Robotics I

February 3, 2017

Exercise 1

For the NAO humanoid robot in Fig. 1, consider only the *left* arm down to the wrist yaw joint, which is frozen together with the remaining joints of all fingers. The left arm has thus four degrees of freedom (dofs). We provide separately a two-sided technical sheet with the kinematic data of this part of the robot. For describing internal motions, the robot has a global reference frame placed at the torso center.

- Assign the reference frames for the first four dofs of the left arm according to the Denavit-Hartenberg (DH) convention, so that the positive senses of joint rotations match those shown in the technical sheet. Place the origin of the last frame at the end of the Lower Arm (i.e., at the Hand base).
- Draw the torso frame (with axes relabeled as $\boldsymbol{x}_T$, $\boldsymbol{y}_T$, and $\boldsymbol{z}_T$) and the DH frames on one (or on both) of the two distributed extra sheets, which show respectively a CAD view of the torso/left arm and a view of the upper limbs. Provide the 4×4 homogeneous matrix ${}^T\boldsymbol{A}_0$ from the torso to the DH frame 0.
- Complete the Denavit-Hartenberg table of parameters associated to the frames that have been assigned. Determine the values of the joint angles $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T$ when the robot stretches its left arm forward and horizontally, just like in the picture on the left of Fig. 1.

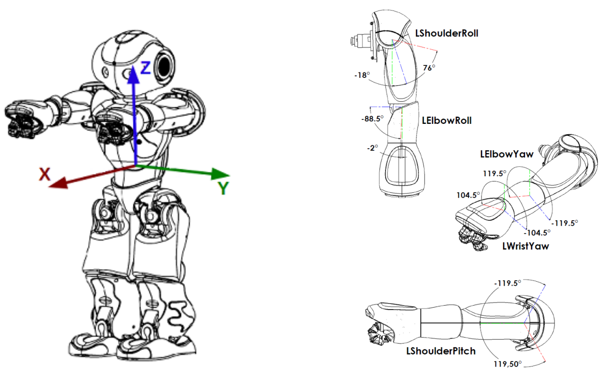


Figure 1: The NAO humanoid robot with the torso frame and three views of its left arm.

Exercise 2

For a planar RP robot with direct kinematics and joint velocity/acceleration limits given respectively by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}, \qquad \boldsymbol{V}_{max} = \begin{pmatrix} 2 \\ 2.5 \end{pmatrix} \text{ [rad/s; m/s]}, \qquad \boldsymbol{A}_{max} = \begin{pmatrix} 3 \\ 1.5 \end{pmatrix} \text{ [rad/s}^2\text{; m/s}^2\text{]},$$

design a minimum time coordinated trajectory, with the end effector moving rest-to-rest between the initial Cartesian position $\boldsymbol{p}_i = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ and the final position $\boldsymbol{p}_f = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$ [m].

- Provide the minimum motion time T and draw the velocity/acceleration profiles of the two joints.
- For the given data, will the Cartesian path followed by the end-effector be a linear one? Will the robot pass through a singularity during motion? Prove your responses.
- Compute the (velocity) manipulability measure and sketch a plot of its value during the planned motion. Are there Cartesian positions during this motion in which the force and velocity manipulability ellipsoids will coincide? Are there configurations at which the velocity ellipsoid becomes a circle? Explain your responses and comment on these situations.

Exercise 3

A planar 3R robot, having links of equal length ℓ, is being controlled by joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. A desired linear Cartesian trajectory $\boldsymbol{p}_d(t)$ is assigned, which starts from point $\boldsymbol{p}_i$ at time $t = 0$ and reaches point $\boldsymbol{p}_f$ at time $t = T$ and has a rest-to-rest motion profile that is continuous for $t \in [0, T]$ up to the acceleration. With reference to Fig. 2, design a single control law for $\dot{\boldsymbol{q}}$ so that the robot end-effector position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ follows, or at least asymptotically tracks the desired trajectory $\boldsymbol{p}_d(t)$. The following behaviors should be simultaneously enforced.

1. Realize exact trajectory following (i.e., $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t) \equiv \boldsymbol{0}$, $\forall t \in [0, T]$) for suitable initial configurations $\boldsymbol{q}(0) = \boldsymbol{q}_{0,e}$.
2. For any generic initial configuration $\boldsymbol{q}(0) = \boldsymbol{q}_0$, achieve trajectory tracking with the position error $\boldsymbol{e}(t)$ that converges exponentially to zero.
3. The error component $e_n(t)$ along the normal direction to the desired linear path should be reduced three times faster than the error component $e_t(t)$ along the tangential direction.
4. Within half of the nominal motion time $T/2$, the norm of the error $\|\boldsymbol{e}(t)\|$ should be reduced at least to one tenth of its initial value.

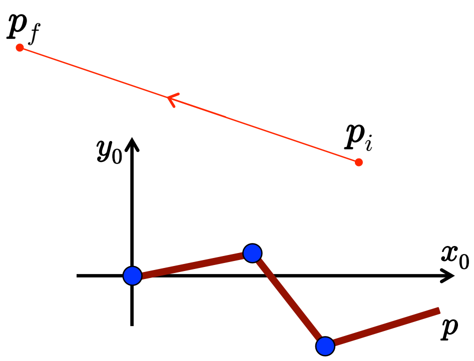


Figure 2: A planar 3R robot and a linear Cartesian trajectory.

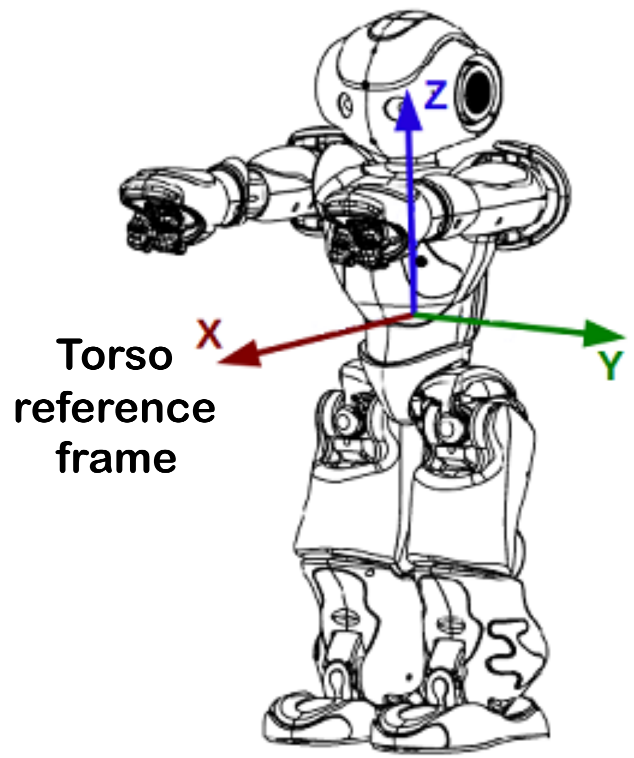
Use the following numerical data:

