

Robotics I

July 11, 2017

Exercise 1

Consider the 5-dof GMF M-100 manipulator sketched in Fig. 1, having a RPP (cylindric) sequence for the first three main joints and two more revolute joints with intersecting axes at the wrist.

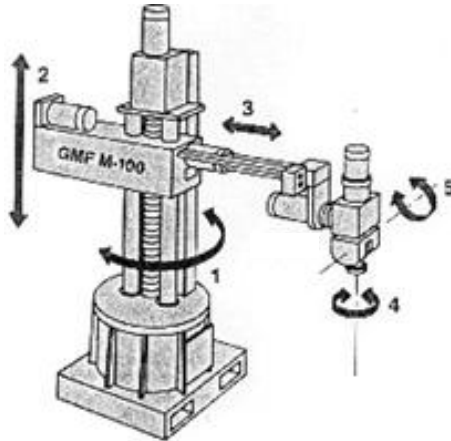


Figure 1: The GMF M-100, a 5-dof manipulator with a RPP-RR sequence of joints.

- Assign the link frames according to the Denavit-Hartenberg convention and derive the associated table of parameters. Place the origin of the reference frame (frame 0) at the robot base on the floor, and choose the origin of the last frame (frame 5) such that $d_5 = 0$.
- Derive the direct kinematics for the position $\mathbf{p} = \mathbf{p}(\mathbf{q}) \in \mathbb{R}^3$ of the origin of the last frame. What if we use cylindrical coordinates $\mathbf{p}_{cyl} = (\phi \ h \ r)^T$ to describe this Cartesian position?
- Derive the explicit expression of the 6×5 geometric Jacobian $\mathbf{J}(\mathbf{q})$ and analyze if and when this matrix loses full rank.

Exercise 2

For a rest-to-rest motion of a robot joint by a given amount Δq , we plan polynomial trajectories $q_i(t)$ of degree i , with $i = 3, 5, 7$. In each case, we impose all possible derivatives to zero at the initial and final instants $t = 0$ and $t = T$. If there is a limit $|\dot{q}_i(t)| \leq V$ on the velocities, which is the fastest possible trajectory among the three? Which are the ratios between the achievable minimum times T_i , for $i = 3, 5, 7$? What if there is instead only a limit $|\ddot{q}_i(t)| \leq A$ on the accelerations?

Exercise 3

- Shortly present (in the form of a table) which are the pros' and cons' of using incremental vs. absolute encoders as position sensors in a robot manipulator.
- What are the dis-/advantages of mounting an optical encoder on the motor side rather than on the link side of a motion transmission/reduction element in a robot joint?
- Describe as many as possible direct or indirect ways to measure the current position of the tip of a tool mounted as end effector of a robot.

[180 minutes, open books but no computer or smartphone]

Solution

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Exercise 1

A Denavit-Hartenberg (DH) frame assignment for the 5-dof GMF M-100 manipulator is shown in Fig. 2. The associated parameters are given in Tab. 1.

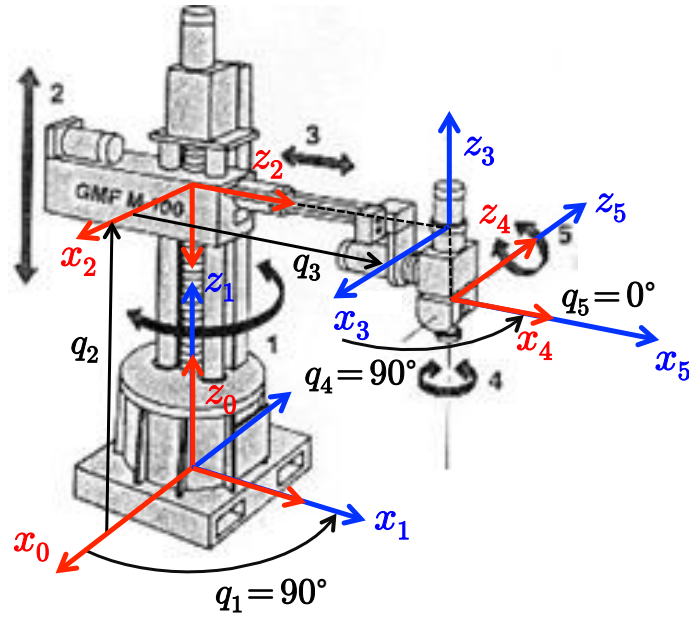


Figure 2: A DH frame assignment for the GMF M-100 manipulator, with the associated joint variables (numerical values are shown for the revolute joints in the this configuration).

i	α_i	a_i	d_i	θ_i
1	0	0	0	q_1
2	$-\pi/2$	0	q_2	$-\pi/2$
3	$\pi/2$	0	q_3	0
4	$-\pi/2$	0	$d_4 < 0$	q_4
5	0	0	0	q_5

Table 1: Table of DH parameters of the frame assignment in Fig. 2 for the GMF M-100 manipulator.

