Robotics I

September 11, 2015

Exercise 1

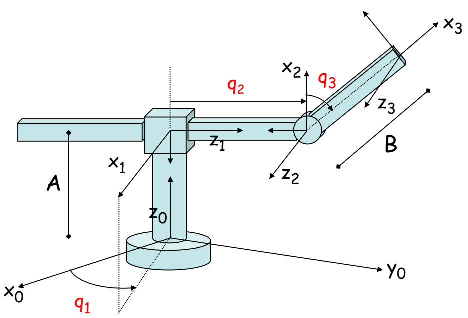
The kinematics of the spherical wrist of a 6R robot is described by the Denavit-Hartenberg parameters in Tab. 1.

i	α_i	a_i	d_i	θ_i
4	$\pi/2$	0	0	q_4
5	$\pi/2$	0	0	q_5
6	0	0	0	q_6

Table 1: Denavit-Hartenberg parameters for a spherical wrist

- Provide the differential mapping between the (wrist) joint velocity $\dot{\boldsymbol{q}}_W = \begin{pmatrix} \dot{q}_4 & \dot{q}_5 & \dot{q}_6 \end{pmatrix}^T$ and the angular velocity of the end-effector $\boldsymbol{\omega} = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}^T$ when the first three joints of the robot do not move. Vector $\boldsymbol{\omega}$ is expressed in the Denavit-Hartenberg frame 3 of the robot.
- In the wrist configuration $\boldsymbol{q}_W = \begin{pmatrix} q_4 & q_5 & q_6 \end{pmatrix}^T = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$ rad, determine a joint velocity vector $\dot{\boldsymbol{q}}_W$ that generates the desired angular velocity $\boldsymbol{\omega}_d = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}^T$ rad/s.

Exercise 2



i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	A	q_1
2	$-\pi/2$	0	q_2	$-\pi/2$
3	0	B	0	q_3

Figure 1: A spatial RPR robot with its Denavit-Hartenberg frames and associated table

For the spatial RPR robot in Fig. 1, the direct kinematics map for the position $\boldsymbol{p}$ of the end-effector (i.e., the origin of the last frame 3) is given by

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} -(q_2 - B\sin q_3)\sin q_1 \\ (q_2 - B\sin q_3)\cos q_1 \\ A + B\cos q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}). \tag{1}$$

Solve the inverse kinematics problem $\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p})$ in closed analytical form, assuming unlimited joint ranges. How many inverse kinematics solutions exist in the generic case?

(continues with Exercise 3)

Exercise 3

Consider a 3R planar robot having equal and unitary link lengths, with kinematics described in terms of the standard Denavit-Hartenberg variables. A task is specified at the differential level by a desired $\dot{\boldsymbol{r}} = \begin{pmatrix} v_x & v_y & \omega_z \end{pmatrix}^T$, namely in terms of linear velocity of the robot end-effector on the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ and of the (scalar) angular velocity of the end-effector frame around $\boldsymbol{z}_0$. Find all singular configurations of the mapping from $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to $\dot{\boldsymbol{r}} \in \mathbb{R}^3$. At a singularity, characterize the directions spanning the range space and the null space of the associated Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$.

[210 minutes; open books]

Solution

September 11, 2015

Exercise 1

From Table 1 of DH parameters, we get the following rotation matrices associated to the three joints of the spherical wrist:

$$
{}^3\boldsymbol{R}_4(q_4) = \begin{pmatrix} \cos q_4 & 0 & \sin q_4 \\ \sin q_4 & 0 & -\cos q_4 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^4\boldsymbol{R}_5(q_5) = \begin{pmatrix} \cos q_5 & 0 & \sin q_5 \\ \sin q_5 & 0 & -\cos q_5 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
{}^5\boldsymbol{R}_6(q_6) = \begin{pmatrix} \cos q_6 & -\sin q_4 & 0 \\ \sin q_4 & \cos q_4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

With these matrices, one can proceed in two alternative (and equivalent) ways.