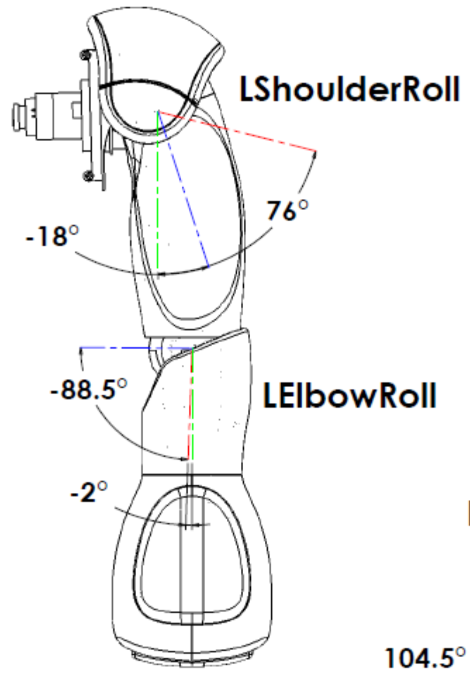
$$\ell = 2 \text{ [m]}, \qquad \boldsymbol{p}_i = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \; \boldsymbol{p}_f = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \text{ [m]}, \qquad T = 4 \text{ [s]}.$$

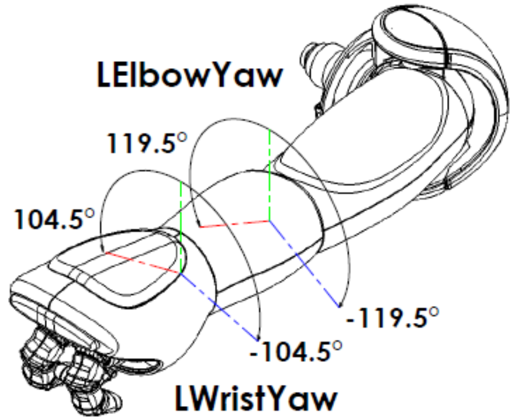
- Determine a possible expression of the desired linear Cartesian trajectory $\boldsymbol{p}_d(t)$.
- Assuming that kinematic singularities are never encountered, provide the explicit symbolic expression of all terms in the control law and the proper numerical values of the control gains.
- Find one possible initial configuration $\boldsymbol{q}_{0,e}$ that leads to exact trajectory following, and explain how to obtain such configurations in general.
- When the robot starts from the configuration $\boldsymbol{q}_0 = \begin{pmatrix} -\pi/2 & 0 & \pi/2 \end{pmatrix}^T$ [rad], compute the numerical value of the joint velocity command $\dot{\boldsymbol{q}}(0)$ at the initial time $t = 0$ with the designed control law.

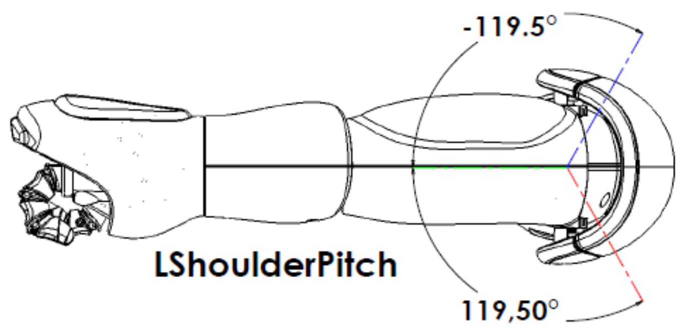
[240 minutes, open books but no computer or smartphone]

Left arm of NAO H25









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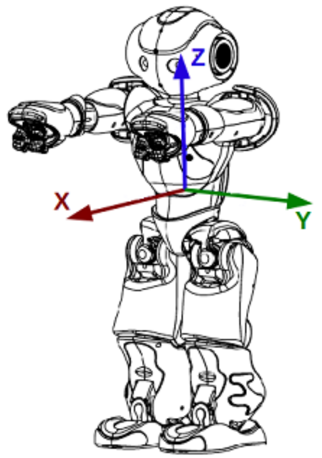
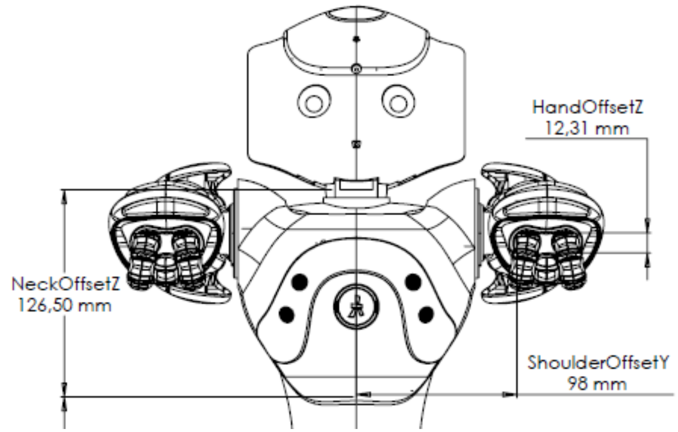
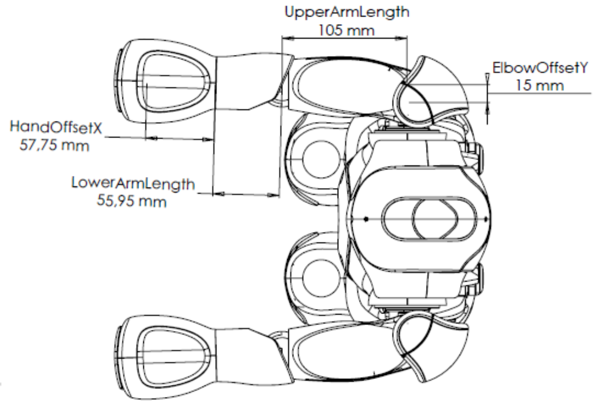


