# Robotics I

**January 11, 2012**

## Exercise 1

Consider a *planar* 3R robot having link lengths $\ell_1 = 1$, $\ell_2 = 0.5$, and $\ell_3 = 0.25$ [m].

- Draw the primary workspace in the plane $(x_0, y_0)$.
- Draw the secondary workspace, i.e., the set of all points that can be reached by the end-effector with any admissible approach angle in the plane $(x_0, y_0)$.
- Following the Denavit-Hartenberg convention for the definition of the joint variables $\boldsymbol{q}$, find all singular configurations of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relating the joint velocity vector $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ to the end-effector velocity vector $\boldsymbol{v} = \begin{pmatrix} v_x & v_y \end{pmatrix}^T \in \mathbb{R}^2$.
- Give the explicit expression of the manipulability index $H = \sqrt{\det(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}))}$ in a form that shows its independence from the variable $q_1$.

## Exercise 2

For the same robot of Exercise 1, consider a *rest-to-rest* motion task for the end-effector from point $\boldsymbol{p}_A = \begin{pmatrix} 1.75 & 0 \end{pmatrix}^T$ [m], with orientation (with respect to the axis $x_0$) $\phi_A = 0$, to point $\boldsymbol{p}_B = \begin{pmatrix} 1.25 & 0.5 \end{pmatrix}^T$ [m], with orientation $\phi_B = 0$.

- For this motion task, define a minimum time trajectory under the joint velocity and acceleration bounds
$$|\dot{q}_i| \leq V_i, \qquad |\ddot{q}_i| \leq A_i, \qquad i = 1, 2, 3,$$
with the following numerical data:
$$V_1 = 90°/\text{s}, \qquad V_2 = 120°/\text{s}, \qquad V_3 = 60°/\text{s},$$
$$A_1 = 150°/\text{s}^2, \qquad A_2 = 160°/\text{s}^2, \qquad A_3 = 240°/\text{s}^2.$$
Provide also the minimum motion time $T$.
- Sketch the velocity and acceleration profiles of each joint for the minimum time trajectory that has been found.
- Is the obtained minimum time trajectory unique for this problem? Motivate the answer.

**[150 minutes; open books]**

## Bonus Exercise (30 additional minutes available)

Write a Matlab program that plots in 3D the manipulability index $H$ of the considered planar 3R robot as a function of $q_2$ and $q_3$. Using such a program, determine (at least approximately) a configuration that globally maximizes this index and provide the associated value of $H$.

# Solutions

**January 11, 2012**

### Exercise 1

The primary workspace is the set of positions that can be reached by the robot end-effector, independently from its orientation. For a planar 3R robot, the primary workspace is an annulus (*it: corona circolare*). The outer radius is simply $R_1 = \ell_1 + \ell_2 + \ell_3$. For the inner radius, the following general formula holds

$$r_1 = \max\left\{|\ell_1 - \ell_2| - \ell_3, |\ell_2 - \ell_3| - \ell_1, |\ell_1 - \ell_3| - \ell_2, 0\right\},$$

which takes into account all possible cases of (non-negative) lengths for the three links. Note that the inner 'hole' would vanish in particular (but not only) for equal $\ell_1 = \ell_2 = \ell_3$. With the given numerical data, it is

$$r_1 = 0.25\,[\mathrm{m}], \qquad R_1 = 1.75\,[\mathrm{m}].$$

Figure 1 shows in gray the primary workspace and the robot arm in a stretched (on the right) and in a folded configuration (on the left), in which the end-effector reaches respectively the outer and inner boundaries of the primary workspace.

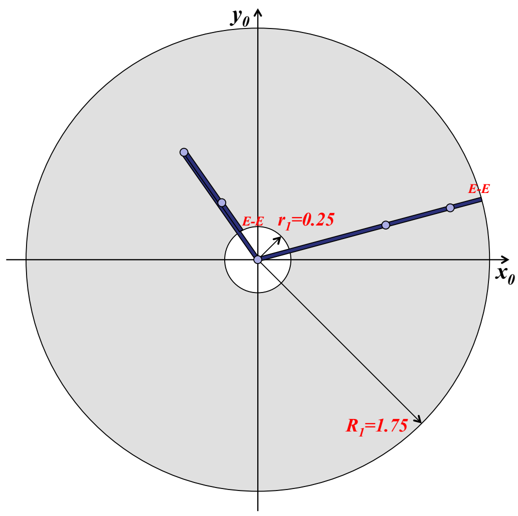


Figure 1: Primary workspace of the planar 3R robot

The secondary workspace is the set of positions that can be reached by the robot end-effector with any of its admissible orientations. For a planar robot, all admissible orientations are defined as rotations by an angle $\phi$ around an axis orthogonal to the plane of motion. The angle $\phi$ can take any value in $(-\pi, \pi]$. The secondary workspace for the considered planar 3R robot is again an annulus with inner and outer radii given by

