{rank}\left\{\begin{pmatrix} 0 & a_4 & 1 & a_4 & 1 \\ L & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -2 \end{pmatrix}\right\} = \text{rank}\left\{\begin{pmatrix} \boldsymbol{J}_0 & \boldsymbol{V} \end{pmatrix}\right\},$$

the given vector $\boldsymbol{V}$ will not belong to the image of $\boldsymbol{J}_0$. Thus, there is no joint velocity $\dot{\boldsymbol{q}}_b \in \mathbb{R}^4$ that will instantaneously realize $\boldsymbol{V}$ (i.e., $\boldsymbol{J}_0\dot{\boldsymbol{q}}_b \neq \boldsymbol{V}$, $\forall \dot{\boldsymbol{q}}_b$).

- A generalized Cartesian force $\boldsymbol{F}_c = \begin{pmatrix} \boldsymbol{f}^T & \boldsymbol{m}^T \end{pmatrix}^T \in \mathbb{R}^6$ applied at the end-effector is statically balanced by zero joint forces/torques $\boldsymbol{\tau} \in \mathbb{R}^4$, i.e., $\boldsymbol{\tau} = \boldsymbol{J}_0^T \boldsymbol{F}_c = \boldsymbol{0}$ if and only if it belongs to the null space of $\boldsymbol{J}_0^T$. Since the Cartesian task has dimension $m = 6$ and $\text{rank}\left\{\boldsymbol{J}_0^T\right\} = \rho_0 = 4$, the null space of $\boldsymbol{J}_0^T$ will have dimension $m - \rho_0 = 2$. A basis for this null space is given by

$$\mathcal{N}\{\boldsymbol{J}_0^T\} = \text{range}\left\{\begin{pmatrix} 0 & 0 \\ 0 & -1/L \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \text{range}\left\{\begin{pmatrix} \boldsymbol{F}_{c1} & \boldsymbol{F}_{c2} \end{pmatrix}\right\}.$$

Therefore, one will obtain $\boldsymbol{\tau} = \boldsymbol{J}_0^T\left(\alpha_1 \boldsymbol{F}_{c1} + \alpha_2 \boldsymbol{F}_{c2}\right) = \boldsymbol{0}$ for any value of the scalars α_1 and α_2.

Exercise 3

The rest-to-rest trajectory planning problem for the planar RP robot can be tackled in the joint space. Through the inverse kinematics of this robot[1]

$$q_1 = \text{ATAN2}\left\{p_y, p_x\right\}, \qquad q_2 = \sqrt{p_x^2 + p_y^2},$$

we obtain for the initial and final Cartesian points, respectively

$$\boldsymbol{p}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ [m]} \quad \Rightarrow \quad \boldsymbol{q}_A = \begin{pmatrix} 0.6435 \\ 5 \end{pmatrix} \text{ [rad, m]} \quad \left(= \begin{pmatrix} 36.87 \\ 5 \end{pmatrix} \text{ [°, m]} \right),$$

to be assumed at the initial time $t = 0$ with zero initial velocity $\dot{\boldsymbol{q}}_A = \boldsymbol{0}$, and

$$\boldsymbol{p}_B = \begin{pmatrix} -3.5355 \\ 3.5355 \end{pmatrix} \text{ [m]} \quad \Rightarrow \quad \boldsymbol{q}_B = \begin{pmatrix} 2.3562 \\ 5 \end{pmatrix} \text{ [rad, m]} \quad \left(= \begin{pmatrix} 135 \\ 5 \end{pmatrix} \text{ [°, m]} \right),$$

to be assumed at the final time $t = T$ with zero initial velocity $\dot{\boldsymbol{q}}_B = \boldsymbol{0}$. Note that the same value $q_{A,2} = q_{B,2} = 5$ [m] has been obtained for the prismatic joint.

Case a. When seeking the minimization of the motion time T under the joint acceleration limits (2) only, we can proceed separately for each joint. Since the second (prismatic) joint doesn't need to move

[1] One could have used also the second solution to the inverse kinematics

$$q_1 = \text{ATAN2}\left\{-p_y, -p_x\right\}, \qquad q_2 = -\sqrt{p_x^2 + p_y^2}.$$

The following developments would have been the same, modulo a change of sign for the joint motions. In any event, the objective of minimizing motion time suggests that the same solution class of the inverse kinematics should be used for both Cartesian points $\boldsymbol{p}_A$ and $\boldsymbol{p}_B$.

