Robotics I

January 11, 2018

Exercise 1

The Panda by Franka Emika shown in Fig. 1 is an innovative lightweight robot intended for friendly and safe human-robot interaction. The robot has seven revolute joints and its kinematics is characterized by a spherical shoulder, an elbow with two offsets, and a non-spherical wrist. This combination allows eliminating unaccessible 'holes' close to the robot base, thus increasing the robot workspace.

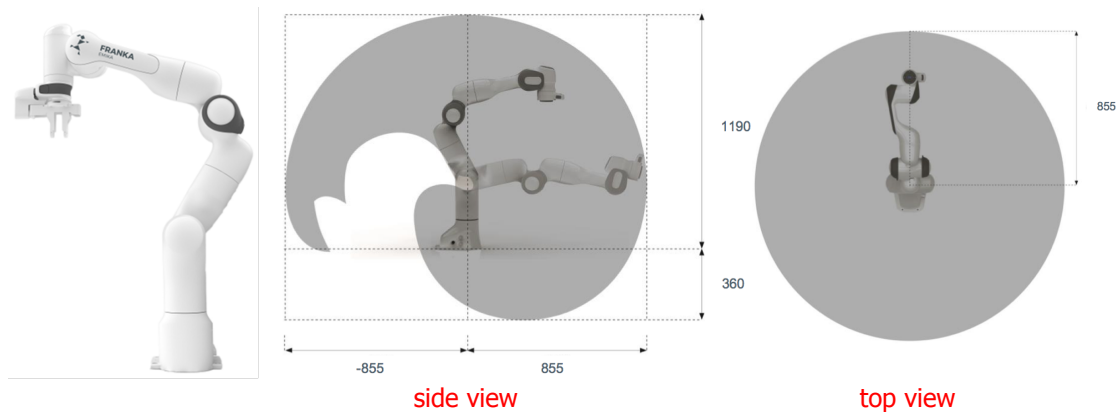side view   top view

Figure 1: The 7R Panda robot by Franka Emika and two views of its workspace.

- Assign the link frames according to the Denavit-Hartenberg (DH) convention and complete the associated symbolic table of parameters, specifying also the signs of the non-zero constant parameters. Draw the frames and fill in the table directly on the extra sheet #1 provided separately. Therein, the two DH frames 0 and 7 are already assigned and should not be modified. [Please, make clean drawings and return the completed sheet with your name written on it.]

- Write explicitly the seven resulting DH homogeneous transformation matrices ${}^0\boldsymbol{A}_1(q_1)$ to ${}^6\boldsymbol{A}_7(q_7)$. [Do NOT attempt to write the direct kinematics in symbolic form!]

- Assume that all constant DH parameters have been specified numerically. While the robot is moving, the actual position $\boldsymbol{p} \in \mathbb{R}^3$ of the end-effector (i.e., of the origin O_7 of frame 7) should be computed in real time using the measurements of $\boldsymbol{q} \in \mathbb{R}^7$ collected by the encoders at each sampling instant (say, every 400 μs). Provide an efficient scheme for this computation, and determine the total number of elementary operations (evaluation of trigonometric functions, of products $\times$, and of sums $+$) required at each sampling step. You may proceed without exploiting the specific structure of the DH matrices, or customize the procedure avoiding unnecessary elementary operations for this robot.

Exercise 2

A number of statements are reported on the extra sheet #2, regarding the Newton and the Gradient methods for the numerical solution of inverse kinematics problems. Check if each statement is **True** or **False**, providing also a *very short* motivation/explanation for your answer. [Return the sheet with your answers and your name written on it.]

Exercise 3

With reference to the setup in Fig. 2, two identical planar 3R manipulators, a master robot M and a slave robot S having link lengths $\ell_1 = \ell_2 = 0.5$ and $\ell_3 = 0.25$ [m], should perform a Cartesian motion task in coordination. The base frames of the robots are displaced by $\boldsymbol{p}_{MS} = (\Delta x, \Delta y, 0) = (1.6, 0.9, 0)$ [m] and rotated by $\alpha_{MS} = \pi$ [rad] around the common $\boldsymbol{z}_0$ axis. The desired Cartesian motion starts at $t = t_0$ from the position $\boldsymbol{p}_M(t_0) \in \mathbb{R}^2$ assumed by the end-effector of the master robot in the configuration $\boldsymbol{q}_M(t_0) = (\pi/2, -\pi/3, 0)$ [rad] and will proceed along a straight line path, which is specified by the initial direction of the end-effector velocity $\boldsymbol{v}_M = \dot{\boldsymbol{p}}_M(t_0) \in \mathbb{R}^2$ resulting from $\dot{\boldsymbol{q}}_M(t_0) = (-\pi/6, 0, -\pi/2)$ [rad/s]. The slave robot should execute the same Cartesian motion in position, while keeping its end-effector always oriented orthogonally to the linear path (more specifically, rotated by a constant angle $\beta = -\pi/2$ [rad] with respect to the vector $\boldsymbol{v}_M$).

- Determine an initial configuration $\boldsymbol{q}_S(t_0)$ of the slave robot such that its end-effector position $\boldsymbol{p}_S(t_0) \in \mathbb{R}^2$ and orientation are initially matched with those required by the motion task.
- Determine the initial joint velocity $\dot{\boldsymbol{q}}_S(t_0) \in \mathbb{R}^3$ of the slave robot, in order to match also the initial desired Cartesian velocity (i.e., $\boldsymbol{v}_S = \dot{\boldsymbol{p}}_S(t_0)$ is equal to $\boldsymbol{v}_M$).
- If the initial configuration of the slave robot is not matched with the desired Cartesian motion, how can this robot still perform the task after an initial transient, with its task error decreasing exponentially to zero?
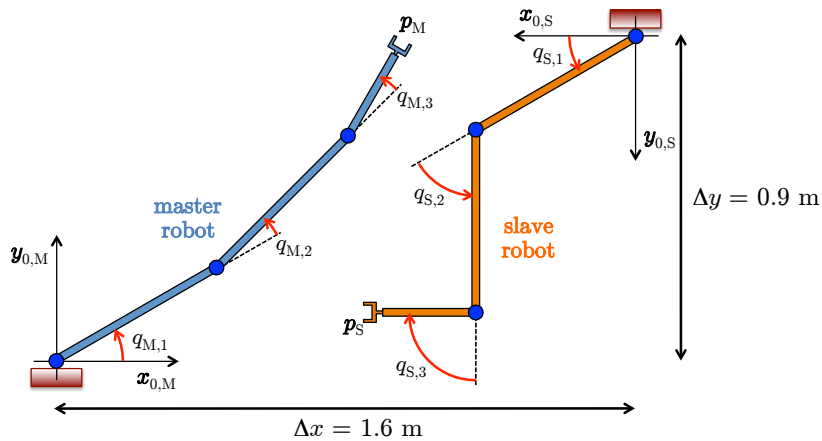

Figure 2: The relative placement of the bases of the two planar 3R robots that should perform a Cartesian motion task in coordination.