$$r_2 = \ell_1 - \ell_2 + \ell_3 = 0.75\,[\mathrm{m}], \qquad R_2 = \ell_1 + \ell_2 - \ell_3 = 1.25\,[\mathrm{m}],$$

as shown by the dark gray area in Fig. 2. The geometric construction of this secondary workspace is rather straightforward. Take any point $A$ in the primary workspace (by the central symmetry with respect to $q_1$, it is sufficient to consider only points lying on a ray starting from the origin) and draw a circle of radius $\ell_3$ centered in $A$. In order for the third link to assume any orientation in the plane when the end-effector is placed in $A$, all points on this circle should be reachable by the tip of the sub-arm made by the first two links. Therefore, we are interested in the (primary) workspace of this auxiliary two-link robot (with link lengths $\ell_1$ and $\ell_2$). This auxiliary workspace is an annulus of inner and outer radii $r_a = |\ell_1 - \ell_2| = 0.5$ [m] and $R_a = \ell_1 + \ell_2 = 1.5$ [m], as represented by the dashed circles in Fig. 2. The two boundaries of the secondary workspace for the 3R robot are obtained when folding the third link (of length $\ell_3 = 0.25$ [m]) by $\pm\pi$ while the tip of the two-link sub-arm is on the boundary of the auxiliary workspace, as exemplified by the two robot configurations reported in Fig. 2.

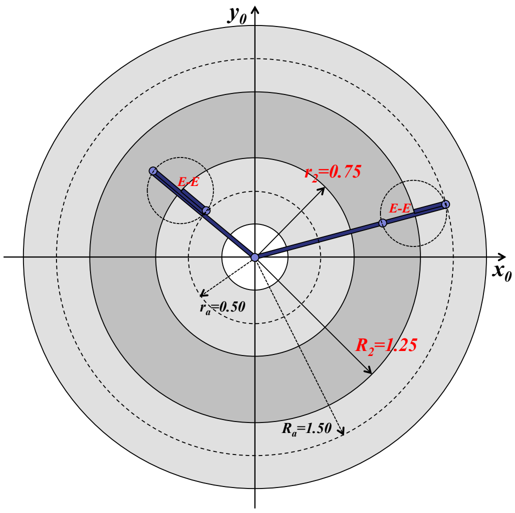


Figure 2: Secondary workspace of the planar 3R robot

Let $\boldsymbol{q} = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \end{pmatrix}^T$ be the joint configuration vector of this robot, with $\theta_i$ $(i = 1, 2, 3)$ defined according to the Denavit-Hartenberg convention. The direct kinematics is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} \\ \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} \end{pmatrix},$$

with the usual shorthand notations (e.g., $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$). The Jacobian $\boldsymbol{J}(\boldsymbol{q})$ of interest in the velocity relation

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

is the $(2 \times 3)$ matrix obtained by differentiation of the direct kinematics:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123}) & -(\ell_2 s_{12} + \ell_3 s_{123}) & -\ell_3 s_{123} \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_3 c_{123} & \ell_2 c_{12} + \ell_3 c_{123} & \ell_3 c_{123} \end{pmatrix} = \begin{pmatrix} j_{11}(\boldsymbol{q}) & j_{12}(\boldsymbol{q}) & j_{13}(\boldsymbol{q}) \\ j_{21}(\boldsymbol{q}) & j_{22}(\boldsymbol{q}) & j_{23}(\boldsymbol{q}) \end{pmatrix}.$$