Front view



Top view

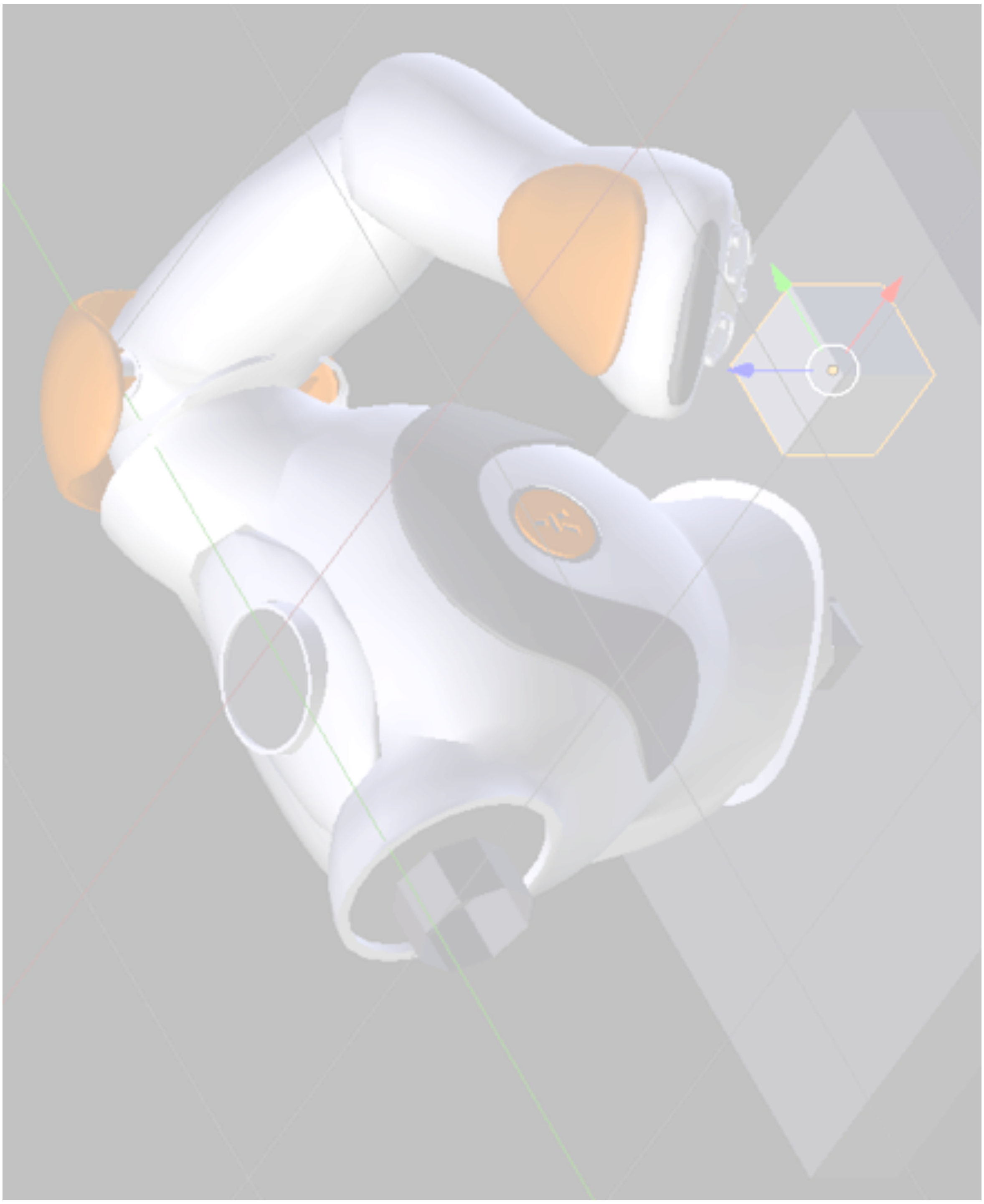


Arms

From ...	To ...	X (mm)	Y (mm)	Z (mm)
Torso	LShoulderPitch	0.00	98.00	100.00
LShoulderPitch	LShoulderRoll	0.00	0.00	0.00
LShoulderRoll	LElbowYaw	105.00	15.00	0.00
LElbowYaw	LElbowRoll	0.00	0.00	0.00
LElbowRoll	LWristYaw	55.95	0.00	0.00

Main length (mm)	
ShoulderOffsetY	98.00
ElbowOffsetY	15.00
UpperArmLength	105.00
LowerArmLength	55.95
ShoulderOffsetZ	100.00
HandOffsetX	57.75
HandOffsetZ	12.31

Name ______________________________



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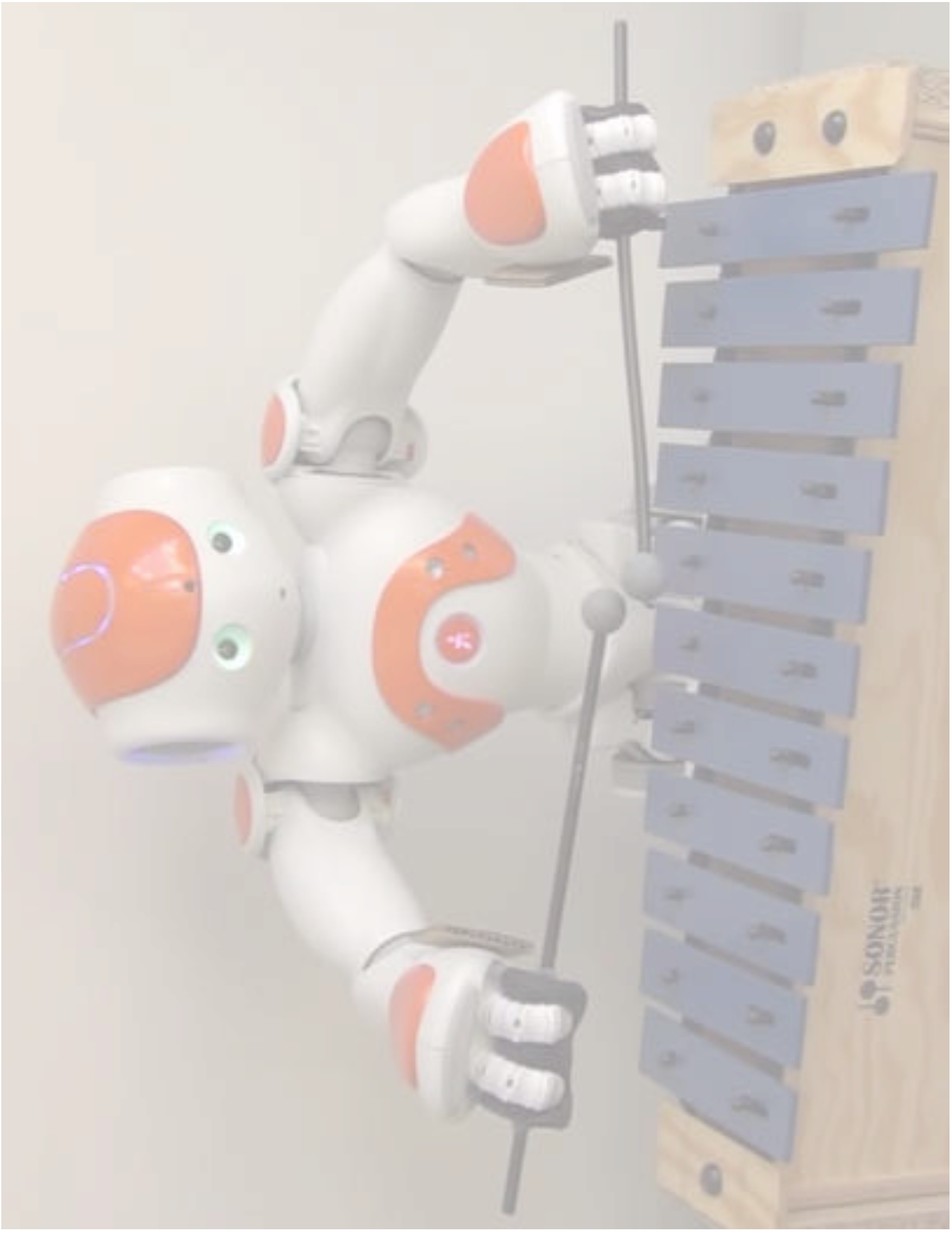