Exercise 4

Consider the planning of a smooth trajectory for a planar RP robot (with unlimited joint ranges) between the configurations $\boldsymbol{q}(0) = (\pi/4, -1)$ [rad,m] at $t = 0$ and $\boldsymbol{q}(T) = (-\pi/2, 1)$ [rad,m] at $t = T$. The initial and final joint velocity and acceleration should be zero, and the acceleration should be continuous in the entire time interval $[0, T]$. The following joint velocity and acceleration bounds are also present:

$$|\dot{q}_1| \le V_1 = 120°/\text{s}, \quad |\dot{q}_2| \le V_2 = 180 \text{ cm/s}, \quad |\ddot{q}_1| \le A_1 = 150°/\text{s}^2, \quad |\ddot{q}_2| \le A_2 = 200 \text{ cm/s}^2. \quad (1)$$

Define a suitable class of trajectories and choose a final time $T = 3$ s. Will the resulting robot motion be feasible with respect to the bounds in (1)? Using uniform time scaling, find the minimum feasible motion time T^* to perform the desired reconfiguration along the chosen trajectory. Sketch a plot of the resulting joint velocity and acceleration profiles. Will the robot cross a singular configuration during its motion?

[240 minutes, open books but no computer or smartphone]

7R Panda robot by Franka Emika – DH frames assignment and table

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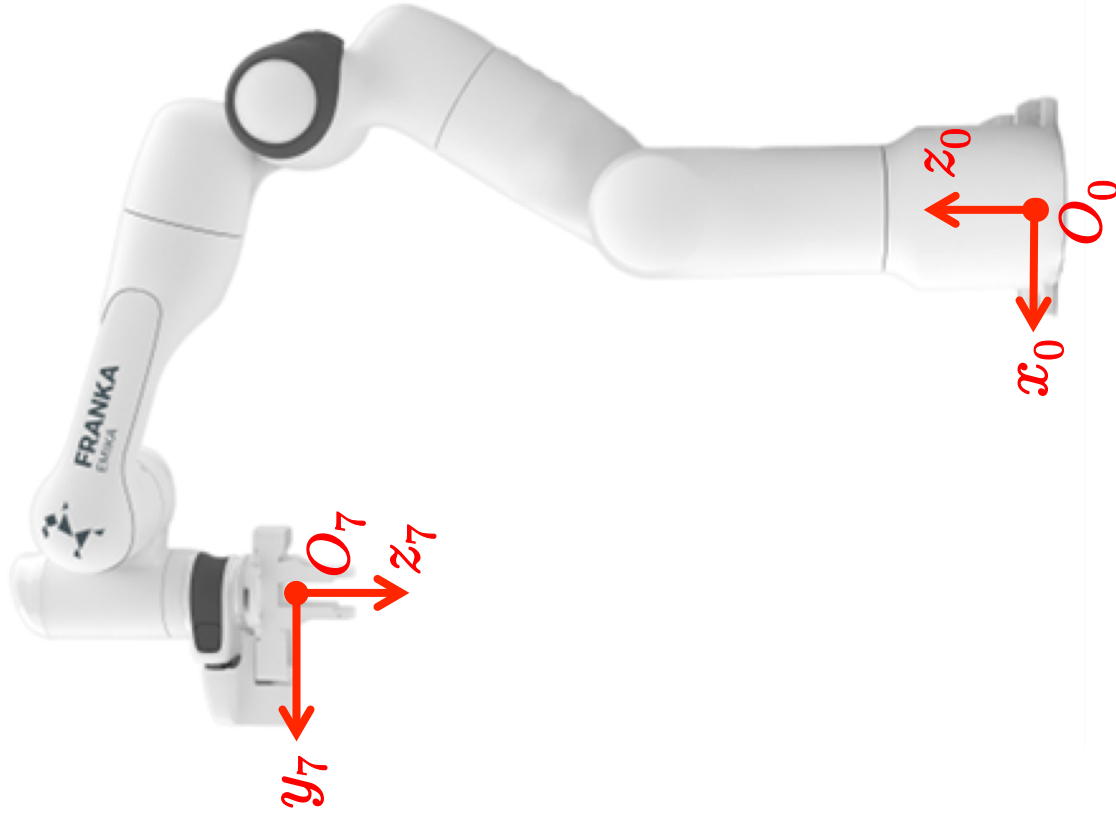
i	α_i	a_i	d_i	θ_i
1				
2				
3				
4				
5				
6				
7				

Robotics I - Sheet for Exercise 2

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Name: _____

Consider the basic algorithms of the two main numerical methods used for solving inverse kinematics problems, denoted here as **N** (Newton method) and **G** (Gradient method). Check if each of the following statements is **True** or **False**, and provide a *very short* motivating/explanation sentence.