The singularities of the Jacobian correspond to the zeroing of the manipulability index $H$. In order to obtain this index, we compute

$$\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}) = \begin{pmatrix} j_{11}^2(\boldsymbol{q}) + j_{12}^2(\boldsymbol{q}) + j_{13}^2(\boldsymbol{q}) & j_{11}(\boldsymbol{q})j_{21}(\boldsymbol{q}) + j_{12}(\boldsymbol{q})j_{22}(\boldsymbol{q}) + j_{13}(\boldsymbol{q})j_{23}(\boldsymbol{q}) \\ j_{11}(\boldsymbol{q})j_{21}(\boldsymbol{q}) + j_{12}(\boldsymbol{q})j_{22}(\boldsymbol{q}) + j_{13}(\boldsymbol{q})j_{23}(\boldsymbol{q}) & j_{21}^2(\boldsymbol{q}) + j_{22}^2(\boldsymbol{q}) + j_{23}^2(\boldsymbol{q}) \end{pmatrix}$$

and from this, after some trigonometric simplifications[1],

$$H = \sqrt{\det(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}))} = \sqrt{(\ell_1(\ell_2 s_2 + \ell_3 s_{23}))^2 + (\ell_3(\ell_1 s_{23} + \ell_2 s_3))^2 + (\ell_2 \ell_3 s_3)^2} \geq 0, \qquad (1)$$

which is only a function of $q_2$ and $q_3$, as expected (and required). A singularity occurs when all three addends under the square root are simultaneously zero[2]. This happens if and only if (proceeding from the last addend to the first)

$$s_3 = 0 \quad \text{AND} \quad s_{23} = 0 \quad \text{AND} \quad s_2 = 0.$$

As a result, four types of singularities are present (at any $q_1$) for the pair $\{q_2, q_3\}$: $\{0, 0\}$, $\{0, \pm\pi\}$, $\{\pm\pi, 0\}$, and $\{\pm\pi, \pm\pi\}$. The robot is in a singular configuration whenever the links are stretched or folded along a ray starting from the origin. Note that the singularities remain the same for all (positive) values of the link lengths $\ell_i$, $i = 1, 2, 3$.

**Exercise 2**

The desired motion task can be formulated using the Cartesian vector $\boldsymbol{r} = \begin{pmatrix} p_x & p_y & \phi \end{pmatrix}^T \in \mathbb{R}^3$. However, the problem is best addressed in the joint space, since the bounds are directly defined in this space and there are no requirements on the Cartesian path to be followed by the end-effector during motion. Therefore, we will plan a joint trajectory such that the robot end-effector moves from the initial Cartesian task vector at time $t = 0$,

$$\boldsymbol{r}_A = \boldsymbol{r}(0) = \begin{pmatrix} \boldsymbol{p}_A \\ \phi_A \end{pmatrix} = \begin{pmatrix} 1.75 \\ 0 \\ 0 \end{pmatrix},$$

to the final Cartesian task vector at time $t = T$,

$$\boldsymbol{r}_B = \boldsymbol{r}(T) = \begin{pmatrix} \boldsymbol{p}_B \\ \phi_B \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 0 \end{pmatrix}.$$

We use inverse kinematics to associate initial and final joint configurations to $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$. Note that the planar 3R robot is *not* redundant for a planar position and orientation task ($m = n = 3$). With the given link lengths, there is a unique joint configuration $\boldsymbol{q}_A$ associated to $\boldsymbol{r}_A$,

$$\boldsymbol{q}_A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{q}(0),$$

[1] One can use the *Symbolic Toolbox* of Matlab to obtain this compact expression. However, simplifications by hand are easier in this case in view of the specific recurrent structure of the terms in $\boldsymbol{J}$. A convenient alternative way is to express the Jacobian in the reference frame attached to the first (or even second!) link. Let

$${}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}$$

be the planar rotation matrix characterizing the orientation of the first frame with respect to the zero frame. It is easy to verify that ${}^0\boldsymbol{R}_1^T(q_1)\boldsymbol{J}(\boldsymbol{q}) = {}^1\boldsymbol{J}(q_2, q_3)$ is independent from $q_1$. Moreover, this matrix can be used as well for the computation of $H$ since

$$H = \sqrt{\det\left(\boldsymbol{J}\boldsymbol{J}^T\right)} = \sqrt{\det {}^0\boldsymbol{R}_1^T} \cdot \sqrt{\det\left(\boldsymbol{J}\boldsymbol{J}^T\right)} \cdot \sqrt{\det {}^0\boldsymbol{R}_1} = \sqrt{\det\left({}^0\boldsymbol{R}_1^T \boldsymbol{J}\right)\left(\boldsymbol{J}^T\, {}^0\boldsymbol{R}_1\right)} = \sqrt{\det\left({}^1\boldsymbol{J}\, {}^1\boldsymbol{J}^T\right)}.$$