($q^*_{a,2}(t) \equiv 0$), it does not impose any lower bound on the motion time. The problem is solved by looking just at the first joint motion. In the absence of a joint velocity limit, the time-optimal motion of the first joint will be a bang-bang profile in acceleration and, accordingly, a triangular profile in velocity. Since a positive displacement is requested for q_1, we have

$$\ddot{q}^*_{a,1}(t) = \begin{cases} A_1, & t \in \left[0, \dfrac{T}{2}\right) \\ -A_1, & t \in \left[\dfrac{T}{2}, T\right] \end{cases},$$

and thus

$$\dot{q}^*_{a,1}(t) = \begin{cases} A_1 t, & t \in \left[0, \dfrac{T}{2}\right) \\ A_1 \dfrac{T}{2} - A_1 \left(t - \dfrac{T}{2}\right), & t \in \left[\dfrac{T}{2}, T\right] \end{cases}$$

and[2]

$$q^*_{a,1}(t) = \begin{cases} q_{A,1} + \dfrac{1}{2} A_1 t^2, & t \in \left[0, \dfrac{T}{2}\right) \\ q_{A,1} + \dfrac{A_1 T^2}{8} + A_1 \dfrac{T}{2}\left(t - \dfrac{T}{2}\right) - \dfrac{1}{2} A_1 \left(t - \dfrac{T}{2}\right)^2, & t \in \left[\dfrac{T}{2}, T\right] \end{cases}.$$

By symmetry, half of the total displacement $\Delta q_1 = |q_{B,1} - q_{A,1}|$ will be completed at the midtime of motion. Therefore, from the equality

$$q^*_{a,1}\left(\frac{T}{2}\right) = q_{A,1} + \frac{A_1 T^2}{8} = q_{A,1} + \frac{q_{B,1} - q_{A,1}}{2},$$

we obtain

$$T^*_a = \sqrt{\frac{4\,|q_{B,1} - q_{A,1}|}{A_1}} = 1.401 \text{ [s]}, \tag{9}$$

where $A_1 = 200\,[^\circ/\text{s}^2]$ and $q_{B,1} - q_{A,1} = 98.13^\circ$ have been used. Moreover, the peak velocity of joint 1 will be

$$V_1 = \dot{q}^*_{a,1}\left(\frac{T^*_a}{2}\right) = A_1 \frac{T^*_a}{2} = \sqrt{A_1\,|q_{B,1} - q_{A,1}|} = 140.1\,[^\circ/\text{s}] = 2.4451\,[\text{rad/s}]. \tag{10}$$

The plots of the joint positions, velocities and accelerations are shown in Fig. 4. The resulting Cartesian path traced by the end-effector along the time-optimal joint trajectory $\boldsymbol{q}^*_a$ is indeed an arc of a circle of radius $r = q_{A,2} = 5$ [m], as shown in Fig. 5.

Indeed the solution found for this case is not at all unique. Joint 2 may in fact move in an arbitrary way, as long as it goes back to the same initial position with zero final velocity within the instant of time T^*_a, and without violating its acceleration bound A_2 during the interval $[0, T^*_a]$.

Case b. In order to verify whether the previous solution $\boldsymbol{q}^*_a(t)$ satisfies the additional Cartesian bound (3), we need to compute the end-effector acceleration $\ddot{\boldsymbol{p}}$ differentiating twice the expression (1) of the direct kinematics. We obtain

$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{q}_2 \cos q_1 - \dot{q}_1 q_2 \sin q_1 \\ \dot{q}_2 \sin q_1 + \dot{q}_1 q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

[2]Formulas are written so as to highlight how they were obtained via integration. The expressions for the velocity and position in in the second half of the motion can be rewritten also as

$$\dot{q}^*_{a,1}(t) = A_1 T - A_1 t, \qquad q^*_{a,1}(t) = q_{A,1} - \frac{A_1 T^2}{4} + A_1 T\, t - \frac{A_1}{2}\, t^2, \qquad t \in \left[\frac{T}{2}, T\right].$$