The first way is to recognize that the requested mapping is given by part of the geometric Jacobian of the robot, namely the lower right 3×3 matrix in the orientation rows,

$$
\boldsymbol{\omega} = \boldsymbol{J}_O(\boldsymbol{q}_W)\,\dot{\boldsymbol{q}}_W = \begin{pmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{pmatrix} \begin{pmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix},
$$

where $\boldsymbol{z}_{i-1}$ is the unitary vector along joint i, and all vectors should be expressed here in the DH frame 3. We obtain:

$$
\boldsymbol{z}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{z}_4 = {}^3\boldsymbol{R}_4(q_4) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin q_4 \\ -\cos q_4 \\ 0 \end{pmatrix}, \quad \boldsymbol{z}_5 = {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos q_4 \sin q_5 \\ \sin q_4 \sin q_5 \\ -\cos q_5 \end{pmatrix}.
$$

The second way uses the time derivative of rotation matrices, and is a bit longer. The orientation of the end-effector (with respect to the DH frame 3) is given by

$$
\begin{aligned}
{}^3\boldsymbol{R}_6 &= {}^3\boldsymbol{R}_4(q_4)\,{}^4\boldsymbol{R}_5(q_5)\,{}^5\boldsymbol{R}_6(q_6) \\
&= \begin{pmatrix} \sin q_4 \sin q_6 + \cos q_4 \cos q_5 \cos q_6 & \cos q_6 \sin q_4 - \cos q_4 \cos q_5 \sin q_6 & \cos q_4 \sin q_5 \\ \cos q_5 \cos q_6 \sin q_4 - \cos q_4 \sin q_6 & -\cos q_4 \cos q_6 - \cos q_5 \sin q_4 \sin q_6 & \sin q_4 \sin q_5 \\ \cos q_6 \sin q_5 & -\sin q_5 \sin q_6 & -\cos q_5 \end{pmatrix}.
\end{aligned}
$$

Using the known differential relation $\dot{\boldsymbol{R}}\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{\omega})$ applied to $\boldsymbol{R} = {}^3\boldsymbol{R}_6$, one obtains the skew-symmetric matrix

$$
\begin{aligned}
\boldsymbol{S}(\boldsymbol{\omega}) &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ \omega_y & -\omega_z & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dot{q}_6 \cos q_5 - \dot{q}_4 & \dot{q}_6 \sin q_4 \sin q_5 - \dot{q}_5 \cos q_4 \\ \dot{q}_4 - \dot{q}_6 \cos q_5 & 0 & -\dot{q}_5 \sin q_4 - \dot{q}_6 \cos q_4 \sin q_5 \\ \dot{q}_5 \cos q_4 - \dot{q}_6 \sin q_4 \sin q_5 & \dot{q}_5 \sin q_4 + \dot{q}_6 \cos q_4 \sin q_5 & 0 \end{pmatrix},
\end{aligned}
$$

from which the angular velocity vector (still expressed in the DH frame 3, i.e., as ${}^3\boldsymbol{\omega}$) can be extracted:

$$
\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \sin q_4 & \cos q_4 \sin q_5 \\ 0 & -\cos q_4 & \sin q_4 \sin q_5 \\ 1 & 0 & -\cos q_5 \end{pmatrix} \begin{pmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix} = \boldsymbol{J}_O(q_4, q_5)\,\dot{\boldsymbol{q}}_W.
$$

The determinant of matrix $\boldsymbol{J}_O(q_4, q_5)$ is $\det \boldsymbol{J}_O = \sin q_5$. When $\boldsymbol{q}_W = \begin{pmatrix} 0 & \pi/2 & 0 \end{pmatrix}^T$, the wrist is not in a singular configuration. Therefore, we can determine the unique solution to the requested task as

$$\boldsymbol{J}_O(0, \pi/2)\, \dot{\boldsymbol{q}}_W = \boldsymbol{\omega}_d \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \dot{\boldsymbol{q}}_W = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_W = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Exercise 2