[2] The three terms are the square of the minors $(1, 2)$, $(1, 3)$, and $(2, 3)$ of matrix $\boldsymbol{J}$. In fact, the singularities can be found equivalently by imposing the simultaneous zeroing of these three minors.

i.e., at $t = 0$ the arm is fully stretched along the $x_0$ axis. On the other hand, there are two inverse kinematic solutions associated to $\boldsymbol{r}_B$, namely[3]

$$\boldsymbol{q}'_B = \begin{pmatrix} 0 \\ 90° \\ -90° \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{q}''_B = \begin{pmatrix} 2\arctan 0.5 \\ -\dfrac{\pi}{2} \\ -\left(2\arctan 0.5 - \dfrac{\pi}{2}\right) \end{pmatrix} \text{[rad]} = \begin{pmatrix} 53.13° \\ -90° \\ 36.87° \end{pmatrix},$$

which can be chosen in alternative as the final desired $\boldsymbol{q}(T)$. Therefore, we should plan two minimum time trajectories, one from $\boldsymbol{q}_A$ to $\boldsymbol{q}'_B$ and another from $\boldsymbol{q}_A$ to $\boldsymbol{q}''_B$, and then choose the fastest one.

The motion task is rest-to-rest, and thus we should have $\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \boldsymbol{0}$. Since joint coordination is not required (the joints can complete their displacements at different time instants), the solution is obtained by planning the fastest possible motion independently for each joint. In view of the presence of joint velocity and acceleration bounds, the optimal solution will have a bang-coast-bang acceleration profile (or it is a sub-case of this) for all joints.

For a generic joint $i$, let $T_i$ be the minimum motion time, $\Delta_i = |q_i(T_i) - q_i(0)|$ the required displacement (in absolute value), and $T_{s,i}$ the duration of the acceleration (and, symmetrically, of the deceleration) phase. We have the known relations

$$T_{s.i} = \frac{V_i}{A_i}, \qquad T_i = \frac{\Delta_i}{V_i} + \frac{V_i}{A_i} = \frac{\Delta_i A_i + V_i^2}{A_i V_i}, \tag{2}$$

which are valid provided that

$$\Delta_i \geq \frac{V_i^2}{A_i} \qquad \Leftrightarrow \qquad T_I \geq 2T_{s,i}. \tag{3}$$

When strict inequalities hold in (3), we denote this as *case a* (the standard one). When the relations (3) hold as equalities (*case b*), the coast phase collapses and the acceleration profile will be a bang-bang one: the absolute velocity reaches its maximum value $V_i$ only at the midpoint of the trajectory. When the relations (3) are violated (*case c*), a specific motion profile with symmetric phases at maximum acceleration/deceleration and no coast phase can be used: a peak (absolute) velocity $\bar{V}_i$ will be reached only at the midpoint of the trajectory, with $\bar{V}_i < V_i$. The value of $\bar{V}_i$ is found from

$$\Delta_i = \frac{\bar{V}_i^2}{A_i} \qquad \Rightarrow \qquad \bar{V}_i = \sqrt{\Delta_i A_i},$$

which expresses the fact that the area of the triangular velocity profile should be equal to the required displacement $\Delta_i$. In this situation, the formulas (2) should be replaced by

$$T_{s.i} = \frac{\bar{V}_i}{A_i} = \sqrt{\frac{\Delta_i}{A_i}}, \qquad T_i = 2T_{s,i} = 2\sqrt{\frac{\Delta_i}{A_i}}. \tag{4}$$

In particular, the expressions (4) hold also for the limit situation of a joint requiring no displacement ($\Delta_i = 0$): its minimum motion time $T_i$ will be equal to 0 (*case d*). No matter which case occurs, the minimum time $T$ for completing the desired robot motion task will always be the largest of the computed $T_i$'s, or

$$T = \max\{T_1, T_2, T_3\}.$$

[3] These are obtained by applying the formulas of Sect. 2.12.1 of the textbook. In this case, however, both solutions can be found by simple geometric inspection, noting also that it should be $\phi_B = q_{1,B} + q_{2,B} + q_{3,B} = 0$.

For the present problem, the above computations are repeated for the two possible robot displacements in the joint space, leading to $T'$ and $T''$.

