Robotics I

September 10, 2012

A 3R robot manipulator has the following Denavit-Hartenberg table:

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	$a_1 > 0$	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

Table 1: DH table of a 3R robot

1. Sketch the kinematic structure of the robot and place the D-H frames according to Table 1.
2. Draw the robot in the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$ [rad].

Assume now the numerical data $a_1 = 0.2$, $a_2 = 0.5$, and $a_3 = 0.5$ [m] and let the robot be in the configuration specified at step 2.

3. Given a desired velocity $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 0.5 \end{pmatrix}^T$ [m/s] for the robot end-effector (the origin O_3 of frame 3), determine the instantaneous joint velocity vector $\dot{\boldsymbol{\theta}}$ that realizes $\boldsymbol{v}$.
4. With the solution $\dot{\boldsymbol{\theta}}$ found at step 3, compute the associated angular velocity $\boldsymbol{\omega}$ of the robot end-effector frame.
5. Let the value $\boldsymbol{\omega}$ found at step 4 be the desired angular velocity for the robot end-effector frame. Characterize *all* instantaneous joint velocities $\dot{\boldsymbol{\theta}}$ that realize $\boldsymbol{\omega}$ at the given robot configuration.
6. What is the structure of all feasible $\boldsymbol{\omega}$ that can be realized by this robot in a *generic* configuration $\boldsymbol{\theta}$? What can we say about the differential mapping $\dot{\boldsymbol{\theta}} \rightarrow \boldsymbol{\omega}$?

[120 minutes; open books]

Solution

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The robot has a kinematic structure similar to that of the first three joints of the KUKA KR5 robot (the industrial robot in our Robotics Laboratory). Figures 1 and 2 provide, respectively, a sketch of the kinematic structure, with associated D-H frames, and the robot posture at the specified $\boldsymbol{\theta}$.

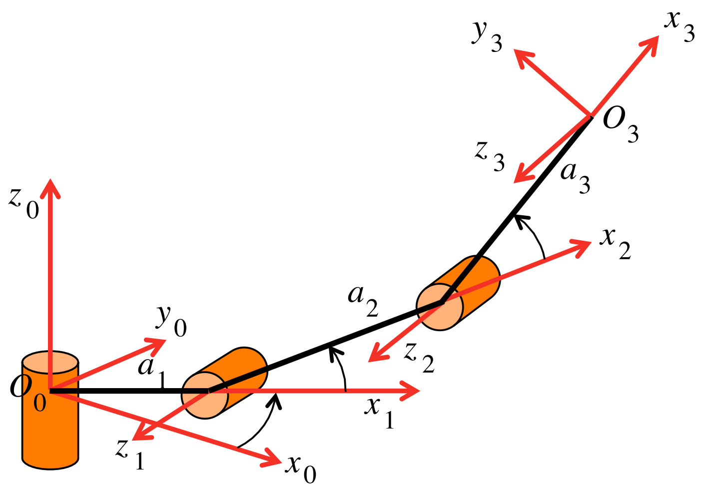

Figure 1: Kinematic structure and D-H frames

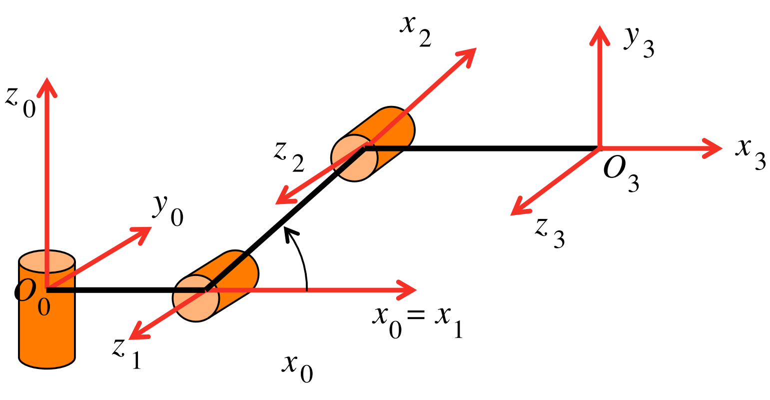

Figure 2: The robot at the configuration $\boldsymbol{\theta} = (0 \quad \pi/4 \quad -\pi/4)^T$

For steps 3-6, we need to compute the robot Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$. For the linear part, $\boldsymbol{J}_L(\boldsymbol{\theta})$, we may use either the vector product computations of the geometric Jacobian or simply differentiate analytically the positional direct kinematics. From the product of the homogeneous matrices associated to the D-H table 1, it follows

$$\boldsymbol{p}_{hom} = \begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(\theta_1)\,{}^1\boldsymbol{A}_2(\theta_2)\,{}^2\boldsymbol{A}_3(\theta_3) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix} = \begin{pmatrix} (a_1 + a_2c_2 + a_3c_{23})\,c_1 \\ (a_1 + a_2c_2 + a_3c_{23})\,s_1 \\ a_2s_2 + a_3s_{23} \\ 1 \end{pmatrix}.$$

