Robotics I

September 12, 2016

Exercise 1

The last three revolute joints (labeled from 4 to 6) of the 6-dof Universal Robot UR10 constitute a non-spherical wrist and are described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i (mm)	θ_i
4	$-\pi/2$	0	$d_4 = 163.9$	q_4
5	$\pi/2$	0	$d_5 = 115.7$	q_5
6	0	0	$d_6 = 92.2$	q_6

Table 1: Denavit-Hartenberg parameters of the non-spherical wrist of the UR10 robot.

- Provide the analytic expressions of the inverse kinematic mapping, which takes as input a desired orientation of the (end-effector) frame 6, as expressed by a rotation matrix $\boldsymbol{R}$, and provides as output *all* solutions for the wrist angles (q_4, q_5, q_6) in the regular case. Characterize also the singular cases, and explain what happens in such situations.
- Apply your formulas to solve the inverse kinematics for the UR10 robot wrist, given the following numerical input:

$$\boldsymbol{R} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

Exercise 2

Consider the planar RPR robot in Fig. 1. The prismatic axis of the second joint is skewed by an angle $\beta = 45°$ with respect to the first link.

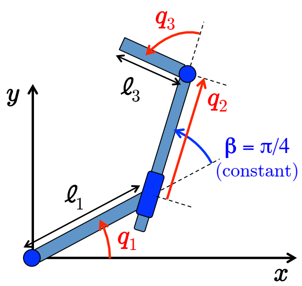

Figure 1: A planar RPR robot with its joint coordinates q_1, q_2 and q_3.

- Using the coordinates shown, provide the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ that relates $\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{pmatrix}^T$ to the velocity $\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{p}_x & \dot{p}_y \end{pmatrix}^T$ of the end effector and find the singularities of this mapping.
- Let the robot be at $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0.2 & -\pi/4 \end{pmatrix}^T$ [rad,m,rad], with kinematic data $\ell_1 = 1$ and $\ell_3 = 0.5$ [m]. For a desired end-effector velocity $\dot{\boldsymbol{p}}_d = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$ [m/s], determine numerically
  - the minimum norm (least squares) solution $\dot{\boldsymbol{q}}_{LS}$;
  - another solution $\dot{\boldsymbol{q}}_0 \neq \dot{\boldsymbol{q}}_{LS}$, such that $\boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{p}}_d$.

[120 minutes; open books]

Solution

September 12, 2016

Exercise 1

From Tab. 1, we build the rotation matrices

$$
{}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & -\sin q_4 \\ \sin q_4 & 0 & \cos q_4 \\ 0 & -1 & 0 \end{pmatrix}, \qquad {}^4\boldsymbol{R}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 \\ \sin q_5 & 0 & -\cos q_5 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} \cos q_6 & -\sin q_6 & 0 \\ \sin q_6 & \cos q_6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Using the usual compact notation for trigonometric functions, the orientation of the end-effector frame expressed w.r.t. frame 3 of the UR10 robot (which is taken here as reference frame for the wrist kinematics) is given by

$$
{}^4\boldsymbol{R}_6(\boldsymbol{q}) = {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5)\,{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} c_4c_5c_6 - s_4s_6 & -s_4c_6 - c_4c_5s_6 & c_4s_5 \\ c_4s_6 + s_4c_5c_6 & c_4c_6 - s_4c_5s_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{pmatrix}, \tag{1}
$$

where $\boldsymbol{q} = \begin{pmatrix} q_4 & q_5 & q_6 \end{pmatrix}^T$.

Let R_{ij} $(i,j = 1,2,3)$ be the elements of the desired orientation matrix $\boldsymbol{R}$. We solve then the matrix equation ${}^4\boldsymbol{R}_6(\boldsymbol{q}) = \boldsymbol{R}$ by inspecting the structure of the scalar elements in (1). It is easy to see that

$$
q_5 = \text{ATAN2}\left\{\pm\sqrt{R_{13}^2 + R_{23}^2}, R_{33}\right\}, \tag{2}
$$

providing in the regular case two solutions q_5^+ and q_5^- (with equal modulus and opposite signs). Provided that $R_{13}^2 + R_{23}^2 = \sin q_5 \neq 0$, namely that $q_5 \neq 0$ and $\neq \pi$ as a result of (2), we can solve for the other two angles in an unique way as

$$
q_4 = \text{ATAN2}\left\{\frac{R_{23}}{\sin q_5^\pm}, \frac{R_{13}}{\sin q_5^\pm}\right\}, \qquad q_6 = \text{ATAN2}\left\{\frac{R_{32}}{\sin q_5^\pm}, \frac{-R_{31}}{\sin q_5^\pm}\right\}, \tag{3}
$$

yielding the two pairs (q_4^+, q_6^+) and (q_4^-, q_6^-), associated respectively to the two choices q_5^+ and q_5^- in (2).

In the singular case, $\sin q_5 = 0, \cos q_5 = \pm 1$, only the sum or the difference of the two other joint angles will be defined.