1. **N** and **G** always fail at singularities.
   True ☐   **False** ☐

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2. **G** stops when a singularity is encountered.
   True ☐   **False** ☐

---

3. Out of singularities, **N** finds always a solution faster than **G**.
   True ☐   **False** ☐

---

4. **N** can be used only when there is a single global solution to the problem.
   True ☐   **False** ☐

---

5. Both **N** and **G** need knowledge of the analytic Jacobian of the task.
   True ☐   **False** ☐

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6. For a non-square Jacobian, the pseudoinverse should replace the Jacobian transpose in **G**.
   True ☐   **False** ☐

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7. Close to a solution, it is computationally faster to evaluate an iteration of **N** than one of **G**.
   True ☐   **False** ☐

---

8. **G** works better for linear problems, **N** for quadratic ones.
   True ☐   **False** ☐

---

9. Neither **N** nor **G** would terminate without the use of a small tolerance on the final error.
   True ☐   **False** ☐

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10. Beside matrix operations with the Jacobian and the error, **G** needs an extra choice to be made.
    True ☐   **False** ☐

Solution

January 11, 2018

Exercise 1

To help in defining DH frames for the Panda robot by Franka Emika, Figure 3 shows preliminarily the arm decomposition into the series of its links and the definition of the seven joint axes (with a few comments).
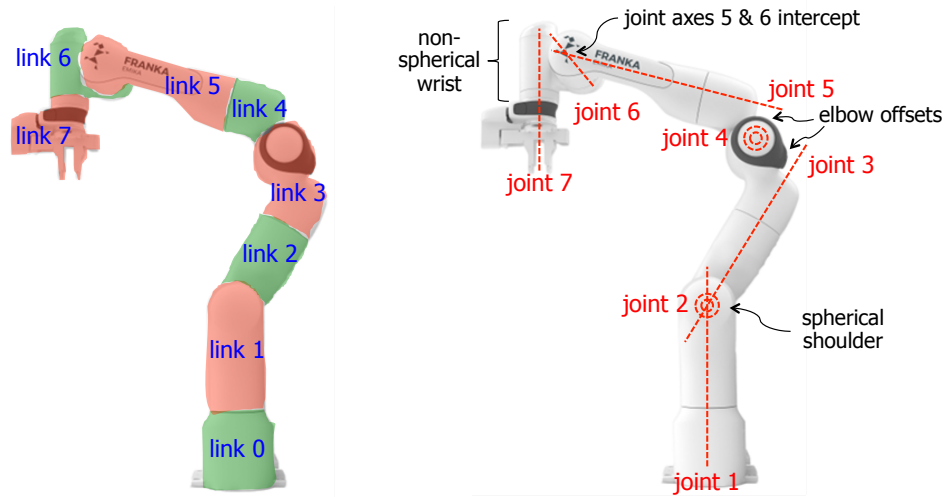

Figure 3: Decomposition of the Panda robot into links and definition of the seven joint axes.

A possible DH frame assignment and the associated table of parameters are reported in Fig. 4 and Tab. 1, respectively, together with their signs. The definition of the constant non-zero DH parameters d_j and a_k and of the DH variables θ_i $(i = 1, \ldots, 7)$ at the current configuration is illustrated in Fig. 5.
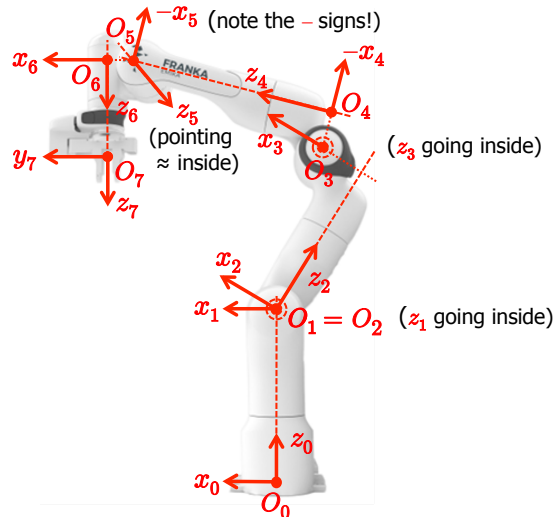

Figure 4: A possible DH frame assignment for the Panda robot by Franka Emika.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	$d_1 > 0$	q_1
2	$-\pi/2$	0	0	q_2
3	$\pi/2$	$a_3 > 0$	$d_3 > 0$	q_3
4	$-\pi/2$	$a_4 < 0$	0	q_4
5	$\pi/2$	0	$d_5 > 0$	q_5
6	$\pi/2$	$a_6 > 0$	0	q_6
7	0	0	$d_7 > 0$	q_7

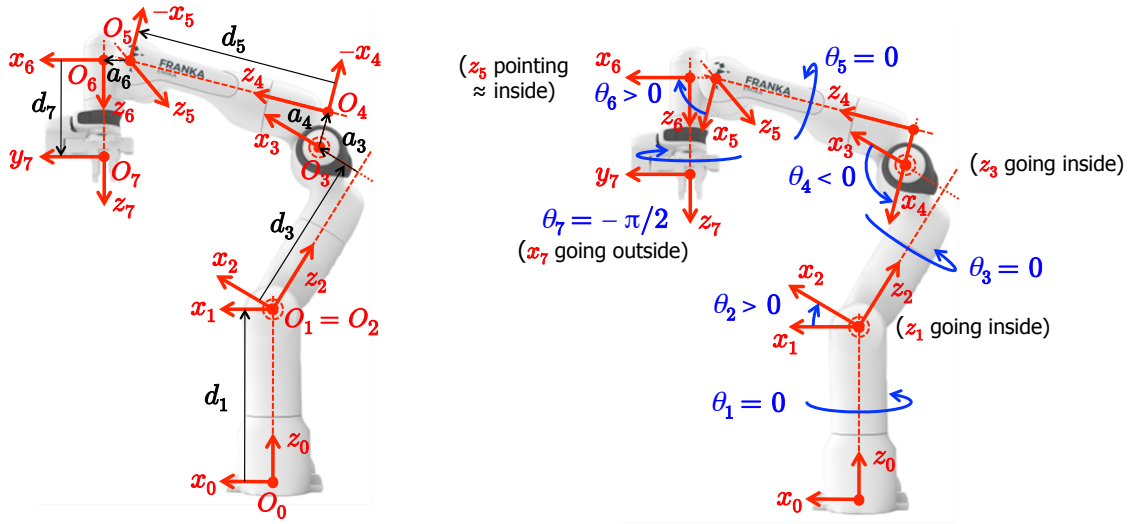Table 1: Parameters associated to the DH frames in Fig. 4.

Figure 5: Definition of the (non-zero) constant and variable DH parameters for the Panda robot.