Accordingly, the homogeneous transformation matrices are:

$$\begin{aligned}
{}^0\mathbf{A}_1(q_1) &= \begin{pmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
{}^1\mathbf{A}_2(q_2) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & {}^2\mathbf{A}_3(q_3) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
{}^3\mathbf{A}_4(q_4) &= \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 & 0 \\ \sin q_4 & 0 & \cos q_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & {}^4\mathbf{A}_5(q_5) &= \begin{pmatrix} \cos q_5 & -\sin q_5 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

From these, we compute

$$\begin{aligned}
\mathbf{p}_{hom} &= \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = {}^0\mathbf{A}_1(q_1) \left({}^1\mathbf{A}_2(q_2) \left({}^2\mathbf{A}_3(q_3) \left({}^3\mathbf{A}_4(q_4) \left({}^4\mathbf{A}_5(q_5) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \right) \right) \right) \\
&= {}^0\mathbf{A}_1(q_1) \left({}^1\mathbf{A}_2(q_2) \left({}^2\mathbf{A}_3(q_3) \left({}^3\mathbf{A}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) \right) \right) \\
&= {}^0\mathbf{A}_1(q_1) \left({}^1\mathbf{A}_2(q_2) \left({}^2\mathbf{A}_3(q_3) \begin{pmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{pmatrix} \right) \right) \\
&= {}^0\mathbf{A}_1(q_1) \left({}^1\mathbf{A}_2(q_2) \begin{pmatrix} 0 \\ -d_4 \\ q_3 \\ 1 \end{pmatrix} \right) \\
&= {}^0\mathbf{A}_1(q_1) \begin{pmatrix} q_3 \\ 0 \\ q_2 + d_4 \\ 1 \end{pmatrix} = \begin{pmatrix} q_3 \cos q_1 \\ q_3 \sin q_1 \\ q_2 + d_4 \\ 1 \end{pmatrix} \Rightarrow \mathbf{p} = \begin{pmatrix} q_3 \cos q_1 \\ q_3 \sin q_1 \\ q_2 + d_4 \end{pmatrix}.
\end{aligned}$$

It is easy to recognize that the expression in cylindrical coordinates is

$$\mathbf{p}_{cyl} = \begin{pmatrix} \phi \\ h \\ r \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 + d_4 \\ q_3 \end{pmatrix}.$$

The geometric Jacobian is also very simple

$$\mathbf{J}(\mathbf{q}) = \begin{pmatrix} \mathbf{J}_L(\mathbf{q}) \\ \mathbf{J}_A(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}} \\ z_0 \quad \mathbf{0} \quad \mathbf{0} \quad z_3 \quad z_4(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} -q_3 \sin q_1 & 0 & \cos q_1 & 0 & 0 \\ q_3 \cos q_1 & 0 & \sin q_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(q_1 + q_3) \\ 0 & 0 & 0 & 0 & \sin(q_1 + q_3) \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix has full (column) rank equal to 5 if and only if $q_3 \neq 0$. When $q_3 = 0$, the rank drops to 4.

Exercise 2

It is convenient to work with doubly normalized expressions of the three polynomial trajectories interpolating q_0 at $t = 0$ with $q_0 + \Delta q$ at $t = T$. Let $\tau = t/T \in [0, 1]$. For the cubic polynomial, imposing zero velocity at the two boundaries, we have

$$q_3(\tau) = q_0 + \Delta q (3\tau^2 - 2\tau^3). \quad (1)$$

For the quintic polynomial, imposing zero velocity and acceleration at the two boundaries, we have

$$q_5(\tau) = q_0 + \Delta q (10\tau^3 - 15\tau^4 + 6\tau^5). \quad (2)$$

Finally, for the 7th-degree polynomial, imposing zero velocity, acceleration, and jerk at the two boundaries, we have

$$q_7(\tau) = q_0 + \Delta q (35\tau^4 - 84\tau^5 + 70\tau^6 - 20\tau^7). \quad (3)$$