Solution

February 3, 2017

Exercise 1

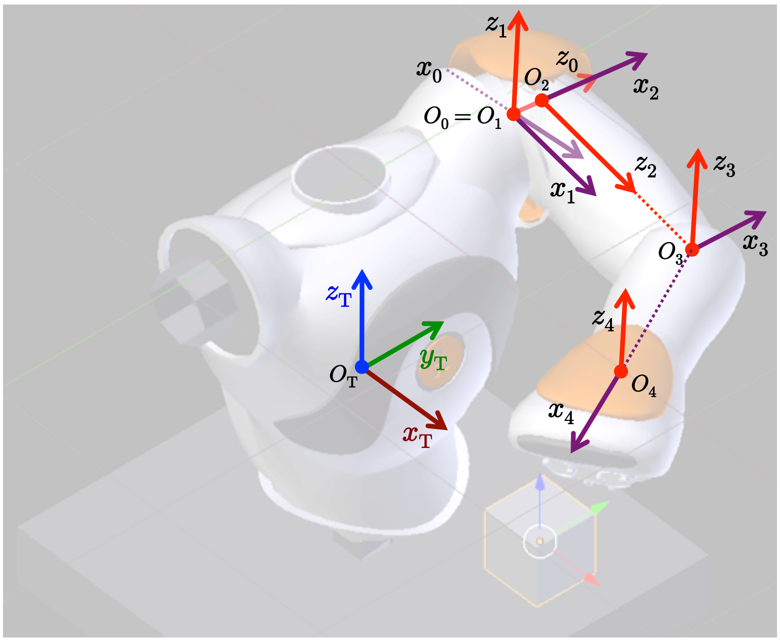


Figure 3: The assigned DH frames on a CAD image of the torso and left arm of the NAO.

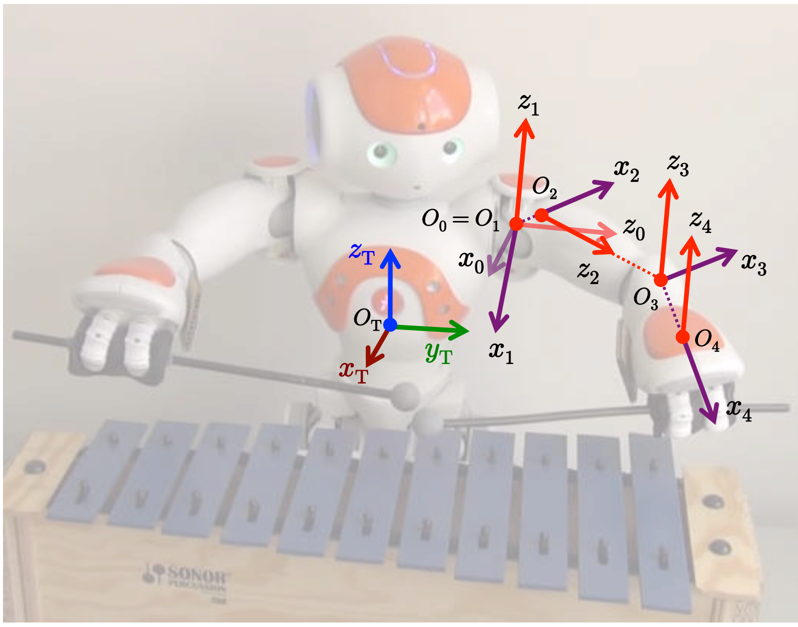
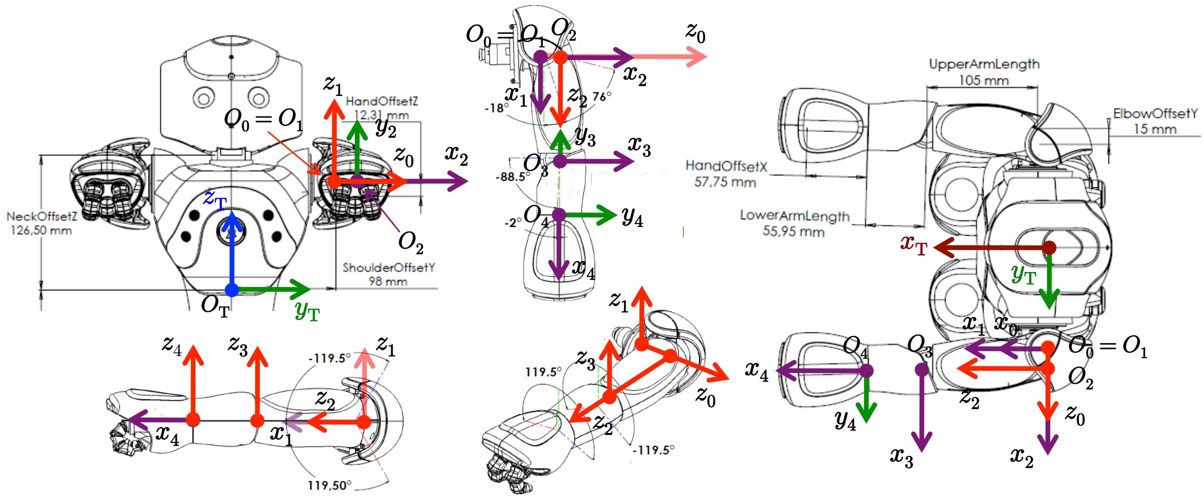


Figure 4: The assigned DH frames displayed on a picture of the upper limbs of the NAO.