When the formulas (2–3) are applied to the desired orientation $\boldsymbol{R}$, they yield the two solutions

$$
\boldsymbol{q}^+ = \begin{pmatrix} \pi/2 \\ 3\pi/4 \\ \pi \end{pmatrix} = \begin{pmatrix} 1.5708 \\ 2.3562 \\ 3.1416 \end{pmatrix}, \qquad \boldsymbol{q}^- = \begin{pmatrix} -\pi/2 \\ -3\pi/4 \\ 0 \end{pmatrix}. \tag{4}
$$

Exercise 2

For a generic skew angle β, the direct kinematics of the RPR planar robot in Fig. 1 is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 \cos q_1 + q_2 \cos(\beta + q_1) + \ell_3 \cos(\beta + q_1 + q_3) \\ \ell_1 \sin q_1 + q_2 \sin(\beta + q_1) + \ell_3 \sin(\beta + q_1 + q_3) \end{pmatrix}$$

and its Jacobian $\boldsymbol{J} = \partial \boldsymbol{f}/\partial \boldsymbol{q}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell_1 \sin q_1 - q_2 \sin(\beta + q_1) - \ell_3 \sin(\beta + q_1 + q_3) & \cos(\beta + q_1) & -\ell_3 \sin(\beta + q_1 + q_3) \\ \ell_1 \cos q_1 + q_2 \cos(\beta + q_1) + \ell_3 \cos(\beta + q_1 + q_3) & \sin(\beta + q_1) & \ell_3 \cos(\beta + q_1 + q_3) \end{pmatrix}. \tag{5}$$

To find the singularities of the differential kinematics, namely the configurations where the resulting matrix $\boldsymbol{J}(\boldsymbol{q})$ loses rank, we compute the three minors obtained by deleting, respectively, the third, second, or first column of $\boldsymbol{J}(\boldsymbol{q})$. We obtain

$$\begin{aligned} \det \boldsymbol{J}_{[-3]} &= -\left(q_2 + \ell_1 \cos\beta + \ell_3 \cos q_3\right), \\ \det \boldsymbol{J}_{[-2]} &= -\ell_3 \left(\ell_1 \sin(\beta + q_3) + q_2 \sin q_3\right), \\ \det \boldsymbol{J}_{[-1]} &= \ell_3 \cos q_3. \end{aligned}$$

All three determinants are simultaneously equal to zero if and only if

$$\cos q_3 = 0, \qquad q_2 = -\ell_1 \cos\beta.$$

When this happens, the rank of the Jacobian $\boldsymbol{J}$ in (5) falls down to 1. If we plug in now the given value $\beta = \pi/4$, we find the singularity at $q_2 = -\ell_1\sqrt{2}/2$, $q_3 = \pm\pi/2$ (for any value of q_1)[1].

Next, at the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & 0.2 & -\pi/4 \end{pmatrix}^T$ and with the kinematic data $\beta = \pi/4$, $\ell_1 = 1$, and $\ell_3 = 0.5$, the Jacobian in (5) becomes

$$\boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -1.6414 & -0.7071 & -0.5 \\ -0.1414 & 0.7071 & 0 \end{pmatrix},$$

which is of full rank. For $\dot{\boldsymbol{p}}_d = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$, the minimum norm solution is obtained using the pseudoinverse of the Jacobian[2]

$$\dot{\boldsymbol{q}}_{LS} = \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.5185 \\ 0.1037 \\ 0.1512 \end{pmatrix}. \tag{6}$$

Other solutions can be obtained in many ways. For instance, when 'freezing' the prismatic joint ($\dot{q}_2 = 0$) we would still have a non-singular sub-Jacobian $\boldsymbol{J}_{[-2]}(\boldsymbol{q}_0)$. Thus, by computing

$$\begin{pmatrix} \dot{q}_{0,1} \\ \dot{q}_{0,3} \end{pmatrix} = \boldsymbol{J}_{[-2]}^{-1}(\boldsymbol{q}_0)\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

[1] Another notable case is when $\beta = \pm\pi/2$, i.e., the prismatic joint is orthogonal to the first link. In that case, the singularity occurs when $q_2 = 0$ and $q_3 = \pm\pi/2$.

[2] Note that the units of the solution vector in (6) are non-homogeneous, namely [rad/s] for the first and third joints and [m/s] for the second joint. In this context, the concept of (unweighted) norm is not a properly defined one. Nonetheless, the use of a pseudoinverse solution is still a common practice even in such cases.

a different feasible solution is obtained as

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} \dot{q}_{0,1} \\ 0 \\ \dot{q}_{0,3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}. \tag{7}$$

Note that only the last joint is eventually used in this case in order to realize the desired end-effector motion. Indeed, the norm of this joint velocity, $\|\dot{\boldsymbol{q}}_0\| = 2$, is larger than the one of the pseudoinverse solution, $\|\dot{\boldsymbol{q}}_{LS}\| = 0.55$.

* * * * *