The three velocities take the expressions

$$\dot{q}_3(\tau) = \frac{\Delta q}{T} (6\tau - 6\tau^2) = \frac{6\Delta q}{T} \tau (1 - \tau), \quad (4)$$

$$\dot{q}_5(\tau) = \frac{\Delta q}{T} (30\tau^2 - 60\tau^3 + 30\tau^4) = \frac{30\Delta q}{T} \tau^2 (1 - 2\tau + \tau^2) = \frac{30\Delta q}{T} \tau^2 (1 - \tau)^2, \quad (5)$$

and

$$\begin{aligned} \dot{q}_7(\tau) &= \frac{\Delta q}{T} (140\tau^3 - 420\tau^4 + 420\tau^5 - 140\tau^6) \\ &= \frac{140\Delta q}{T} \tau^3 (1 - 3\tau + 3\tau^2 - \tau^3) = \frac{140\Delta q}{T} \tau^3 (1 - \tau)^3, \end{aligned} \quad (6)$$

while the accelerations are

$$\ddot{q}_3(\tau) = \frac{\Delta q}{T^2} (6 - 12\tau) = \frac{6\Delta q}{T^2} (1 - 2\tau) \quad (7)$$

$$\ddot{q}_5(\tau) = \frac{\Delta q}{T^2} (60\tau - 180\tau^2 + 120\tau^3) = \frac{60\Delta q}{T^2} \tau (1 - 3\tau + 2\tau^2) = \frac{60\Delta q}{T^2} \tau (1 - \tau) (1 - 2\tau), \quad (8)$$

and

$$\begin{aligned} \ddot{q}_7(\tau) &= \frac{\Delta q}{T^2} (420\tau^2 - 1680\tau^3 + 2100\tau^4 - 840\tau^5) = \frac{420\Delta q}{T^2} \tau^2 (1 - 4\tau + 5\tau^2 - 2\tau^3) \\ &= \frac{420\Delta q}{T^2} \tau^2 (1 - 2\tau + \tau^2) (1 - 2\tau) = \frac{420\Delta q}{T^2} \tau^2 (1 - \tau)^2 (1 - 2\tau). \end{aligned} \quad (9)$$

The maximum velocity is always attained at the trajectory halftime $t = T/2$, or $\tau = 0.5$, where the acceleration is in fact zero in all cases. Indeed, for the cubic trajectory this is the only instant where the acceleration \ddot{q}_3 is zero. For the quintic trajectory, the factorization in (8) shows that the acceleration \ddot{q}_5 has (by construction) a zero also at the boundaries (where the velocity is anyway zero, together with the acceleration). Similarly, for the 7th-degree trajectory, the factorization in (9) shows that the acceleration \ddot{q}_7 has (by construction) a zero of multiplicity 2 also at the boundaries (where the velocity is anyway zero, together with acceleration and jerk). Therefore, in all three cases we evaluate the presence of a symmetric bound V on the velocity as

$$|\dot{q}_3(\tau)| \leq V, \quad \max_{\tau \in [0,1]} |\dot{q}_3(\tau)| = |\dot{q}_3(0.5)| = \frac{3|\Delta q|}{2T} \quad \Rightarrow \quad T_3 (= \min T_V \text{ for } q_3(\tau)) = \frac{3|\Delta q|}{2V}, \quad (10)$$

$$|\dot{q}_5(\tau)| \leq V, \quad \max_{\tau \in [0,1]} |\dot{q}_5(\tau)| = |\dot{q}_5(0.5)| = \frac{15|\Delta q|}{8T} \quad \Rightarrow \quad T_5 (= \min T_V \text{ for } q_5(\tau)) = \frac{15|\Delta q|}{8V}, \quad (11)$$

and

$$|\dot{q}_7(\tau)| \leq V, \quad \max_{\tau \in [0,1]} |\dot{q}_7(\tau)| = |\dot{q}_7(0.5)| = \frac{35|\Delta q|}{16T} \quad \Rightarrow \quad T_7 (= \min T_V \text{ for } q_7(\tau)) = \frac{35|\Delta q|}{16V}. \quad (12)$$

As a result —not really unexpected— the fastest trajectory under a maximum velocity bound is the cubic one, followed by the quintic, and then by the 7th-degree polynomial ($T_3 < T_5 < T_7$). The ratios of the minimum times are independent from Δq and V and equal to:

$$\frac{T_7}{T_5} = \frac{3}{2} = 1.5, \quad \frac{T_5}{T_3} = \frac{5}{4} = 1.25, \quad \frac{T_7}{T_3} = \frac{15}{8} = 1.875. \quad (13)$$