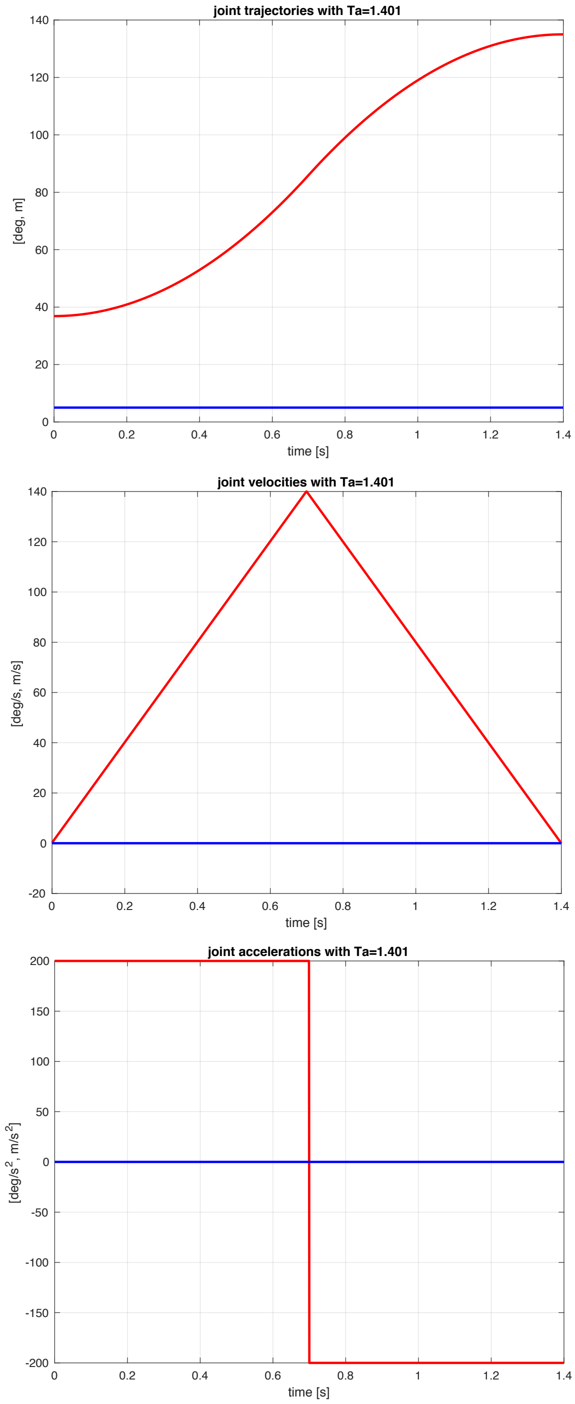

Figure 4: Position, velocity and acceleration profiles of the solution trajectory $\boldsymbol{q}_a^*$ for Case a. Joint 1 = red, joint 2 = blue.

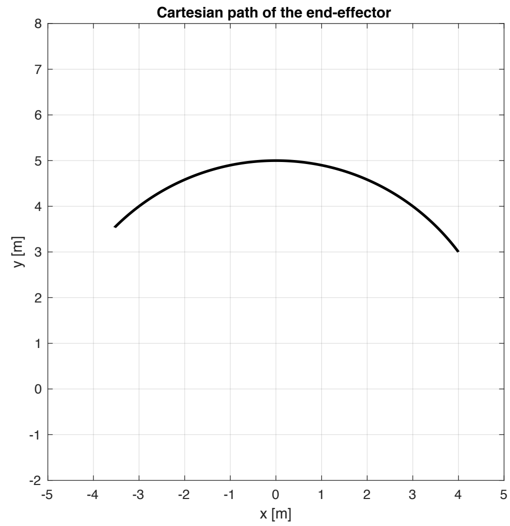

Figure 5: Cartesian path traced by the end-effector along the joint trajectory $\boldsymbol{q}_a^*$ of Fig. 4.

and

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -\left(\dot{q}_1^2\, q_2 \cos q_1 + 2\,\dot{q}_1 \dot{q}_2 \sin q_1\right) \\ -\dot{q}_1^2\, q_2 \sin q_1 + 2\,\dot{q}_1 \dot{q}_2 \cos q_1 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{11}$$

with $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ having a quadratic dependence on the components of $\dot{\boldsymbol{q}}$.