Based on Tab. 1, the seven DH homogeneous transformation matrices are:

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
{}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & 0 & -\sin q_2 & 0 \\ \sin q_2 & 0 & \cos q_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & a_3 \cos q_3 \\ \sin q_3 & 0 & -\cos q_3 & a_3 \sin q_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
{}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 & a_4 \cos q_4 \\ \sin q_4 & 0 & \cos q_4 & a_4 \sin q_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^4\boldsymbol{A}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 & 0 \\ \sin q_5 & 0 & -\cos q_5 & 0 \\ 0 & 1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad
{}^5\boldsymbol{A}_6(q_6) = \begin{pmatrix} \cos q_6 & 0 & \sin q_6 & a_6 \cos q_6 \\ \sin q_6 & 0 & -\cos q_6 & a_6 \sin q_6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

and

$$
{}^6\boldsymbol{A}_7(q_7) = \begin{pmatrix} \cos q_7 & -\sin q_7 & 0 & 0 \\ \sin q_7 & \cos q_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

The numerical computation for obtaining the end-effector position $\boldsymbol{p} \in \mathbb{R}^3$ at a given (measured) configuration $\boldsymbol{q} \in \mathbb{R}^7$ is efficiently organized as recursive matrix-vector operations in the form

$$
\boldsymbol{p}_{\text{hom}} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \left[{}^1\boldsymbol{A}_2(q_2) \left[{}^2\boldsymbol{A}_3(q_3) \left[{}^3\boldsymbol{A}_4(q_4) \left[{}^4\boldsymbol{A}_5(q_5) \left[{}^5\boldsymbol{A}_6(q_6) \left[{}^6\boldsymbol{A}_7(q_7) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] \right] \right] \right] \right] \right]. \tag{2}
$$

The number of elementary operations to be performed is evaluated as follows. We need in any case:

- 14 trigonometric evaluations (2 for each of the 7 DH matrices);
- 6 products within the elements of the DH matrices.

Without exploiting the specific structure of the DH matrices, but taking into account that only the first three rows of each matrix are actually involved in the computations, we have additionally:

- at each step, a product of a 3×4 matrix times a 4-dimensional vector $\Rightarrow 12 \times, 9 +$;
- for the 7 recursive steps $\Rightarrow 84$ products and 63 sums.

The total number of products and sums can be reduced if we proceed by customization, namely by using the known structure of the specific DH matrices and thus avoiding useless products by 0 or ± 1 and additions of null terms. In this case, we have:

- the first step (the product by ${}^6\boldsymbol{A}_7$) can be skipped, and computations can start with the 4-dimensional vector $\begin{pmatrix} 0 & 0 & d_7 & 1 \end{pmatrix}^T$;
- in the second step (the product by ${}^5\boldsymbol{A}_6$), we have only $4 \times$ and $2 +$ actual operations;
- proceeding similarly: in the third step (with ${}^4\boldsymbol{A}_5$) $\Rightarrow 3 \times, 1 +$; in the fourth (etc., …) $\Rightarrow 6 \times, 4 +$; in the fifth $\Rightarrow 7 \times, 5 +$; in the sixth $\Rightarrow 4 \times, 2 +$; in the seventh and last (with ${}^0\boldsymbol{A}_1$) $\Rightarrow 5 \times, 3 +$;
- in total $\Rightarrow 29$ products and 17 sums.

Exercise 2

1. **N** and **G** always fail at singularities.

   False. **G** fails in a singularity only if the error belongs to the null space of $\boldsymbol{J}^T$.

2. **G** stops when a singularity is encountered.

   False. **G** can still move through a singularity if the error $\boldsymbol{e} \notin \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q})\}$.

3. Out of singularities, **N** finds always a solution faster than **G**.

   False. When it converges, **N** is faster than **G**. But **N** may not converge, even out of singularities.

4. **N** can be used only when there is a single global solution to the problem.

   False. N (as well as **G**) can be used in any case, finding only one solution at a time.

5. Both **N** and **G** need knowledge of the analytic Jacobian of the task.

   True. Both methods use $\boldsymbol{J}$ (and would converge to a wrong solution with an approximated one).

6. For a non-square Jacobian, the pseudoinverse should replace the Jacobian transpose in **G**.

   False. The pseudoinverse replaces in this case the Jacobian inverse in **N**. **G** needs no changes.

7. Close to a solution, it is computationally faster to evaluate an iteration of **N** than one of **G**.

   False. (Pseudo-)Inverting the Jacobian matrix (in **N**) needs more time than transposing it (in **G**)!

8. **G** works better for linear problems, **N** for quadratic ones.

   False. Relative performance hard to assess in general, but **N** solves a linear problem in one step.

9. Neither **N** nor **G** would terminate without the use of a small tolerance on the final error.

   True. Numerical methods always require a small final tolerance (due to roundings, etc.).

10. Beside matrix operations with the Jacobian and the error, **G** needs an extra choice to be made.

    True. G needs the choice of a stepsize in the (negative) gradient direction.

Exercise 3

The solution to the problem requires a combination of direct kinematics and differential kinematics for the master robot and of inverse kinematics and inverse differential kinematics for the slave robot. In addition, one should take into account the roto-translation between the base frames of the two robots. This is represented by a constant homogenous transformation matrix used to change the representations of position vectors starting from the two different origins. Some care should be also used in the treatment of the end-effector orientation of the slave robot as required by the task. Note that the slave robot, with its $n = 3$ joints, is not redundant for the given coordinated task, which in fact specifies $m = 3$ variables.

To begin, frame 0 of the slave robot is related to frame 0 of the master robot by

$$\boldsymbol{T}_{MS} = \begin{pmatrix} \boldsymbol{R}_{MS} & \boldsymbol{p}_{MS} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha_{MS} & -\sin \alpha_{MS} & 0 & \Delta x \\ \sin \alpha_{MS} & \cos \alpha_{MS} & 0 & \Delta y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1.6 \\ 0 & -1 & 0 & 0.9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

It follows that

$$\boldsymbol{R}_{SM} = \boldsymbol{R}_{MS}^T = \boldsymbol{R}_{MS}, \qquad \boldsymbol{T}_{SM} = (\boldsymbol{T}_{MS})^{-1} = \boldsymbol{T}_{MS}. \tag{4}$$

Since the problem is planar, we can conveniently work with two-dimensional vectors in the plane (and their three-dimensional homogeneous version), 2×2 rotation matrices (around a unit normal to the motion plane), and associated 3×3 homogeneous transformation matrices. We have in particular

$$\overline{\boldsymbol{T}}_{MS} = \begin{pmatrix} \overline{\boldsymbol{R}}_{MS} & \overline{\boldsymbol{p}}_{MS} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1.6 \\ 0 & -1 & 0.9 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

and similar relations hold for the overlined matrices as those in (4).