The above developments are applied by taking into account the numerical values of $V_i$ and $A_i$ $(i = 1, 2, 3)$, and the absolute displacements $\Delta_i$ $(i = 1, 2, 3)$ for the two different joint trajectories. For the joint trajectory from $\boldsymbol{q}_A$ to $\boldsymbol{q}'_B$, it s

$$T' = \max\{0, 1.5, 1.75\} = 1.75 \,[\text{s}].$$

Note that *case d* applies to the motion of joint 1, *case b* to the motion of joint 2, and *case a* to the motion of joint 3. For the joint trajectory from $\boldsymbol{q}_A$ to $\boldsymbol{q}''_B$, it is

$$T'' = \max\{1.1903, 1.5, 0.8645\} = 1.5 \,[\text{s}].$$

Here, *case c* applies to the motion of joint 1, *case b* to the motion of joint 2, and *case a* to the motion of joint 3. Therefore, the minimum time trajectory satisfying the task goes from $\boldsymbol{q}_A$ (stretched arm) to $\boldsymbol{q}''_B$ and the associated minimum time is $T = \min\{T', T''\} = 1.5$ s.

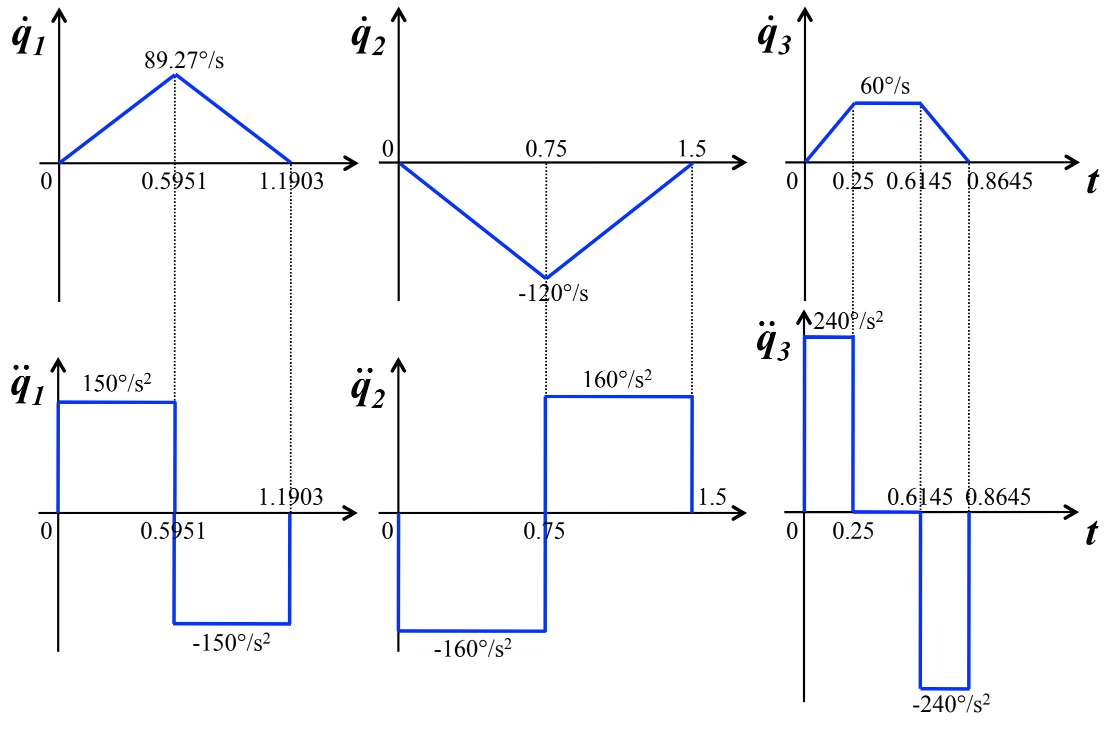


Figure 3: Velocity (top) and acceleration (bottom) profiles for joints 1 to 3 (from left to right)

For the minimum time trajectory thus defined, the velocity and acceleration profiles of the three joints are sketched in Fig. 3. From these, it can be easily checked that the orientation of the end-effector does not remain not constant during the resulting motion (i.e., it is not kept fixed to $\phi = 0$). For the orientation to be constant during motion, the joint velocities should add up to zero at any instant $t \in [0, T]$, which is not the case here.