The placement of the torso frame and the Denavit-Hartenberg (DH) assignment for the left arm are illustrated in the two Figs. 3 and 4, for different arm configurations and from different perspective views. The first axis of the shoulder is the ShoulderPitch axis $\boldsymbol{z}_0$, followed by the ShoulderRoll axis $\boldsymbol{z}_1$. Please note the small offset (ElbowOffsetY = 1.5 cm) between the two incident axes at the robot shoulder and the following axis $\boldsymbol{z}_2$, which provides the ElbowYaw degree of freedom. Without this offset, which would then be added to the value ShoulderOffsetY as a kinematic approximation, the NAO robot would have a spherical shoulder.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	$q_1 = 0$
2	$\pi/2$	ElbowOffsetY = 15 mm	0	$q_2 = \pi/2$
3	$-\pi/2$	0	UpperArmLength = 105 mm	$q_3 = 0$
4	0	LowerArmLength = 55.95 mm	0	$q_4 = -\pi/2$

Figure 5: Front and top views of NAO upper limbs with the arms stretched forward and three views of the left arm [above]; the DH table of parameters [below].

In Fig. 5, the upper limbs of the NAO are shown from the front and top viewpoints, together with three views of the left arm, and the associated DH table is reported. The last column in the table contains the actual values of the joint variables $\boldsymbol{q} = \begin{pmatrix} 0 & \pi/2 & 0 & -\pi/2 \end{pmatrix}^T$ when the left arm is stretched forward and horizontally, as in the picture.

The 4×4 homogeneous matrix ${}^T\!\boldsymbol{A}_0$ from the torso frame to the DH frame 0 is given by

$$
{}^T\!\boldsymbol{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \text{ShoulderOffsetY} \\ 0 & -1 & 0 & \text{ShoulderOffsetZ} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.98 \\ 0 & -1 & 0 & 0.10 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

where lengths are expressed in [m].

Exercise 2

Since the kinematic bounds on motion are imposed on the single joints, it is convenient to plan the interpolating trajectory in the joint space. The minimum transfer time is achieved when each joint executes a bang-bang or bang-coast-bang acceleration trajectory. Coordinated motion of the robot arm is then obtained by uniform time scaling of the faster joint(s), so as to align the final time to that of the joint which has the slowest completion time (due to the values of the velocity and acceleration limits, in combination with the distance to be traveled by that joint).

First, the initial and final Cartesian positions are inverted in the joint space. Considering for simplicity only the inverse solution for the RP robot which has a positive value for the prismatic variable q_2,

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \Rightarrow \qquad q_2 = \sqrt{p_x^2 + p_y^2} > 0, \qquad q_1 = \text{ATAN2}\{p_y, p_x\},$$

we obtain, respectively for $\boldsymbol{p} = \boldsymbol{p}_i$ and $\boldsymbol{p} = \boldsymbol{p}_f$,

$$\boldsymbol{q}_i = \begin{pmatrix} \text{ATAN2}\{3,4\} \\ \sqrt{4^2+3^2} \end{pmatrix} = \begin{pmatrix} 0.6435 \\ 5 \end{pmatrix} \quad \text{and} \quad \boldsymbol{q}_f = \begin{pmatrix} \text{ATAN2}\{1,-1\} \\ \sqrt{1^2+(-1)^2} \end{pmatrix} = \begin{pmatrix} 2.3562 \\ \sqrt{2} \end{pmatrix} \quad \text{[rad; m]},$$

yielding

$$\Delta\boldsymbol{q} = \boldsymbol{q}_f - \boldsymbol{q}_i = \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} 1.7127 \\ -3.5858 \end{pmatrix} \quad \text{[rad; m]}.$$

From the known formula about the existence of a cruising phase in a trapezoidal velocity profile, since

$$1.7127 = |\Delta q_1| > \frac{V_{max,1}^2}{A_{max,1}} = \frac{4}{3} = 1.3333,$$

we have that the acceleration profile for the first joint is bang-coast-bang, with acceleration/deceleration time and minimum transfer time given by

$$T_{a,1} = \frac{V_{max,1}}{A_{max,1}} = 0.6667 \text{ [s]}, \qquad T_1 = \frac{|\Delta q_1|\, A_{max,1} + V_{max,1}^2}{A_{max,1}\, V_{max,1}} = 1.5230 \text{ [s]}.$$

On the other hand, since

$$3.5858 = |\Delta q_2| < \frac{V_{max,2}^2}{A_{max,2}} = \frac{9}{1.5} = 6,$$

we have that the acceleration profile for the second joint is bang-bang, with acceleration/deceleration time, minimum transfer time, and maximum reached velocity given by

$$T_{a,2} = \sqrt{\frac{|\Delta q_2|}{A_{max,2}}} = 1.5461 \text{ [s]}, \qquad T_2 = 2\,T_{a,2} = 3.0923 \text{ [s]}, \qquad \bar{V}_2 = \frac{|\Delta q_2|}{T_{a,2}} = 2.3192 \text{ [m/s]}.$$

Therefore, the minimum time T for the robot motion will be given by the slowest completion time T_i among all joints (i.e., that of joint 2)

$$T = \max\{T_1, T_2\} = 3.0923 \text{ [s]}.$$

To obtain a coordinated joint trajectory, we need to scale down the motion of joint 1 by the factor

$$k = \frac{T}{T_1} = 2.0304 \qquad \Rightarrow \qquad \dot{q}_{s,1}(t) = \frac{\dot{q}_1(t)}{k}, \qquad \ddot{q}_{s,1}(t) = \frac{\ddot{q}_1(t)}{k^2}.$$

