Robotics I

April 1, 2016

Consider a planar 2R robot with links of length $\ell_1 = 1$ and $\ell_2 = 0.5$ [m]. The end-effector should move *smoothly* from an initial point $\boldsymbol{p}_{in}$ to a final point $\boldsymbol{p}_{fin}$ in the robot workspace so that

- the motion starts and ends with zero Cartesian velocity and acceleration;
- at the start, the robot is in the 'right arm' inverse kinematics solution (i.e., with positive q_2), and remains in this type of solution throughout the motion;
- coordinated motion is enforced to the joints;
- symmetric limits on joint velocity, acceleration, and jerk are satisfied:

$$|\dot{q}_i| \leq V_i, \qquad |\ddot{q}_i| \leq A_i, \qquad |\dddot{q}_i| \leq J_i, \qquad i = 1, 2.$$

In order to address this motion task, choose a class of trajectories and determine, within the considered class, a minimum time trajectory, given the following position data

$$\boldsymbol{p}_{in} = \begin{pmatrix} 0.4 \\ 1.2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{p}_{fin} = \begin{pmatrix} -1 \\ -0.2 \end{pmatrix} \text{ [m]}$$

and joint limits

$$\begin{aligned} V_1 &= 1 \text{ [rad/s]}, \quad A_1 = 3 \text{ [rad/s}^2\text{]}, \quad J_1 = 30 \text{ [rad/s}^3\text{]}, \\ V_2 &= 2 \text{ [rad/s]}, \quad A_2 = 7.5 \text{ [rad/s}^2\text{]}, \quad J_2 = 70 \text{ [rad/s}^3\text{]}. \end{aligned}$$

Provide the minimum feasible time T^* obtained and the maximum (absolute) values attained by the velocity and the acceleration at the two joints.

At the trajectory midpoint, $t = T^*/2$, determine the values of the end-effector Cartesian velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$, and draw the robot in its current configuration together with the vectors $\boldsymbol{v}$ and $\boldsymbol{a}$.

[180 minutes; open books]

Solution

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In view of the nature of the given robot motion limits, it is highly recommended to define the trajectory in the joint space.

The direct and inverse kinematics of the 2R planar robot are given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} \\ \ell_1 s_1 + \ell_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q})$$

$$\Rightarrow \quad \boldsymbol{q} = \begin{pmatrix} q_1{}^{+/-} \\ q_2{}^{+/-} \end{pmatrix} = \begin{pmatrix} \text{ATAN2}\{p_y(\ell_1 + \ell_2 c_2) - p_x \ell_2 s_2, p_x(\ell_1 + \ell_2 c_2) + p_y \ell_2 s_2\} \\ \text{ATAN2}\{s_2, c_2\} \end{pmatrix} = \boldsymbol{f}^{-1}(\boldsymbol{p}),$$

with

$$c_2 = \frac{p_x^2 + p_y^2 - \ell_1^2 - \ell_2^2}{2\ell_1 \ell_2}, \qquad s_2 = \pm\sqrt{1 - c_2^2},$$

and where the $+/-$ associated as index to the joint angles q_1 and q_2 mean that for their evaluation the $+$ or, respectively, the $-$ sign has been used in the definition of s_2. Substituting the link lengths and the problem data for $\boldsymbol{p} = \boldsymbol{p}_{in}$ and $\boldsymbol{p} = \boldsymbol{p}_{fin}$, and picking up the solution with $q_2^+ > 0$ yields

$$\boldsymbol{q}_{in} = \begin{pmatrix} 49.83° \\ 69.51° \end{pmatrix} = \begin{pmatrix} 0.8697 \\ 1.2132 \end{pmatrix} \text{ [rad]}, \qquad \boldsymbol{q}_{fin} = \begin{pmatrix} 162.66° \\ 102.12° \end{pmatrix} = \begin{pmatrix} 2.8391 \\ 1.7824 \end{pmatrix} \text{ [rad]}.$$

Taking into account the smoothness requirement and the boundary conditions on velocity and acceleration, we choose a polynomial trajectory of degree 5 for each joint. In the double normalized form, its expression is

$$\boldsymbol{q}(\tau) = \boldsymbol{q}_{in} + \boldsymbol{\Delta q}\left(10\tau^3 - 15\tau^4 + 6\tau^5\right), \qquad \boldsymbol{\Delta q} = \boldsymbol{q}_{fin} - \boldsymbol{q}_{in} = \begin{pmatrix} 1.9712 \\ 0.5693 \end{pmatrix} \text{ [rad]}, \quad \tau = \frac{t}{T} \in [0, 1].$$