As a first step, we characterize and analyze the singular configurations of the robot. In fact, in a non-redundant situation, singular configurations correspond to Cartesian positions where the number of inverse kinematic solutions differs from the generic one. From the direct kinematics (1), we obtain the analytic Jacobian

$$\boldsymbol{J}_L(q) = \begin{pmatrix} -(q_2 - B \sin q_3) \cos q_1 & -\sin q_1 & B \cos q_3 \sin q_1 \\ -(q_2 - B \sin q_3) \cos q_1 & \cos q_1 & -B \cos q_3 \cos q_1 \\ 0 & 0 & -B \sin q_3 \end{pmatrix},$$

whose determinant is

$$\det \boldsymbol{J}_L(q) = B \sin q_3 (q_2 - B \sin_3).$$

Therefore, the robot is in a singularity either when $\sin q_3 = 0$ ($q_3 = \{0, \pi\}$), or when $q_2 - B \sin q_3 = 0$, or when these two conditions are simultaneously satisfied, i.e., when both $\sin q_3 = 0$ and $q_2 = 0$ hold. In the first case ($\sin q_3 = 0$), the last link is vertical (upwards or downwards) and the robot end-effector is on one of the two (top or bottom) horizontal planes defining the boundaries of its (otherwise unlimited) workspace. In correspondence to these boundary points, we shall see that there is is a drop in the number of inverse kinematic solutions (from four to two). In the second case ($q_2 - B \sin q_3 = 0$), the end-effector is placed on the axis of the first joint. For these Cartesian points, it is apparent that any change of q_1 only, will not change the position of the robot end-effector. As a consequence, any value of q_1 can be part of a solution to the inverse kinematics (the number of solutions becomes infinite). In the combined case, the rank of the Jacobian matrix $\boldsymbol{J}_L$ drops down to 1 and a double singularity is obtained. For all other Cartesian positions of the robot end-effector within the primary workspace, we are in the generic case with a constant, finite number of inverse kinematic solutions (namely, four).

With the above in mind, consider the last equation in the direct kinematics (1). If $A - B < p_z < A + B$, we have two solutions for q_3:

$$q_3^{[+]} = \text{acos}\left(\frac{p_z - A}{B}\right), \quad q_3^{[-]} = -q_3^{[+]}, \qquad \text{with } \left\{q_3^{[+]}, q_3^{[-]}\right\} \in (-\pi, +\pi). \tag{2}$$

Equivalently, we could have used the ATAN2 function as follows:

$$c_3 = \frac{p_z - A}{B}, \quad s_3 = \pm\sqrt{1 - c_3^2} \quad \Rightarrow \quad q_3^{[+/-]} = \text{ATAN2}\left\{\pm\, s_3, c_3\right\}.$$

For $p_z > A$ the two solutions $q_3^{[+]}$ and $q_3^{[-]}$ will both be in the interval $(-\pi/2, \pi/2)$, whereas for $p_z < A$ their absolute values will be in the interval $(\pi/2, \pi)$. When $p_z = A$, it is $q_3^{[+]} = +\pi/2$ and $q_3^{[-]} = -\pi/2$ (the third link is horizontal in both cases). For $p_z = A + B$, the two values collapse into $q_3 = 0$ (the third link points upward); similarly, for $p_z = A - B$, there is a single solution $q_3 = \pi$ (the third link points downward). These two cases correspond to singularities at the two boundaries of the workspace. Outside the above closed interval, i.e., when $p_z \notin [A - B, A + B]$, there is no solution for q_3 and thus to the inverse kinematics problem: the requested height of the end-effector is outside the workspace.