With the direct positional kinematics of the master robot

$$\boldsymbol{p}_M(\boldsymbol{q}_M) = \begin{pmatrix} \ell_1 \cos q_{M,1} + \ell_2 \cos(q_{M,1} + q_{M,2}) + \ell_3 \cos(q_{M,1} + q_{M,2} + q_{M,3}) \\ \ell_1 \sin q_{M,1} + \ell_2 \sin(q_{M,1} + q_{M,2}) + \ell_3 \sin(q_{M,1} + q_{M,2} + q_{M,3}) \end{pmatrix}, \tag{6}$$

the end-effector position is evaluated in $\boldsymbol{q}_M(t_0) = (\pi/2, -\pi/3, 0)$ [rad] $= (90°, -60°, 0)$ at time $t = t_0$,

$$\boldsymbol{p}_M(\boldsymbol{q}_M(t_0)) = \begin{pmatrix} 0.6495 \\ 0.8750 \end{pmatrix} = {}^M\boldsymbol{p}_M, \tag{7}$$

where the superscript on the last term reminds us of the frame in which this (planar) vector is expressed. Similarly, using the 2×3 Jacobian of the master robot (written with the usual compact notation)

$$\boldsymbol{J}_M(\boldsymbol{q}_M) = \frac{\partial \boldsymbol{p}_M(\boldsymbol{q}_M)}{\partial \boldsymbol{q}_M} = \begin{pmatrix} -(\ell_1 s_{M,1} + \ell_2 s_{M,12} + \ell_3 s_{M,123}) & -(\ell_2 s_{M,12} + \ell_3 s_{M,123}) & -\ell_3 s_{M,123} \\ \ell_1 c_{M,1} + \ell_2 c_{M,12} + \ell_3 c_{M,123} & \ell_2 c_{M,12} + \ell_3 c_{M,123} & \ell_3 c_{M,123} \end{pmatrix}, \tag{8}$$

the initial linear velocity in the plane of the master end-effector is evaluated as

$$\dot{\boldsymbol{p}}_M(\boldsymbol{q}_M(t_0)) = \boldsymbol{J}_M(\boldsymbol{q}_M(t_0))\, \dot{\boldsymbol{q}}_M(t_0) = \begin{pmatrix} -0.8750 & -0.3750 & -0.1250 \\ 0.6495 & 0.6495 & 0.2165 \end{pmatrix} \begin{pmatrix} -\pi/6 \\ 0 \\ -\pi/2 \end{pmatrix} = \begin{pmatrix} 0.6545 \\ -0.6802 \end{pmatrix} = {}^M\boldsymbol{v}_M. \tag{9}$$

In order to obtain the desired initial position of the end-effector of the slave robot and its desired linear velocity, the vectors in (7) and (9) should be expressed in the reference frame of this second robot. Using (3) and (4), we have

$$ {}^S\boldsymbol{p}_{S,hom} = \begin{pmatrix} {}^S\boldsymbol{p}_S \\ 1 \end{pmatrix} = \overline{\boldsymbol{T}}_{SM}\, {}^M\boldsymbol{p}_{M,hom} = \overline{\boldsymbol{T}}_{MS} \begin{pmatrix} {}^M\boldsymbol{p}_M \\ 1 \end{pmatrix} \quad \Rightarrow \quad {}^S\boldsymbol{p}_S = \begin{pmatrix} 0.9505 \\ 0.0250 \end{pmatrix} \text{[m]} \tag{10}$$

and

$$ {}^S\boldsymbol{v}_S = \overline{\boldsymbol{R}}_{SM}\, {}^M\boldsymbol{v}_M = \begin{pmatrix} -0.6545 \\ 0.6802 \end{pmatrix} \text{[m/s]}. \tag{11}$$

The angular direction of vector ${}^S\boldsymbol{v}_S$ prescribes also the desired orientation of the slave end-effector as

$$\phi_S = q_{S,1} + q_{S,2} + q_{S,3} = \text{ATAN2}\left\{ {}^S\boldsymbol{v}_{S,y}, {}^S\boldsymbol{v}_{S,x} \right\} + \beta = 2.3370 - \frac{\pi}{2} = 0.7662 \text{ [rad]} = 43.89°. \tag{12}$$

The above quantities, together with other parts of the solution, are conveniently illustrated in Fig. 6.

In order to solve the inverse kinematics of the slave robot for the desired ${}^S\boldsymbol{p}_S$ and ϕ_S, given respectively by (10) and (12), we compute first the associated position of the tip of the second link

$$\boldsymbol{p}_{S2} = \begin{pmatrix} p_{S2,x} \\ p_{S2,y} \end{pmatrix} = {}^S\boldsymbol{p}_S - \begin{pmatrix} \ell_3 \cos\phi_S \\ \ell_3 \sin\phi_S \end{pmatrix} = \begin{pmatrix} 0.7703 \\ -0.1483 \end{pmatrix} \text{[m]}. \tag{13}$$

The solution for the first two joints follows then from the standard inverse formulas for a planar 2R robot:

$$c_2 = \frac{p_{S2,x}^2 + p_{S2,y}^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2} = 0.2308 \quad \text{(should belong to the interval } [-1, +1]\text{)}$$

$$s_2 = -\sqrt{1 - c_2^2} = -0.9730 \quad \text{(the solution with } s_2 < 0 \text{ is chosen here)} \tag{14}$$