Therefore,

$$\boldsymbol{v} = \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}\,\dot{\boldsymbol{\theta}} = \boldsymbol{J}_L(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}}, \text{ with } \boldsymbol{J}_L(\boldsymbol{\theta}) = \begin{pmatrix} -(a_1 + a_2c_2 + a_3c_{23})\,s_1 & -(a_2s_2 + a_3s_{23})\,c_1 & -a_3s_{23}c_1 \\ (a_1 + a_2c_2 + a_3c_{23})\,c_1 & -(a_2s_2 + a_3s_{23})\,s_1 & -a_3s_{23}s_1 \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{pmatrix}.$$

For the angular part, $\boldsymbol{J}_A(\boldsymbol{\theta})$, we have by definition (taking into account that velocity vectors are expressed by default in the 0th frame)

$$\boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{z}_1 & {}^0\boldsymbol{z}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{z}_0 & {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{z}_1 & {}^0\boldsymbol{R}_1(\theta_1)\,{}^1\boldsymbol{R}_2(\theta_2)\,{}^2\boldsymbol{z}_2 \end{pmatrix},$$

with ${}^i\boldsymbol{z}_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, for $i = 0, 1, 2$. As a result,

$$\boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}}, \quad \text{with } \boldsymbol{J}_A(\boldsymbol{\theta}) = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Evaluating the two Jacobians at the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$ with the given numerical data yields

$$\boldsymbol{J}_L = \begin{pmatrix} 0 & -0.3536 & 0 \\ 1.0536 & 0 & 0 \\ 0 & 0.8536 & 0.5 \end{pmatrix}, \qquad \boldsymbol{J}_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2}$$

Therefore, for $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 0.5 \end{pmatrix}^T$,

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}_L^{-1}\boldsymbol{v} = \begin{pmatrix} 0.9492 \\ -2.8284 \\ 5.8284 \end{pmatrix} \text{[rad/s]} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A\dot{\boldsymbol{\theta}} = \begin{pmatrix} 0 \\ -3 \\ 0.9492 \end{pmatrix} \text{[rad/s]}. \tag{3}$$

From the general structure of $\boldsymbol{J}_A(\boldsymbol{\theta})$ in (1) we see that this matrix is always singular, having constant rank equal to 2. At a generic configuration (i.e., for a generic value of θ_1), we characterize the following subspaces of interest:

$$\mathcal{R}(\boldsymbol{J}_A(\boldsymbol{\theta})) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}(\boldsymbol{J}_A(\boldsymbol{\theta})) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Therefore, all feasible $\boldsymbol{\omega}$ will have the form

$$\boldsymbol{\omega} \in \mathcal{R}(\boldsymbol{J}_A(\boldsymbol{\theta})) \qquad \Rightarrow \qquad \boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} s_1 \\ -c_1 \\ 0 \end{pmatrix} \beta$$

with $\alpha = \dot{\theta}_1 \in \mathbb{R}$ and $\beta = \dot{\theta}_1 + \dot{\theta}_2 \in \mathbb{R}$. Conversely, given a generic $\dot{\boldsymbol{\theta}}$ generating a $\boldsymbol{\omega}$, the same value of end-effector angular velocity is obtained by adding a joint velocity vector $\dot{\boldsymbol{\theta}}_0 \in \mathcal{N}(\boldsymbol{J}_A(\boldsymbol{\theta}))$, or

$$\dot{\boldsymbol{\theta}} + \gamma\,\dot{\boldsymbol{\theta}}_0 = \dot{\boldsymbol{\theta}} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} = \boldsymbol{J}_A(\boldsymbol{\theta})\left(\dot{\boldsymbol{\theta}} + \gamma\,\dot{\boldsymbol{\theta}}_0\right).$$

for any $\gamma \in \mathbb{R}$.

Particularizing this general result to the specific configuration $\boldsymbol{\theta} = \begin{pmatrix} 0 & \pi/4 & -\pi/4 \end{pmatrix}^T$, with $\boldsymbol{J}_A$ given in (2), all joint velocities that generate the same value $\boldsymbol{\omega}$ as in (3) are given by

$$\dot{\boldsymbol{\theta}}_\gamma = \begin{pmatrix} 0.9492 \\ -2.8284 \\ 5.8284 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \text{for any } \gamma \in \mathbb{R} \qquad \Rightarrow \qquad \boldsymbol{\omega} = \boldsymbol{J}_A \dot{\boldsymbol{\theta}}_\gamma = \begin{pmatrix} 0 \\ -3 \\ 0.9492 \end{pmatrix}.$$

Note that the minimum norm joint velocity $\dot{\boldsymbol{\theta}}^*$ realizing this value of $\boldsymbol{\omega}$ is obtained by unconstrained minimization of $\|\dot{\boldsymbol{\theta}}_\gamma\|^2$ with respect to γ. This yields

$$\gamma = -\frac{\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}_0}{\dot{\boldsymbol{\theta}}_0^T \dot{\boldsymbol{\theta}}_0} = 4.3284 \qquad \Rightarrow \qquad \dot{\boldsymbol{\theta}}^* = \begin{pmatrix} 0.9492 \\ 1.5 \\ 1.5 \end{pmatrix},$$

with $\|\dot{\boldsymbol{\theta}}^*\| = 2.3240$ —as opposed to $\|\dot{\boldsymbol{\theta}}\| = 6.5476$ for the value $\dot{\boldsymbol{\theta}}$ computed in (3). As could be expected, the minimum norm solution balances the effort between the velocities of joints 2 and 3.

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