Squaring and summing the first two equations in (1), we obtain also

$$p_x^2 + p_y^2 = (q_2 - B s_3)^2. \tag{3}$$

When $p_x^2 + p_y^2 \neq 0$ (again, out of the singularity associated to points along the first joint axis), we can solve for q_1 from the first two equations in (1) and find again two solutions. The first solution

$$q_1^{[+]} = \text{ATAN2}\left\{-p_x, p_y\right\} \tag{4}$$

has the robot facing the desired point (p_x, p_y), while the second solution[1]

$$q_1^{[-]} = \text{ATAN2}\{p_x, -p_y\} \quad (= q_1^{[+]} \pm \pi) \tag{5}$$

has the robot back directed toward the desired point (p_x, p_y). For $p_x^2 + p_y^2 = 0$, the angle q_1 remains undefined (singular case).

Finally, the first two equations in (1) can be combined as follows

$$-\sin q_1 p_x + \cos q_1 p_y = q_2 - B \sin q_3 = 0 \qquad \Rightarrow \qquad q_2 = \cos q_1 p_y - \sin q_1 p_x + B \sin q_3.$$

Taking into account all four combinations of solutions for q_1 and q_3 as given by eqs. (2), (4), and (5), we obtain the four associated solutions for q_2 as

$$\begin{aligned}
q_2^{[++]} &= \cos q_1^{[+]} p_y - \sin q_1^{[+]} p_x + B \sin q_3^{[+]} \\
q_2^{[+-]} &= \cos q_1^{[+]} p_y - \sin q_1^{[+]} p_x + B \sin q_3^{[-]} \\
q_2^{[-+]} &= \cos q_1^{[-]} p_y - \sin q_1^{[-]} p_x + B \sin q_3^{[+]} \\
q_2^{[--]} &= \cos q_1^{[-]} p_y - \sin q_1^{[-]} p_x + B \sin q_3^{[-]},
\end{aligned} \tag{6}$$

with an obvious choice for the notation of sign labels. Note that these solutions can also be rewritten as

$$\begin{aligned}
q_2^{[++]} &= \sqrt{p_x^2 + p_y^2} + B \sin q_3^{[+]} \\
q_2^{[+-]} &= \sqrt{p_x^2 + p_y^2} + B \sin q_3^{[-]} \\
q_2^{[-+]} &= -\sqrt{p_x^2 + p_y^2} + B \sin q_3^{[+]} \\
q_2^{[--]} &= -\sqrt{p_x^2 + p_y^2} + B \sin q_3^{[-]}.
\end{aligned}$$

Therefore, in the generic case (i.e., out of singularities and inside the workspace) there is a total of four distinct inverse kinematics solutions:

$$\left\{q_1^{[+]}, q_2^{[++]}, q_3^{[+]}\right\}, \qquad \left\{q_1^{[+]}, q_2^{[+-]}, q_3^{[-]}\right\}, \qquad \left\{q_1^{[-]}, q_2^{[-+]}, q_3^{[+]}\right\}, \qquad \left\{q_1^{[-]}, q_2^{[--]}, q_3^{[-]}\right\}.$$

For example, assume that $A = B = 1$ [m]. We obtain the following joint solutions for two specific desired Cartesian positions $\boldsymbol{p}_d$

$$\boldsymbol{p}_d = \begin{pmatrix} 1.5 \\ 0 \\ 1 + \frac{\sqrt{3}}{2} \end{pmatrix} \text{[m]} \quad \Rightarrow \quad \left\{ \begin{pmatrix} -90° \\ 2 \\ 30° \end{pmatrix}, \begin{pmatrix} -90° \\ 1 \\ -30° \end{pmatrix}, \begin{pmatrix} 90° \\ -1 \\ 30° \end{pmatrix}, \begin{pmatrix} 90° \\ -2 \\ -30° \end{pmatrix} \right\}$$

$$\boldsymbol{p}_d = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \text{[m]} \quad \Rightarrow \quad \left\{ \begin{pmatrix} 180° \\ 3 \\ 90° \end{pmatrix}, \begin{pmatrix} 180° \\ 1 \\ -90° \end{pmatrix}, \begin{pmatrix} 0° \\ -1 \\ 90° \end{pmatrix}, \begin{pmatrix} 0° \\ -3 \\ -90° \end{pmatrix} \right\},$$

where angles q_1 and q_3 have been expressed in degrees, while the translation variable q_2 is in meters.