The new trapezoidal velocity profile of joint 1 will have a coordinated motion time $T_{s,1}$ and a reduced cruise velocity $\bar{V}_1$ given by

$$T_{s,1} = k\,T_1 = T = 3.0923 \text{ [s]}, \qquad \bar{V}_1 = \frac{V_{max,1}}{k} = 0.9850 \text{ [rad/s]},$$

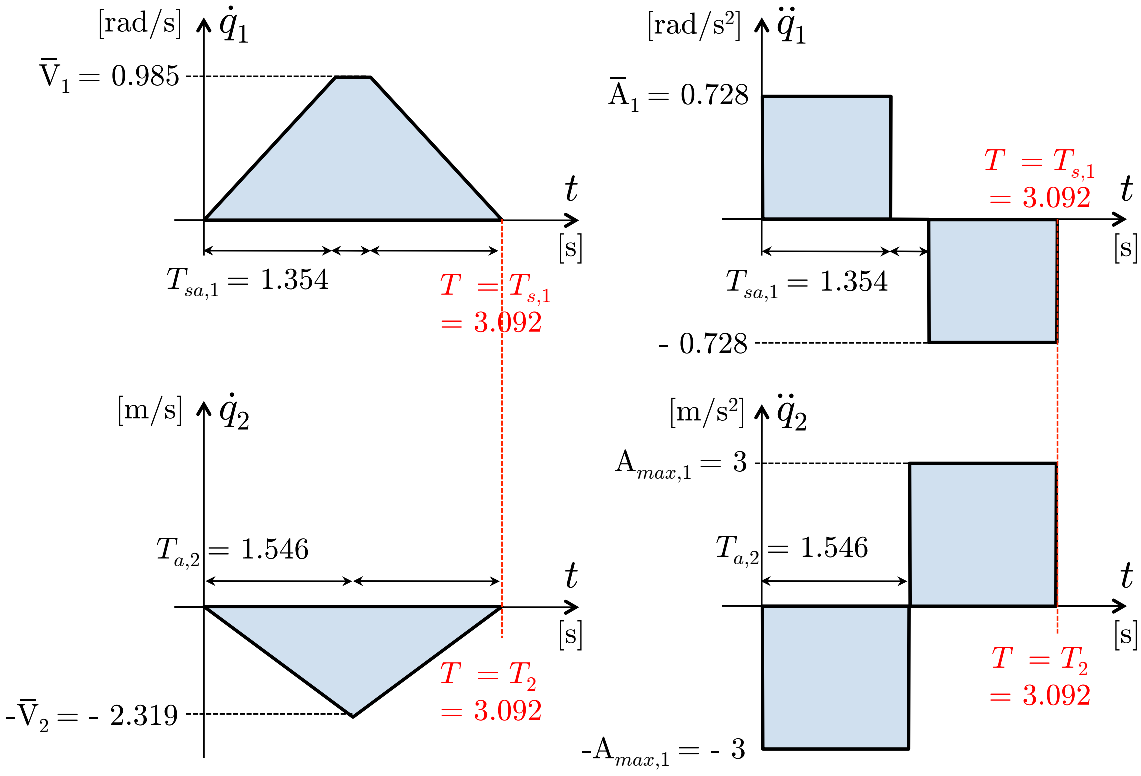


Figure 6: Coordinated minimum time velocity and acceleration profiles of joint 1 [above] and of joint 2 [below].

while the new acceleration profile of joint 1 will have a scaled acceleration/deceleration time $T_{sa,1}$ and a reduced maximum acceleration $\bar{A}_1$ given by

$$T_{sa,1} = k\, T_{a,1} = 1.3536 \text{ [s]}, \qquad \bar{A}_1 = \frac{A_{max,1}}{k^2} = 0.7277 \text{ [rad/s}^2\text{]}.$$

The final velocity and acceleration profiles of the two joints are drawn in Fig. 6. Note that the velocity profile of joint 2 is negative because $\Delta q_2 < 0$.

The Cartesian path corresponding to the designed joint trajectory will *not* be a linear segment joining $\boldsymbol{p}_i$ to $\boldsymbol{p}_f$. To prove this, it is sufficient to show that there exists at least one configuration $\boldsymbol{q}_m = \begin{pmatrix} q_{m1} & q_{m2} \end{pmatrix}^T$ belonging to the joint trajectory that maps (via direct kinematics) into a Cartesian point $\boldsymbol{p}_m = \begin{pmatrix} p_{mx} & p_{my} \end{pmatrix}^T$ which is not on this segment; namely, its Cartesian coordinates will not satisfy

$$\frac{p_x - p_{i,x}}{p_{f,x} - p_{i,x}} = \frac{p_y - p_{i,y}}{p_{f,y} - p_{i,y}} \qquad \Rightarrow \qquad \frac{p_x - 4}{5} = \frac{p_y - 3}{2} \qquad \Leftrightarrow \qquad 2p_x - 5p_y + 7 = 0,$$

which is the equation of a line passing through the two points $\boldsymbol{p}_i = \begin{pmatrix} p_{i,x} & p_{i,y} \end{pmatrix}^T = \begin{pmatrix} 4 & 3 \end{pmatrix}^T$ and $\boldsymbol{p}_f = \begin{pmatrix} p_{f,x} & p_{f,y} \end{pmatrix}^T = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$. A simple choice is to pick the midpoint of the joint motion. We have

$$\boldsymbol{q}_m = \frac{\boldsymbol{q}_i + \boldsymbol{q}_f}{2} = \begin{pmatrix} 1.4998 \\ 3.2071 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{p}_m = \begin{pmatrix} q_{m2} \cos q_{m1} \\ q_{m2} \sin q_{m1} \end{pmatrix} = \begin{pmatrix} 0.2273 \\ 3.1990 \end{pmatrix}$$

$$\Rightarrow \qquad 2 \cdot 0.2273 - 5 \cdot 3.1990 + 7 = -8.5405 \neq 0,$$

which shows that the Cartesian point is not on the linear path from $\boldsymbol{p}_i$ to $\boldsymbol{p}_f$. Moreover, since the analytic Jacobian of the RP robot and its determinant are

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \det \boldsymbol{J}(\boldsymbol{q}) = -q_2,$$

and the motion of the second joint is confined between its initial and final values, i.e., $q_2(t) \in \left[\sqrt{2}, 5\right]$ for all $t \in [0, T]$, then q_2 will never be zero and the robot will not pass through a singularity during motion.