$$q_{S,2} = \text{ATAN2}\{s_2, c_2\} = 0.4787 \text{ [rad]} = 27.42°,$$

and

$$c_1 = p_{S2,x}(\ell_1 + \ell_2 c_2) + p_{S2,y}\ell_2 s_2$$

$$s_1 = p_{S2,y}(\ell_1 + \ell_2 c_2) - p_{S2,x}\ell_2 s_2 \tag{15}$$

$$q_{S,1} = \text{ATAN2}\{s_1, c_1\} = -1.3378 \text{ [rad]} = -76.65°.$$

Finally,

$$q_{S,3} = \phi_S - q_{S,1} - q_{S,2} = -1.6253 \text{ [rad]} = 93.12°. \tag{16}$$
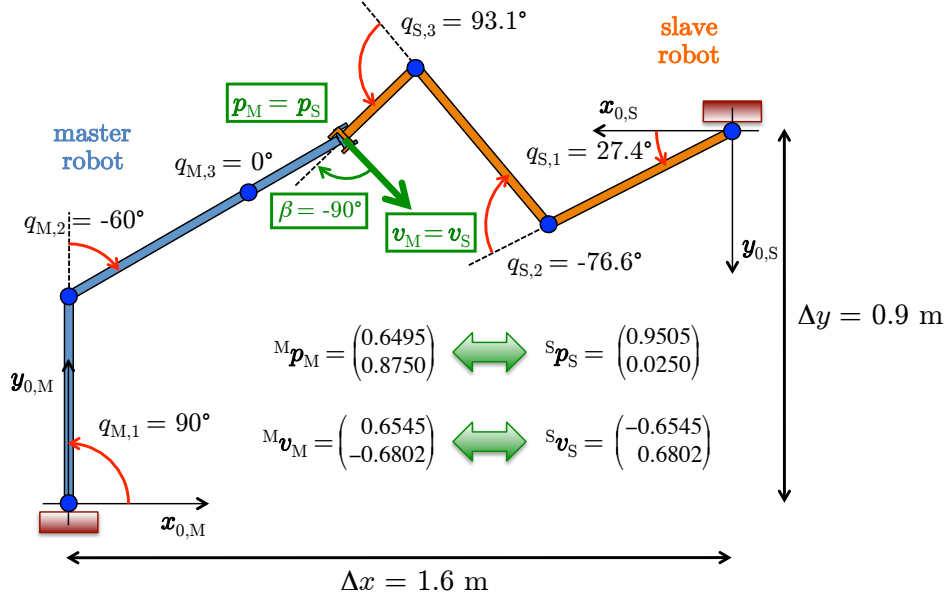
9

Figure 6: The desired initial configurations of the master and slave robots and the direction of the Cartesian velocity (not in scale) for the coordinated task.

As just seen, the coordination task for the slave robot is specified by three variables which can be jointly described as

$$\boldsymbol{r}_S = \begin{pmatrix} {}^S\boldsymbol{p}_S \\ \phi_S \end{pmatrix} \in \mathbb{R}^3, \qquad \dot{\boldsymbol{r}}_S = \begin{pmatrix} {}^S\dot{\boldsymbol{p}}_S \\ \dot{\phi}_S \end{pmatrix} = \begin{pmatrix} {}^S\boldsymbol{v}_S \\ 0 \end{pmatrix} \in \mathbb{R}^3, \tag{17}$$

where we set $\dot{\phi}_S = 0$ since the absolute orientation of the slave end-effector should remain constant along the fixed linear path. Therefore, since

$$\boldsymbol{r}_S(\boldsymbol{q}_S) = \begin{pmatrix} \ell_1 \cos q_{S,1} + \ell_2 \cos(q_{S,1} + q_{S,2}) + \ell_3 \cos(q_{S,1} + q_{S,2} + q_{S,3}) \\ \ell_1 \sin q_{S,1} + \ell_2 \sin(q_{S,1} + q_{S,2}) + \ell_3 \sin(q_{S,1} + q_{S,2} + q_{S,3}) \\ q_{S,1} + q_{S,2} + q_{S,3} \end{pmatrix}, \tag{18}$$

the 3×3 Jacobian of the slave robot is defined as

$$\boldsymbol{J}_S(\boldsymbol{q}_S) = \frac{\partial \boldsymbol{r}_S(\boldsymbol{q}_S)}{\partial \boldsymbol{q}_S} = \begin{pmatrix} -(\ell_1 s_{S,1} + \ell_2 s_{S,12} + \ell_3 s_{S,123}) & -(\ell_2 s_{S,12} + \ell_3 s_{S,123}) & -\ell_3 s_{S,123} \\ \ell_1 c_{S,1} + \ell_2 c_{S,12} + \ell_3 c_{S,123} & \ell_2 c_{S,12} + \ell_3 c_{S,123} & \ell_3 c_{S,123} \\ 1 & 1 & 1 \end{pmatrix}. \tag{19}$$

Substituting the values of $\boldsymbol{q}_S$ from (14–16) yields

$$\boldsymbol{J}_S = \begin{pmatrix} -0.0250 & 0.2053 & -0.1733 \\ 0.9505 & 0.5067 & 0.1801 \\ 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

The initial joint velocity of the slave robot is then computed as

$$\dot{\boldsymbol{q}}_S = \boldsymbol{J}_S^{-1} \dot{\boldsymbol{r}}_S = \begin{pmatrix} -1.3424 & 1.5566 & -0.5131 \\ 3.1669 & -0.6098 & 0.6588 \\ -1.8245 & -0.9468 & 0.8543 \end{pmatrix} \begin{pmatrix} -0.6545 \\ 0.6802 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.9374 \\ -2.4875 \\ 0.5501 \end{pmatrix} \text{[rad/s]} = \begin{pmatrix} 111.00 \\ -142.52 \\ 31.52 \end{pmatrix} \text{[°/s]}. \tag{21}$$

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Finally, if there is an initial mismatch (because the slave robot is not in one of the two possible correct initial configurations, the other being that obtained with $s_2 < 0$ in (14)), a kinematic feedback control law driven by the (Cartesian) task error should be applied. In order to have the task error converge to zero exponentially and in a decoupled way for each component of the task error, the required control law is

$$\dot{q}_S = J_S^{-1}(q_S) \left(\dot{r}_{S,d} + K(r_{S,d} - r_S(q_S)) \right), \tag{22}$$

where a subscript d has been added to denote the desired task variables and their first time derivatives, and with $K > 0$ being a diagonal 3×3 gain matrix that specifies the rate of convergence of the errors.