Since the norm of a vector is invariant with respect to a rotation, $\|\ddot{\boldsymbol{p}}\| = \|\boldsymbol{R}\,\ddot{\boldsymbol{p}}\|$, to ease computations it is convenient to express the acceleration (11) in a frame rotated with q_1 on the plane (x, y), namely

$$\begin{aligned} {}^1\ddot{\boldsymbol{p}} &= \begin{pmatrix} \cos q_1 & \sin q_1 \\ -\sin q_1 & \cos q_1 \end{pmatrix} \ddot{\boldsymbol{p}} = \boldsymbol{R}^T(q_1)\, \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{R}^T(q_1)\, \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -\dot{q}_1^2\, q_2 \\ 2\,\dot{q}_1 \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \ddot{q}_2 - \dot{q}_1^2\, q_2 \\ q_2\, \ddot{q}_1 + 2\,\dot{q}_1 \dot{q}_2 \end{pmatrix}. \end{aligned} \tag{12}$$

As a result, we have from (12) the closed-form expression

$$\|\ddot{\boldsymbol{p}}\| = \|{}^1\ddot{\boldsymbol{p}}\| = \sqrt{\left(\ddot{q}_2 - \dot{q}_1^2\, q_2\right)^2 + \left(q_2\, \ddot{q}_1 + 2\,\dot{q}_1 \dot{q}_2\right)^2}. \tag{13}$$

When (13) is evaluated along the joint trajectory $\boldsymbol{q}_a^*(t)$, since $q_2 = q_{A,2}$ (constant) and $\dot{q}_2 = \ddot{q}_2 = 0$, we have

$$\|\,\ddot{\boldsymbol{p}}\,\|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = \sqrt{q_{A,2}^2\, (\dot{q}_1^4 + \ddot{q}_1^2)}. \tag{14}$$

Taking the maximum of $\dot{q}_1$ and $\ddot{q}_1$ (which occur both at the midtime of motion), using (10), and converting $A_1 = 200 \cdot \pi/180 = 3.4906$ [rad/s^2] yields

$$\max \|\ddot{\boldsymbol{p}}\|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = \sqrt{q_{A,2}^2\, (V_1{}^4 + A_1^2)} = \sqrt{(5)^2\, [(2.4451)^4 + (3.4906)^2]} = 34.58 > 10 = A_c. \tag{15}$$

Thus, $\|\ddot{\boldsymbol{p}}\|$ reaches a peak which is about 3.5 times higher than the bound A_c in (3). The evolution of the norm of the Cartesian acceleration (14) is shown in Fig. 6 (this plot is obtained using Matlab).

Therefore, in the presence of this additional bound, the previous trajectory should be made considerably slower. This can be achieved in many ways, leading to different solutions for the new trajectory $\boldsymbol{q}_b(t)$,

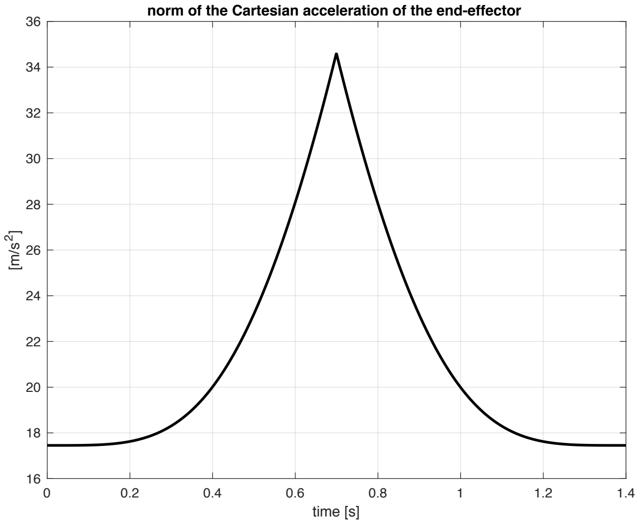

Figure 6: Norm of the end-effector acceleration $\ddot{\boldsymbol{p}}$ along the joint trajectory $\boldsymbol{q}_a^*$ of Fig. 4.

each with a possibly different motion time T_b. Indeed, among all feasible $\boldsymbol{q}_b(t)$, the optimal solution $\boldsymbol{q}_b^*(t)$ will have the least completion time $T_b^* \leq T_b$. However, finding the minimum time in this situation is not straightforward. Below we give some clues on how to proceed, al least for generating trajectories that satisfy all constraints.