Since the RP robot encounters no singular configurations during motion, the Jacobian will always have full rank and the identity $\boldsymbol{J}^{\#}(\boldsymbol{q})^T \boldsymbol{J}^{\#}(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}$ holds. Thus, the manipulability ellipsoid in velocity and the manipulability measure w are given by

$$\boldsymbol{v}^T \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} \boldsymbol{v} = \boldsymbol{v}^T \begin{pmatrix} q_2^2 \sin^2 q_1 + \cos^2 q_1 & \left(1 - q_2^2\right) \sin q_1 \cos q_1 \\ \left(1 - q_2^2\right) \sin q_1 \cos q_1 & q_2^2 \cos^2 q_1 + \sin^2 q_1 \end{pmatrix} \boldsymbol{v} = 1 \tag{1}$$

and

$$w = \sqrt{\det\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)} = |q_2| \,.$$

As a result, during motion the manipulability w will decrease linearly (no need to plot this!) from the initial value $q_{i,2} = 5$ to the final value $q_{f,2} = \sqrt{2}$. On the other hand, by observing the expression of $\boldsymbol{J}\boldsymbol{J}^T$ in (1), we can immediately see that $\boldsymbol{J}\boldsymbol{J}^T = \boldsymbol{I}$ for $q_2 = 1$, and the manipulability ellipsoid becomes a circle. In this situation, which is not encountered in this particular planned motion, there is an *isotropic* behavior for the transformation of velocities as well as for the transformation of forces.

Exercise 3

The desired Cartesian trajectory $\boldsymbol{p}_d(t)$ can be defined using decomposition in space (tracing a linear path) and time (moving with a quintic polynomial) as

$$\boldsymbol{p}_d(s) = \boldsymbol{p}_i + s\left(\boldsymbol{p}_f - \boldsymbol{p}_i\right), \quad s \in [0,1], \qquad s(t) = 6\left(\frac{t}{T}\right)^5 - 15\left(\frac{t}{T}\right)^4 + 10\left(\frac{t}{T}\right)^3, \quad t \in [0,T], \tag{2}$$

where the six coefficients of a quintic polynomial are necessary and sufficient to impose the required rest-to-rest motion with zero initial velocity and acceleration (which produces a continuous acceleration profile also at the initial and final instants). The desired Cartesian velocity is then

$$\dot{\boldsymbol{p}}_d(t) = \frac{d\boldsymbol{p}_d(s)}{ds}\,\dot{s}(t) = \frac{30\left(\boldsymbol{p}_f - \boldsymbol{p}_i\right)}{T}\left(\left(\frac{t}{T}\right)^4 - 2\left(\frac{t}{T}\right)^3 + \left(\frac{t}{T}\right)^2\right). \tag{3}$$

For later use, we note that the numerical value of $\dot{\boldsymbol{p}}_d(t)$ at the initial time $t = 0$ is indeed $\dot{\boldsymbol{p}}_d(0) = \boldsymbol{0}$.

In view of the requirements, the control problem should be attacked at the Cartesian level. Since the task is two-dimensional (position tracking in the plane), we consider the following direct kinematics of the 3R robot

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell\,(c_1 + c_{12} + c_{123}) \\ \ell\,(s_1 + s_{12} + s_{123}) \end{pmatrix}, \tag{4}$$

with the usual shorthand notation, e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$. The associated 2×3 Jacobian is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123}) & -\ell\,(s_{12} + s_{123}) & -\ell\, s_{123} \\ \ell\,(c_1 + c_{12} + c_{123}) & \ell\,(c_{12} + c_{123}) & \ell\, c_{123} \end{pmatrix}. \tag{5}$$

Standard inversion of the non-square Jacobian $\boldsymbol{J}$ is indeed impossible. Rather, we should use the pseudoinverse $\boldsymbol{J}^{\#}$ (or any other form of generalized inversion) in the kinematic control law. Having assumed that the robot Jacobian remains of full rank during the whole execution of the task, we will evaluate the pseudoinverse as

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{6}$$

being $\boldsymbol{J}\boldsymbol{J}^{\#} = \boldsymbol{I}$.

We remind that a Cartesian kinematic control law of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\boldsymbol{e}\right), \qquad \boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad \boldsymbol{K} > 0, \text{ diagonal}, \tag{7}$$

would satisfy most of the problem requirements, except for the specification of the transient error dynamics along the tangent and normal directions to the Cartesian path. In fact, when setting $\boldsymbol{K} = \mathrm{diag}\{k_i\}$ in the present planar case, the error dynamics along the two orthogonal directions $\boldsymbol{x}$ and $\boldsymbol{y}$ would become linear and decoupled, and the two scalar gains $k_i > 0$, $i = x, y$, in (7) can be chosen so as to yield the desired error decay, i.e.,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}\boldsymbol{e}\right) = -\boldsymbol{K}\boldsymbol{e} \qquad \Rightarrow \qquad \begin{aligned} \dot{e}_x &= -k_x e_x \\ \dot{e}_y &= -k_y e_y. \end{aligned}$$

However, when using (7), this linear and decoupled dynamics is not displayed along other directions in the plane.

In order to achieve a similar decoupled behavior along the two orthogonal directions $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ which are, respectively, tangent and normal to the linear path, we need to rotate the Cartesian error $\boldsymbol{e} = {}^0\boldsymbol{e}$ into the task frame attached to the path, react to the rotated error ${}^t\boldsymbol{e} = \begin{pmatrix} e_t & e_n \end{pmatrix}^T$ in a decoupled way (so that the two components of ${}^t\boldsymbol{e}$ independently decay at the specified exponential rates), and then map back this control action into a velocity command expressed in the original Cartesian frame (where the robot Jacobian in (5) is also expressed). The kinematic control law (7) becomes then

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\dot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t\,\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\boldsymbol{e}\right), \qquad \boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t), \qquad \boldsymbol{K} = \begin{pmatrix} k_t & 0 \\ 0 & k_n \end{pmatrix} > 0, \tag{8}$$

with the constant 2×2 (planar) rotation matrix ${}^0\boldsymbol{R}_t$ defined from the linear path as

$$\boldsymbol{x}_t = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \boldsymbol{y}_t = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \qquad \Rightarrow \qquad {}^0\boldsymbol{R}_t = \begin{pmatrix} \boldsymbol{x}_t & \boldsymbol{y}_t \end{pmatrix}.$$