In order to obtain the maximum values reached along this trajectory by the velocity, acceleration, and jerk, which should satisfy the given limits, we compute the first four time derivatives:

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T}\left(30\tau^2 - 60\tau^3 + 30\tau^4\right) \\ \ddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^2}\left(60\tau - 180\tau^2 + 120\tau^3\right) \\ \dddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^3}\left(60 - 360\tau + 360\tau^2\right) \\ \ddddot{\boldsymbol{q}} &= \frac{\boldsymbol{\Delta q}}{T^4}\left(-360 + 720\tau\right). \end{aligned}$$

We analyze the constraints imposed by the joint limits starting with the one with highest differential order. We will work now with *scalar* quantities, i.e., joint by joint, dropping for simplicity the joint index. The maximum jerk in the closed interval $\tau \in [0, 1]$ occurs either at the boundaries or where the fourth derivative is zero:

$$\dddot{q}(0) = \dddot{q}(1) = 60\,\frac{\Delta q}{T^3}, \qquad \ddddot{q}(\tau) = 0 \quad @\,\tau^* = 0.5 \quad \Rightarrow \quad \dddot{q}(0.5) = -30\,\frac{\Delta q}{T^3}.$$

Thus, the minimum motion time T that satisfies the jerk limit is given by

$$|\dddot{q}(\tau)| \leq J \quad \Rightarrow \quad T \geq \sqrt[3]{\frac{60|\Delta q|}{J}} =: T_J.$$

The maximum acceleration occurs where the third derivative is zero (no need to check the value at the boundaries, since we have $\ddot{q}(0) = \ddot{q}(1) = 0$ by construction):

$$\dddot{q}(\tau) = 0 \quad \Longleftrightarrow \quad 1 - 6\tau + 6\tau^2 = 0 \quad @\,\tau^* = 0.5 \pm \frac{\sqrt{3}}{6} \quad \Rightarrow \quad \ddot{q}(\tau^*) = \pm\,5.7735\,\frac{\Delta q}{T^2}.$$

The minimum motion time T that satisfies the acceleration limit is given by

$$|\ddot{q}(\tau)| \leq A \quad \Rightarrow \quad T \geq \sqrt{\frac{5.7735|\Delta q|}{A}} =: T_A.$$

Similarly, the maximum velocity occurs where the second derivative is zero (again, no need to check the value at the boundaries, since $\dot{q}(0) = \dot{q}(1) = 0$):

$$\ddot{q}(\tau) = 0 \quad \Longleftrightarrow \quad \tau(1 - 3\tau + 2\tau^2) = 0 \quad @\,\tau^* = \{0, 0.5, 1\} \quad \Rightarrow \quad \dot{q}(0.5) = \frac{30}{16}\,\frac{\Delta q}{T},$$

and thus

$$|\dot{q}(\tau)| \leq V \quad \Rightarrow \quad T \geq \frac{30}{16}\,\frac{|\Delta q|}{V} =: T_V.$$

As a result, the minimum feasible motion time T^* is obtained as

$$T^* = \max\{T_J, T_A, T_V\} = \max\left\{3.9148\,\sqrt[3]{\frac{|\Delta q|}{J}}, 2.4028\,\sqrt{\frac{|\Delta q|}{A}}, 1.8750\,\frac{|\Delta q|}{V}\right\}.$$

Using the data (all in radians) of the problem at hand, we compute the minimum motion time for the first joint as

$$T_1^* = \max\{T_{J,1}, T_{A,1}, T_{V,1}\} = \max\{1.5792, 1.9468, 3.6926\} = 3.6926 \text{ [s]},$$

where the velocity limit is the most constraining one. Similarly, for the second joint it is

$$T_2^* = \max\{T_{J,2}, T_{A,2}, T_{V,2}\} = \max\{0.7872, 0.6619, 0.5336\} = 0.7872 \text{ [s]}$$

and the jerk will be the variable reaching first its limit. Since coordinated motion of the joints should be enforced, the common minimum motion time will be

$$T^* = \max\{T_1^*, T_2^*\} = T_1^* = 3.6926 \text{ [s]},$$

with the second joint traveling much slower than it could in principle. The trajectory profiles of position, velocity, acceleration, and jerk of the two joints are shown in Figs. 3–2.