Exercise 3

Using the standard DH variables and shorthand notations for trigonometric quantities, the task kinematics is given by

$$\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \\ q_1 + q_2 + q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \tag{7}$$

[1] The choice of signs in the additional expression in (5) is made as follows: if $q_1^{[+]}$ is in the first or second quadrant, i.e., $q_1^{[+]} \in (0, \pi)$, then $q_1^{[-]} = q_1^{[+]} - \pi$; if $q_1^{[+]}$ is in the third or fourth quadrant, $q_1^{[+]} \in (-\pi, 0)$, then $q_1^{[-]} = q_1^{[+]} + \pi$.

and thus

$$\dot{\boldsymbol{r}} = \begin{pmatrix} v_x \\ v_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Note that we can factorize the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and redefine the joint velocity as follows

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} \\ c_1 & c_{12} & c_{123} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}_a(\boldsymbol{q})\,\boldsymbol{T}, \qquad \dot{\boldsymbol{q}}_a = \boldsymbol{T}\,\dot{\boldsymbol{q}}.$$

Indeed, $\dot{\boldsymbol{r}} = \boldsymbol{J}_a\dot{\boldsymbol{q}}_a$ and $\boldsymbol{q}_a$ is the vector of *absolute* joint angles w.r.t. the $\boldsymbol{x}_0$ axis of the world frame (the components of $\boldsymbol{q}_a$ appear also as arguments of the trigonometric functions in matrix $\boldsymbol{J}_a$).

It is easy to recognize that all task singularities occur when

$$\det \boldsymbol{J}\, (= \det \boldsymbol{J}_a) = \sin q_2 = 0 \qquad \Leftrightarrow \qquad q_2 = \{0, \pi\}.$$

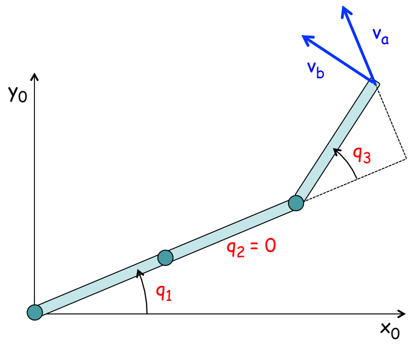


Figure 2: The planar 3R robot in a singular configuration for the three-dimensional task $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$ specified in (7), with two range space vectors defined in the text.

Consider for example the case $q_2 = 0$. Then,

$$\bar{\boldsymbol{J}}_a = \boldsymbol{J}_a(\boldsymbol{q})|_{q_2=0} = \begin{pmatrix} -s_1 & -s_1 & -s_{13} \\ c_1 & c_1 & c_{13} \\ 0 & 0 & 1 \end{pmatrix}.$$

In this singularity, the range of instantaneous motions covered in the task space is characterized by

$$\mathcal{R}(\bar{\boldsymbol{J}}_a) = \text{span}\left\{ \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -s_{13} \\ c_{13} \\ 1 \end{pmatrix} \right\}.$$

With reference to Fig. 2, the first basis vector in the task space is associated to a pure linear motion of the end-effector position in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ —see the unit vector $\boldsymbol{v}_a = \begin{pmatrix} -s_1 & c_1 \end{pmatrix}^T$. The second basis vector implies always a combined roto-translation, with the linear part given by the unitary vector $\boldsymbol{v}_b = \begin{pmatrix} -s_{13} & c_{13} \end{pmatrix}^T$ (projection of the second basis vector on the $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ plane) and with the angular part of unitary value as well.

As for the null space motions in the considered singularity, we have

$$\mathcal{N}(\bar{\boldsymbol{J}}_a) = \alpha\,\dot{q}_{a,N} = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

and thus

$$\mathcal{N}(\bar{\boldsymbol{J}}) = \alpha\, \boldsymbol{T}^{-1} \dot{q}_{a,N} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

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