1. **Uniform scaling.** The simplest way to recover feasibility is to scale uniformly the motion time by a factor $k > 1$, i.e., $T_{b,1} = kT_a^*$, reducing thus both the joint velocity (by a factor k) and the joint acceleration (by a factor k^2). Using again eqs. (14–15), we find the minimum value k by imposing the equality

$$\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_{b,1}^*} = \sqrt{q_{A,2}^2 \left[\left(\frac{V_1}{k} \right)^4 + \left(\frac{A_1}{k^2} \right)^2 \right]} = \frac{1}{k^2} \sqrt{q_{A,2}^2 \left(V_1^4 + A_1^2 \right)} = \frac{1}{k^2} \max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_a^*} = A_c, \tag{16}$$

and thus

$$k = \sqrt{\frac{\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_a^*}}{A_c}} = 1.8596 \qquad \Rightarrow \qquad T_{b,1} = kT_a^* = 2.605\,[\text{s}]. \tag{17}$$

The trajectory profile will be the same as before, with the second joint always at rest and the first (revolute) joint having a bang-bang acceleration with $A_{max} = A_1/k^2 = 57.84$ [°/s²] and a triangular velocity with peak $V_{max} = V_1/k = 75.34$ [°/s]. The plots are similar to those in Fig. 4.

2. **Including a cruise phase.** A different strategy, still following the same Cartesian path as in Fig. 5, would be to apply a smaller acceleration to joint 1 (with $A_{t,max} < A_1$) until reaching some cruise speed $V_{t,max} < V_1$ at $t = T_t$, travel at that speed for a suitable time, and then decelerate for an interval T_t until the final stop at $T_{b,2}$, while complying at all times with the bound on the norm of $\ddot{\boldsymbol{p}}$. The resulting trajectory $\boldsymbol{q}_{b,2}^*$ would be bang-coast-bang in acceleration, i.e., with a symmetric trapezoidal profile in velocity —the reason for the subscript 't' in the above quantities. As usual in such cases, from

$$T_t = \frac{V_{t,max}}{A_{t,max}} \text{ and } \Delta q_1 = (T_{b,2} - T_t)V_{t,max} \quad \Rightarrow \quad T_{b,2} = \frac{\Delta q_1}{V_{t,max}} + \frac{V_{t,max}}{A_{t,max}} = \frac{\Delta q_1 A_{t,max} + V_{t,max}^2}{V_{t,max} A_{t,max}}. \tag{18}$$

On the other hand, the reaching of the Cartesian acceleration limit implies from (14)

$$\max \| \ddot{\boldsymbol{p}} \|_{|\boldsymbol{q}=\boldsymbol{q}_{b,2}^*} = \sqrt{q_{A,2}^2 \left(V_{t,max}^4 + A_{t,max}^2 \right)} = A_c. \tag{19}$$

This is imposed at $t = T_t$, namely at the end of the acceleration phase, where also the velocity has reached its maximum. Solving for $V_{t,max}$ from (19)

$$V_{t,max} = \sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}, \qquad \text{(where } A_{t,max} < \frac{A_c}{q_{A,2}} \text{ is being assumed)}$$

and substituting this within $T_{b,2}$ in (18) gives

$$T_{b,2} = \frac{\Delta q_1}{\sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}} + \frac{\sqrt[4]{\left(\frac{A_c}{q_{A,2}}\right)^2 - A_{t,max}^2}}{A_{t,max}} = f(A_{t,max}), \qquad \text{for } 0 < A_{t,max} < \frac{A_c}{q_{A,2}}. \tag{20}$$

From the functional dependence in (20), it is clear that the minimum of $T_{b,2}$ will not occur neither for very small values of $A_{t,max}$ nor close to its upper limit, as the function f goes to infinity in both cases. Figure 7 plots the value of $T_{b,2}$ as a function of $A_{t,max}$ in its interval of definition. It can be seen that a minimum is (approximately) found for

$$A_{t,max} = 1.59\,[\text{rad/s}^2] = 91.1\,[^\circ/\text{s}^2] \qquad \Rightarrow \qquad T_{b,2} = 2.2475\,[\text{s}].$$

Thus, the motion time $T_{b,2}$ found when using a trapezoidal profile is smaller that the value $T_{b,1}$ found by uniform scaling by about 14%. This reflects the better adaptability of this new trajectory.