The peak velocity of the two joints is reached at $t = T^*/2 = 1.8463$ s

$$\max_{t\in[0,T^*]} \dot{q}_1(t) = \dot{q}_1(1.8463) = 1 \text{ [rad/s]}, \qquad \max_{t\in[0,T^*]} \dot{q}_2(t) = \dot{q}_2(1.8463) = 0.2890 \text{ [rad/s]},$$

while the peak acceleration (in module) is attained at $t = \left(0.5 \pm \sqrt{3}/6\right) T^*$, namely at $t = 0.7803$ s (max positive acceleration) and $t = 2.9123$ s (max negative acceleration = max deceleration)

$$\max_{t\in[0,T^*]} |\ddot{q}_1(t)| = \ddot{q}_1(0.7803) = 0.8339 \text{ [rad/s}^2\text{]}, \qquad \max_{t\in[0,T^*]} |\ddot{q}_2(t)| = \ddot{q}_2(0.7803) = 0.2410 \text{ [rad/s}^2\text{]}.$$

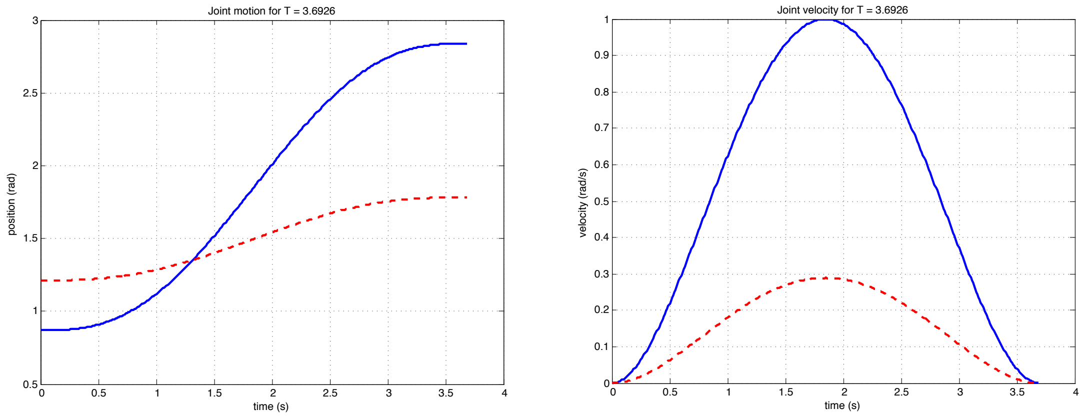

Figure 1: Position [left] and velocity [right] of joint 1 (blue, solid) and joint 2 (red, dashed)

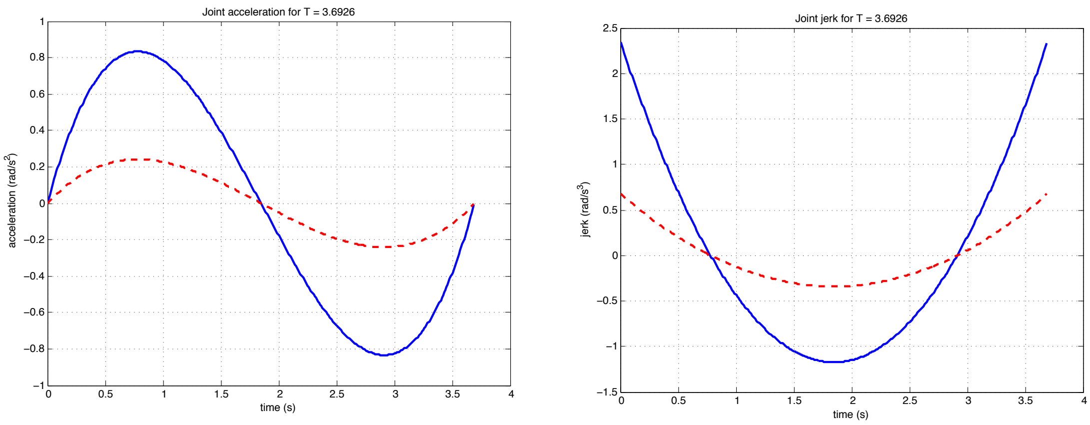

Figure 2: Acceleration [left] and jerk [right] of joint 1 (blue, solid) and joint 2 (red, dashed)

We note also that coordinated motion is symmetric w.r.t. to the total time (for all T, and thus also for T^*). Therefore, the configuration reached at $t = T^*/2$ will simply be

$$\boldsymbol{q}^* := \boldsymbol{q}\left(\frac{T^*}{2}\right) = \boldsymbol{q}_{in} + \frac{\Delta \boldsymbol{q}}{2} = \frac{\boldsymbol{q}_{in} + \boldsymbol{q}_{fin}}{2} = \begin{pmatrix} 106.25^\circ \\ 85.82^\circ \end{pmatrix} = \begin{pmatrix} 1.8544 \\ 1.4978 \end{pmatrix} \text{ [rad]}.$$