Exercise 4

In view of the smoothness requirement, it is convenient to choose quintic polynomials as the class of motion trajectories for each joint. These polynomials can be used to impose also the initial and final values for the joint velocity and acceleration, in particular all zero as in the present case. Being the required joint displacements

$$\boldsymbol{\Delta q} = \boldsymbol{q}(T) - \boldsymbol{q}(0) = \begin{pmatrix} -\dfrac{3\pi}{4} \\ 2 \end{pmatrix} \text{[rad; m]} = \begin{pmatrix} -135 \\ 200 \end{pmatrix} \text{[°; cm]}, \tag{23}$$

the double normalized expression of the trajectory in the joint space is

$$\boldsymbol{q}(\tau) = \boldsymbol{q}(0) + \boldsymbol{\Delta q} \left(6\tau^5 - 15\tau^4 + 10\tau^3 \right), \qquad \tau = \frac{t}{T} \in [0, 1]. \tag{24}$$

In order to find the maximum velocity and acceleration reached along this trajectory, which should satisfy the bounds (1), the first three time derivatives are needed:

$$\dot{\boldsymbol{q}}(\tau) = 30 \frac{\boldsymbol{\Delta q}}{T} \left(\tau^4 - 2\tau^3 + \tau^2 \right), \quad \ddot{\boldsymbol{q}}(\tau) = 60 \frac{\boldsymbol{\Delta q}}{T^2} \left(2\tau^3 - 3\tau^2 + \tau \right), \quad \dddot{\boldsymbol{q}}(\tau) = 60 \frac{\boldsymbol{\Delta q}}{T^3} \left(6\tau^2 - 6\tau + 1 \right). \tag{25}$$

The maximum acceleration occurs where the third derivative is zero (no need to check the value at the boundaries $t = \tau = 0$ and $t = T$ ($\tau = 1$), since we have $\ddot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{q}}(1) = \boldsymbol{0}$ by construction):

$$\dddot{\boldsymbol{q}}(\tau) = \boldsymbol{0} \quad \Longleftrightarrow \quad 6\tau^2 - 6\tau + 1 = 0 \quad @ \tau_a = 0.5 \pm \frac{\sqrt{3}}{6} \quad \Rightarrow \quad \ddot{\boldsymbol{q}}(\tau_a) = \pm 5.7735 \frac{\boldsymbol{\Delta q}}{T^2}. \tag{26}$$

Similarly, the maximum velocity occurs where the second derivative is zero (again, no need to check the value at the boundaries, since $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(1) = 0$):

$$\ddot{\boldsymbol{q}}(\tau) = \boldsymbol{0} \quad \Longleftrightarrow \quad \tau \left(2\tau^2 - 3\tau + 1 \right) = 0 \quad @ \tau_v = \{0, 0.5, 1\} \quad \Rightarrow \quad \dot{\boldsymbol{q}}(0.5) = \frac{30}{16} \frac{\boldsymbol{\Delta q}}{T}. \tag{27}$$

For the given motion time $T = 3$ [s], the maximum values are:

$$
\begin{aligned}
&|\dot{q}_1(0.5)| = |-84.375| < 120 = V_1 \text{ [°/s]}, & &|\dot{q}_2(0.5)| = 125 < 180 = V_2 \text{ [cm/s]}, \\
&|\ddot{q}_1(\tau_a)| = 86.6025 < 150 = A_1 \text{ [°/s}^2\text{]}, & &|\ddot{q}_2(\tau_a)| = 128.3 < 200 = A_2 \text{ [cm/s}^2\text{]}.
\end{aligned} \tag{28}
$$

It follows that the original trajectory is feasible. Figure 7 shows the originally planned trajectory profiles.

As a result, we can speed up motion by considering a uniform time scaling. In order to find the minimum motion time T^* that still produces feasible trajectories, we compute

$$k_1 = \min \left\{ \frac{V_1}{|\dot{q}_1(0.5)|}, \sqrt{\frac{A_1}{|\ddot{q}_1(\tau_a)|}} \right\} = 1.3161, \qquad k_2 = \min \left\{ \frac{V_2}{|\dot{q}_2(0.5)|}, \sqrt{\frac{A_2}{|\ddot{q}_2(\tau_a)|}} \right\} = 1.2485 \tag{29}$$

$$\Rightarrow \qquad k = \min\{k_1, k_2\} = 1.2485 \qquad \Rightarrow \qquad T^* = \frac{T}{k} = \frac{3}{1.2485} = 2.4028 \text{ [s]}. \tag{30}$$

The scaled trajectory and its first two derivatives are shown in Fig. 8. The new maximum values are:

$$
\begin{aligned}
&|\dot{q}_1(0.5)| = |-105.34| < 120 = V_1 \text{ [°/s]}, & &|\dot{q}_2(0.5)| = 156.06 < 180 = V_2 \text{ [cm/s]}, \\
&|\ddot{q}_1(\tau_a)| = 135 < 150 = A_1 \text{ [°/s}^2\text{]}, & &|\ddot{q}_2(\tau_a)| = 200 = A_2 \text{ [cm/s}^2\text{]}.
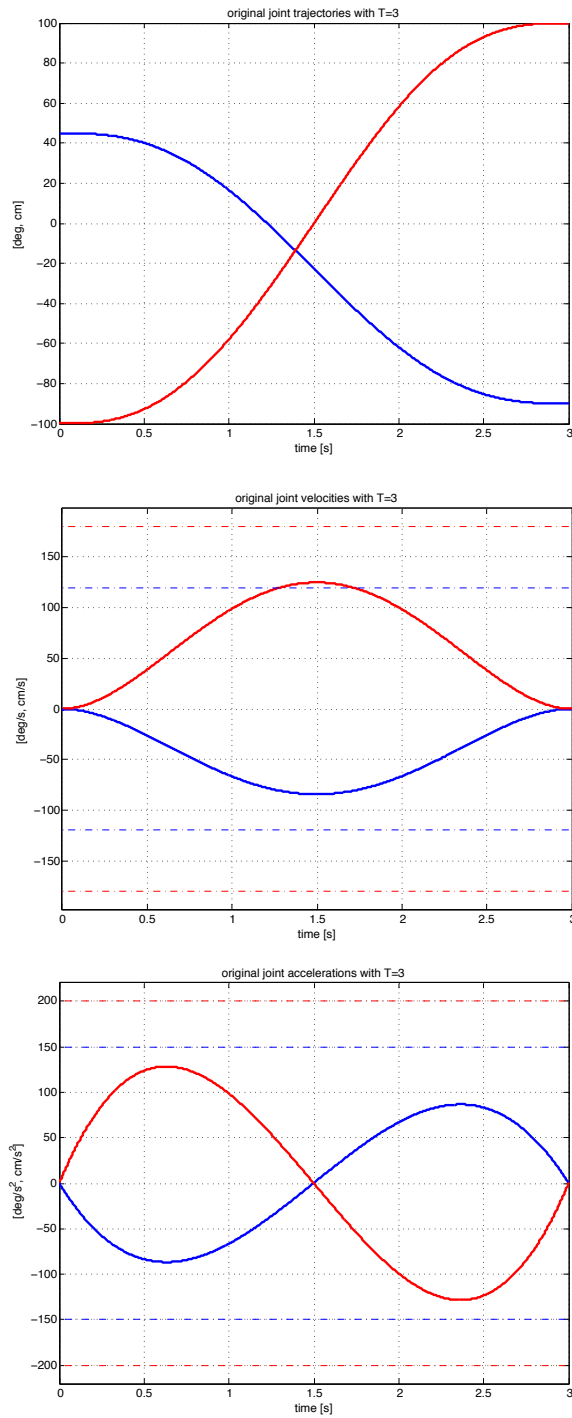\end{aligned} \tag{31}
$$