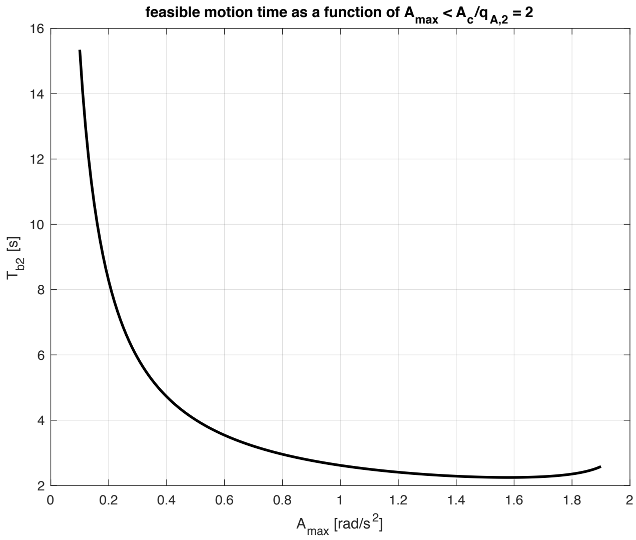

Figure 7: Dependence of the motion time $T_{b,2}$ on $A_{t,max}$ for a feasible trapezoidal velocity profile in Case b.

3. **Moving also joint 2.** Another alternative is to explore the use of an extra motion of the (prismatic) joint 2, which should allow a faster displacement of joint 1 when the Cartesian acceleration bound is limiting the completion time. Note first that, when using (14) in a motion with constant $q_2 = q_{2,0}$, the value of $\|\ddot{\boldsymbol{p}}\|$ will decrease linearly with $q_{2,0}$. Thus, one could try the following three-phase trajectory.

   I. With the first joint kept fixed, retract the second joint to a value $q_{2,0} < q_{A,2}$, using a bang-bang (negative-positive) acceleration profile with maximum acceleration equal to the minimum between A_c and A_2. Let T_I be the time needed for this rest-to-rest motion of joint 2.

II. Perform the displacement Δq_1 with the first joint, just like in Case a. but now with $q_2 = q_{2,0}$. By a judicious choice of $q_{2,0}$, this motion can be executed using a bang-bang acceleration profile with maximum acceleration equal to A_1. In fact, the value of $q_{2,0}$ could be such that the norm of the Cartesian acceleration always satisfies its bound, and reaches the maximum value A_c at least in one instant. Let T_{II} be the needed motion time for this phase.

III. Reverse the motion of phase I so as to move joint 2 from $q_{2,0}$ back to $q_{B,2} = q_{A,2}$, using a bang-bang (now, positive-negative) acceleration profile. Indeed, the time needed for this phase is $T_{III} = T_I$.

The minimum retraction of joint 2 that will guarantee the saturation of the bound (3) mentioned in phase II is evaluated by equating

$$\max \|\ddot{\boldsymbol{p}}\| = \sqrt{q_{20}^2 \left(V_1{}^4 + A_1^2\right)} = A_c \qquad \Rightarrow \qquad q_{20} = \frac{A_c}{\sqrt{V_1{}^4 + A_1^2}} = 1.4445\,[\mathrm{m}]. \tag{21}$$

This implies a net displacement $\Delta q_2 = |q_{20} - q_{A,2}| = 3.5555$ [m] for joint 2. Thus, being $A_2 = 5 < 10 = A_c$, the motion times of the three phases are computed as

$$T_I = \sqrt{\frac{4|q_{20} - q_{A,2}|}{A_2}} = 1.6865, \qquad T_{II} = \sqrt{\frac{4|q_{B,1} - q_{A,1}|}{A_1}} = 1.401\ (= T_a^*),$$

and

$$T_{III} = \sqrt{\frac{4|q_{B,2} - q_{20}|}{A_2}} = T_I = 1.6865.$$

Thus, the total motion time is

$$T_{b,3} = 2T_I + T_{II} = 4.774\,[s].$$

As a result, in this case there is no benefit with such approach with respect to the two previous methods. While some time reduction could still be achieved by moving both joints in this way simultaneously (rather than in alternating sequence), the analysis would become far too complex —and perhaps the outcome would still not be worth of it.

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