To show the resulting closed-loop behavior, let the rotated error be ${}^t\boldsymbol{e} = {}^t\boldsymbol{R}_0\,{}^0\boldsymbol{e} = {}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e}$. Being ${}^0\dot{\boldsymbol{R}}_t = \boldsymbol{0}$ and using (8), the dynamics of ${}^t\boldsymbol{e}$ is

$${}^t\dot{\boldsymbol{e}} = {}^t\boldsymbol{R}_0\left(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}\right) = {}^t\boldsymbol{R}_0\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) = {}^t\boldsymbol{R}_0\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}\boldsymbol{J}^{\#}\left(\dot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t\,\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e}\right)\right) = -\boldsymbol{K}\,{}^0\boldsymbol{R}_t^T\,{}^0\boldsymbol{e} = -\boldsymbol{K}\,{}^t\boldsymbol{e},$$

and thus

$$\begin{aligned} \dot{e}_t &= -k_t e_t \\ \dot{e}_n &= -k_n e_n \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} e_t(t) &= e_t(0)\exp(-k_t t) \\ e_n(t) &= e_n(0)\exp(-k_n t). \end{aligned}$$

Having set $T = 4$ [s] for the total motion time, the specification on the transient errors is enforced as follows. Let first $k_t = k_n = k > 0$. Being the norm of a vector invariant w.r.t. rotations ($\|{}^0\boldsymbol{e}\| = \|{}^0\boldsymbol{R}_t\,{}^t\boldsymbol{e}\| = \|{}^t\boldsymbol{e}\|$), it is

$$\|{}^0\boldsymbol{e}(t)\| = \|{}^t\boldsymbol{e}(t)\| = \sqrt{e_t^2(t) + e_n^2(t)} = \exp(-kt)\sqrt{e_t^2(0) + e_n^2(0)} = \exp(-kt)\,\|{}^t\boldsymbol{e}(0)\| = \exp(-kt)\,\|{}^0\boldsymbol{e}(0)\|$$

Thus, from the requested condition at $t = T/2 = 2$,

$$\|{}^0\boldsymbol{e}(2)\| \le \frac{1}{10}\,\|{}^0\boldsymbol{e}(0)\|,$$

it follows

$$\exp(-2k)\,\|{}^0\boldsymbol{e}(0)\| \le \frac{1}{10}\,\|{}^0\boldsymbol{e}(0)\| \qquad \Rightarrow \qquad \exp(2k) \ge 10 \qquad \Rightarrow \qquad k \ge 0.5\ln 10 = 1.1513.$$

To complete the design of the control gains, we set then

$$k_t = k = 1.1513, \qquad \text{and} \qquad k_n = 3\,k_t = 3.4539. \tag{9}$$

Finally, from the problem data, we compute

$$\boldsymbol{x}_t = \frac{\boldsymbol{p}_f - \boldsymbol{p}_i}{\|\boldsymbol{p}_f - \boldsymbol{p}_i\|} = \begin{pmatrix} \dfrac{-6}{\sqrt{40}} \\ \dfrac{2}{\sqrt{40}} \end{pmatrix} \qquad \Rightarrow \qquad {}^0\boldsymbol{R}_t = \begin{pmatrix} \dfrac{-6}{\sqrt{40}} & \dfrac{-2}{\sqrt{40}} \\ \dfrac{2}{\sqrt{40}} & \dfrac{-6}{\sqrt{40}} \end{pmatrix}. \tag{10}$$

Using the designed position trajectory (2) and its velocity (3), the expression of the direct kinematics (4) and of its Jacobian (5), with the pseudoinverse computed numerically as in (6), the gains (9), and the rotation matrix in (10), the control law (8) can be completely evaluated at every configuration $\boldsymbol{q}$.

The control law (8) achieves exact trajectory following, i.e., $\boldsymbol{e}(t) = \boldsymbol{0}$ for all $t \geq 0$, if and only if the initial configuration $\boldsymbol{q}(0) = \boldsymbol{q}_{0,e}$ of the robot is such that $\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \boldsymbol{p}_i - \boldsymbol{f}(\boldsymbol{q}_{0,e}) = \boldsymbol{0}$. Using (4) with $\ell = 2$ [m], one such solution is immediately found as

$$\boldsymbol{q}_{0,e} = \begin{pmatrix} 0 \\ 0 \\ \pi/2 \end{pmatrix} \text{[rad]} \qquad \Rightarrow \qquad \boldsymbol{f}(\boldsymbol{q}_{0,e}) = \begin{pmatrix} \ell\,(1+1+0) \\ \ell\,(0+0+1) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \boldsymbol{p}_i.$$

More in general, due to the redundancy, finding an initial configuration $\boldsymbol{q}_0$ that matches a desired initial end-effector position $\boldsymbol{p}_0$ for the 3R robot requires the use of an iterative numerical algorithm, such as the gradient or the Newton method. In these two methods, we would need the evaluation of $\boldsymbol{J}^T(\boldsymbol{q})$, which is directly available from (5), or, respectively, of $\boldsymbol{J}^{\#}(\boldsymbol{q})$, which is computed as in (6).

When starting at $\boldsymbol{q}(0) = \boldsymbol{q}_0 = \begin{pmatrix} -\pi/2 & 0 & \pi/2 \end{pmatrix}^T$ [rad], the evaluation of (8) yields as joint velocity control command

$$\begin{aligned}
\dot{\boldsymbol{q}}(0) &= \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\, {}^0\boldsymbol{R}_t\, \boldsymbol{K}\, {}^0\boldsymbol{R}_t^T\, (\boldsymbol{p}_i - \boldsymbol{f}(\boldsymbol{q}_0)) \\
&= \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \end{pmatrix}^{\#} \begin{pmatrix} \frac{-6}{\sqrt{40}} & \frac{-2}{\sqrt{40}} \\ \frac{2}{\sqrt{40}} & \frac{-6}{\sqrt{40}} \end{pmatrix} \begin{pmatrix} 1.1513 & 0 \\ 0 & 3.4539 \end{pmatrix} \begin{pmatrix} \frac{-6}{\sqrt{40}} & \frac{2}{\sqrt{40}} \\ \frac{-2}{\sqrt{40}} & \frac{-6}{\sqrt{40}} \end{pmatrix} \left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0.25 & -0.0833 \\ 0 & 0.1667 \\ -0.25 & 0.4167 \end{pmatrix} \begin{pmatrix} 1.3816 & 0.6908 \\ 0.6908 & 3.2236 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.4539 \\ 6.9078 \end{pmatrix} \text{[rad/s]}.
\end{aligned}$$

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