In order to solve the same minimum time problem when only a symmetric bound A is set on the acceleration, we compute first the jerk for the three polynomial trajectories:

$$\ddot{\ddot{q}}_3(\tau) = -\frac{12\Delta q}{T^3} \neq 0, \quad (14)$$

$$\ddot{\ddot{q}}_5(\tau) = \frac{60\Delta q}{T^3} (1 - 6\tau + 6\tau^2) = \frac{60\Delta q}{T^3} \left(1 - \frac{6}{3 - \sqrt{3}} \tau\right) \left(1 - \frac{6}{3 + \sqrt{3}} \tau\right), \quad (15)$$

and

$$\begin{aligned} \ddot{\ddot{q}}_7(\tau) &= \frac{840\Delta q}{T^3} \tau (1 - 6\tau + 10\tau^2 - 5\tau^3) = \frac{840\Delta q}{T^3} \tau (1 - \tau) (1 - 5\tau + 5\tau^2) \\ &= \frac{840\Delta q}{T^3} \tau (1 - \tau) \left(1 - \frac{10}{5 - \sqrt{5}} \tau\right) \left(1 - \frac{10}{5 + \sqrt{5}} \tau\right). \end{aligned} \quad (16)$$

The jerk of the cubic trajectory is constant over the motion interval ($\tau \in [0,1]$), so that the maximum acceleration in this closed interval is at its boundaries. Because of the symmetric behavior, we have from $|\ddot{q}_3(\tau)| \leq A$:

$$\max_{\tau \in [0,1]} |\ddot{q}_3(\tau)| = |\ddot{q}_3(0)| = |\ddot{q}_3(1)| = \frac{6|\Delta q|}{T^2} \quad \Rightarrow \quad T_3' (= \min T_A \text{ for } q_3(\tau)) = \sqrt{\frac{6|\Delta q|}{A}}. \quad (17)$$

From (15), the jerk of the quintic trajectory has two roots, i.e.,

$$\tau_{5,1} = 0.5 - \frac{\sqrt{3}}{6} \quad \text{and} \quad \tau_{5,2} = 0.5 + \frac{\sqrt{3}}{6}, \quad (18)$$

in the interval $\tau \in [0, 1]$, placed in symmetric positions w.r.t. the motion halftime. Since the acceleration is anyway zero at the boundaries, the maximum acceleration in the closed interval occurs only in the two instants specified in (18). Because of the symmetric behavior, we have from $|\ddot{q}_5(\tau)| \leq A$:

$$\max_{\tau \in [0,1]} |\ddot{q}_5(\tau)| = |\ddot{q}_5(\tau_{5,1})| = |\ddot{q}_5(\tau_{5,2})| = \frac{5.7735 |\Delta q|}{T^2} \Rightarrow T'_5 (= \min T_A \text{ for } q_5(\tau)) = \sqrt{\frac{5.7735 |\Delta q|}{A}}. \quad (19)$$

Finally, from (16) the jerk of the 7th-degree trajectory has two internal roots, i.e.,

$$\tau_{7,1} = 0.5 - \frac{\sqrt{5}}{10} \quad \text{and} \quad \tau_{7,2} = 0.5 + \frac{\sqrt{5}}{10}, \quad (20)$$

in the interval $\tau \in [0, 1]$, placed again in symmetric positions w.r.t. the motion halftime, and two other roots coincident with the boundaries, where the acceleration is anyway zero. Thus, the maximum acceleration in the closed interval occurs in the two instants specified in (20). Because of the symmetric behavior, we have from $|\ddot{q}_7(\tau)| \leq A$:

$$\max_{\tau \in [0,1]} |\ddot{q}_7(\tau)| = |\ddot{q}_7(\tau_{7,1})| = |\ddot{q}_7(\tau_{7,2})| = \frac{3.36 |\Delta q|}{T^2} \Rightarrow T'_7 (= \min T_A \text{ for } q_7(\tau)) = \sqrt{\frac{3.36 |\Delta q|}{A}}. \quad (21)$$

As a result —maybe with a certain surprise— the situation is now reversed with respect to the previous case: the fastest trajectory under a maximum acceleration bound is in fact the 7th-degree polynomial, followed by the quintic, and then by the cubic one ($T'_7 < T'_5 < T'_3$). The ratios of the minimum times are again independent from Δq and V and equal to:

$$\frac{T'_3}{T'_5} = \sqrt{\frac{6}{5.7735}} = 1.0194, \quad \frac{T'_5}{T'_7} = \sqrt{\frac{5.7735}{3.36}} = 1.3108, \quad \frac{T'_3}{T'_7} = \sqrt{\frac{6}{3.36}} = 1.3363. \quad (22)$$

Indeed, limiting the acceleration is already a request targeting increased smoothness of the trajectory. This explains *de facto* why a polynomial of higher degree performs better in this case.

Exercise 3

This exercise asks for a free text. Completeness, technical accuracy, and clarity in writing are evaluated.

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