Moreover, it is

$$\dot{\boldsymbol{q}}^* := \dot{\boldsymbol{q}}\left(\frac{T^*}{2}\right) = \frac{30}{16}\frac{\Delta \boldsymbol{q}}{T^*} = \begin{pmatrix} 57.30 \\ 16.56 \end{pmatrix} \text{ [°/s]} = \begin{pmatrix} 1 \\ 0.2890 \end{pmatrix} \text{ [rad/s]}, \quad \ddot{\boldsymbol{q}}^* := \ddot{\boldsymbol{q}}\left(\frac{T^*}{2}\right) = \boldsymbol{0}.$$

The robot analytic Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})$ and its time derivative $\boldsymbol{H}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})$,

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 s_1 + \ell_2 s_{12}) & -\ell_2 s_{12} \\ \ell_1 c_1 + \ell_2 c_{12} & \ell_2 c_{12} \end{pmatrix}, \quad \boldsymbol{H}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\begin{pmatrix} \ell_1 c_1 \dot{q}_1 + \ell_2 c_{12}(\dot{q}_1 + \dot{q}_2) & \ell_2 c_{12}(\dot{q}_1 + \dot{q}_2) \\ \ell_1 s_1 \dot{q}_1 + \ell_2 s_{12}(\dot{q}_1 + \dot{q}_2) & \ell_2 s_{12}(\dot{q}_1 + \dot{q}_2) \end{pmatrix},$$

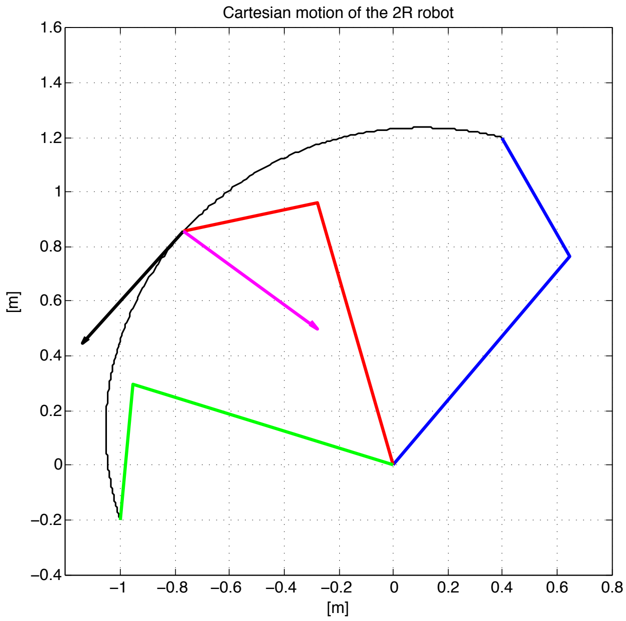

Figure 3: The planar 2R robot arm in its initial configuration (blue), at the midpoint of the trajectory (red), and in the final configuration (green), with the Cartesian path traced by its end-effector (thin blue line). The two arrows placed at the midpoint represent the end-effector velocity (black) and acceleration (magenta), respectively. Their length has been scaled by a factor 2 to fit.

take the numerical values at $\boldsymbol{q} = \boldsymbol{q}^*$, $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^*$

$$\boldsymbol{J}^* := \boldsymbol{J}(\boldsymbol{q}^*) = \begin{pmatrix} -0.8555 & 0.1045 \\ -0.7688 & -0.4890 \end{pmatrix}, \qquad \dot{\boldsymbol{J}}^* := \boldsymbol{H}(\boldsymbol{q}^*, \dot{\boldsymbol{q}}^*) = \begin{pmatrix} 0.9101 & 0.6303 \\ -0.8253 & 0.1347 \end{pmatrix}.$$

Therefore, the required Cartesian velocity and acceleration of the robot end-effector at the trajectory midpoint are

$$\dot{\boldsymbol{p}}^* = \boldsymbol{J}^* \dot{\boldsymbol{q}}^* = \begin{pmatrix} -0.8253 \\ -0.9101 \end{pmatrix} \text{ [m/s]} \qquad \ddot{\boldsymbol{p}}^* = \boldsymbol{J}^* \ddot{\boldsymbol{q}}^* + \dot{\boldsymbol{J}}^* \dot{\boldsymbol{q}}^* = \dot{\boldsymbol{J}}^* \dot{\boldsymbol{q}}^* = \begin{pmatrix} 1.0922 \\ -0.7864 \end{pmatrix} \text{ [m/s}^2\text{]}.$$