Figure 7: Original joint trajectory for a motion time $T = 3$ [s], with velocity and acceleration profiles (joint 1 = blue, joint 2 = red). The bounds (1), shown by dashed lines ($\cdot-$), are satisfied.
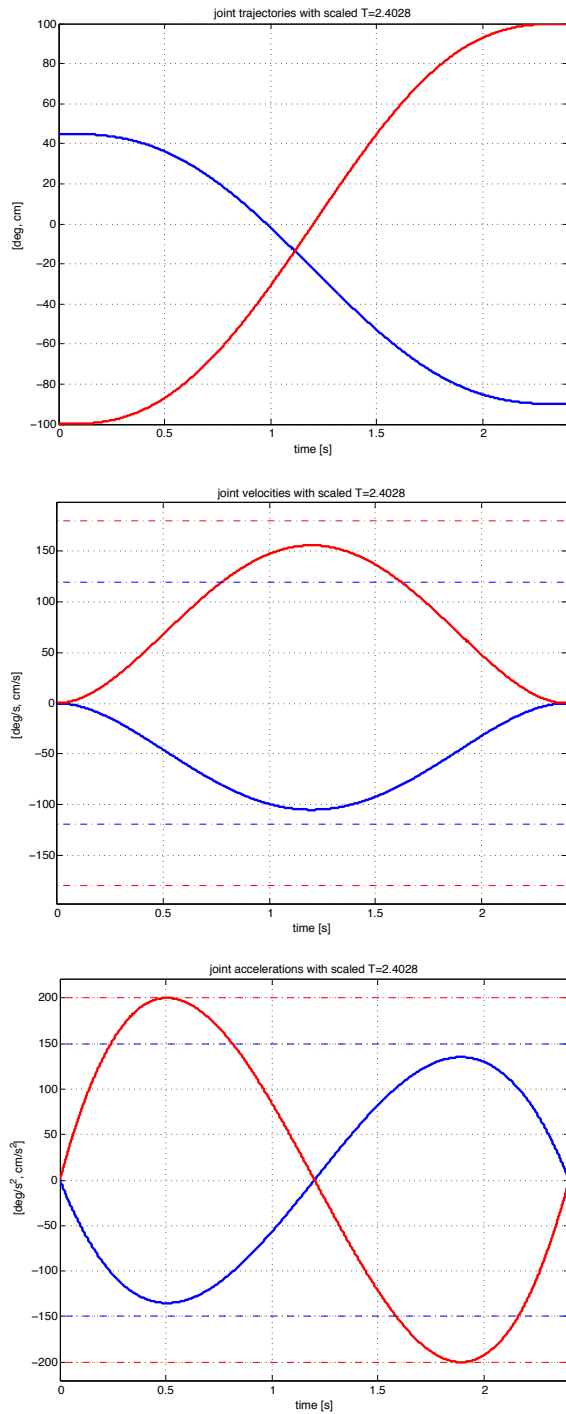
Figure 8: Scaled joint trajectory for a motion time $T^* = 2.4028$ [s], with velocity and acceleration profiles (joint 1 = blue, joint 2 = red). The bound in (1) for the acceleration of the second joint is being reached in two instants that are symmetric w.r.t. the trajectory midpoint.
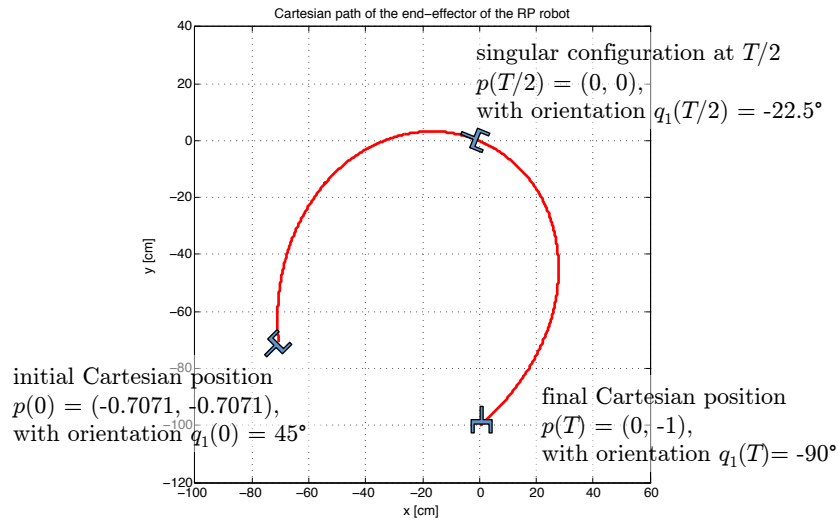
Figure 9: Cartesian path traced by the end-effector of the RP robot. Its orientation is shown at the start and final configurations, and at the intermediate motion instant $t = T/2$ (or $T^*/2$) when the robot crosses the singularity $q_2 = 0$.

During motion, the RP robot will certainly cross a singular configuration with $q_2 = 0$. This will certainly happen for any class of interpolating trajectories, since the initial and final values of q_2 have opposite signs. However, this singularity does not harm when planning (and control) is made directly in the joint space, as in the present case. Figure 9 shows the motion path of the end-effector with the (original or uniformly scaled) quintic polynomial trajectory.

$* * * * *$