Finally, the obtained trajectory is only one of the infinitely many that complete the given task in minimum time. In fact, we could add some extra motion to joints 1 and 3, still reaching their desired final values with zero velocity before or at $T = 1.5$ [s].

**Bonus Exercise**

The following Matlab code plots the 3D profile of the manipulability index $H$ in Fig. 4.

```
% input data (link lengths)

l1=1; l2=0.5; l3=0.25;

% discretization of joint angles (in rads)

delta=0.02; q2=[-pi:delta:pi]; q3=[-pi:delta:pi];

% evaluation of manipulability index H

for J = 1:length(q2), for K = 1:length(q3),
H(J,K)=sqrt((l1*(l2*sin(q2(J))+l3*sin(q2(J)+q3(K))))^2 ...
+(l3*(l1*sin(q2(J)+q3(K))+l2*sin(q3(K))))^2 ...
+(l2*l3*sin(q3(K)))^2 );
end; end;

% mesh plot

[X,Y]=meshgrid(q2,q3);mesh(Y,X,H); % note the reverse order of arguments!
title('Manipulability index H of planar 3R robot');
xlabel('q2');ylabel('q3');zlabel('sqrt (det J*JT)');
```

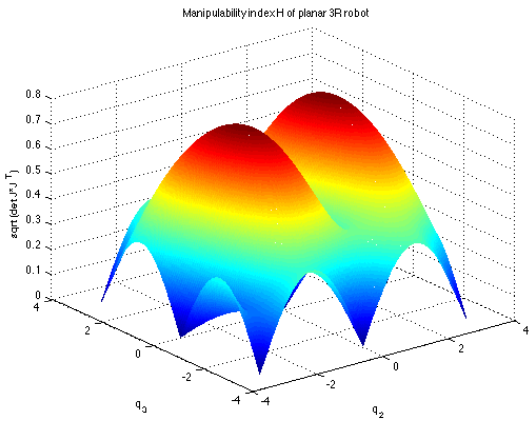


Figure 4: 3D profile of the manipulability index $H$

An additional instruction provides a plot of the level curves $H$ (see Fig. 5).

```
% contour plot

contour(Y,X,H,30); % with 30 level curves (note the reverse order of arguments!)
title('Manipulability index of planar 3R robot');
xlabel('q2');ylabel('q3');
```

From Figs. 4 and 5, it follows that two (symmetric) configurations provide the maximum value $H^*$. In fact, it is easy to check from (1) that $H(q_2, q_3) = H(-q_2, -q_3)$ for any pair $(q_2, q_3)$. Also, we can verify graphically the zeroing of $H$ at the singular configurations already found analytically.

The optimal value $H^*$ and the associated configuration $(q_2^*, q_3^*)$ can be computed (up to the chosen discretization of 0.02 rad) by the following piece of code.

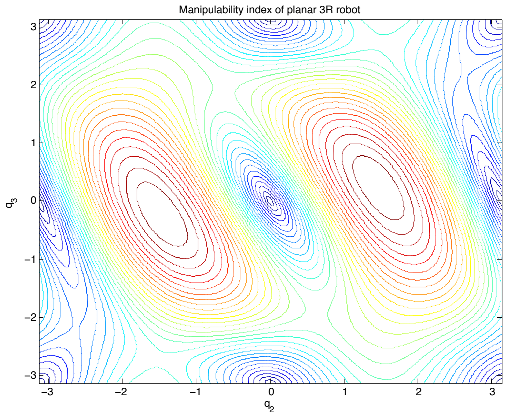


Figure 5: Level curves of the manipulability index $H$

```
% find global maximum value of H and associated configuration (q2,q3)
maxH=0;indexJ=0; indexK=0;
for J = 1:length(q2),
[maxHJ, Kmax]=max(H(J,:)');
if maxHJ > maxH, maxH=maxHJ; indexJ=J; indexK=Kmax; end;
end;
globalmaxH=maxH;
q2maxH=q2(indexJ); % in rads
q3maxH=q3(indexK); % in rads
```

This provides an (approximately) optimal value $H^* = 0.79$, which is obtained at the configuration $(q_2^*, q_3^*) = (1.48, 0.24)$ [rad] $\approx (84.8°, 13.7°)$. For illustration, this optimal configuration (in red) and the symmetric one (in blue) are shown in Fig. 6, choosing the value $q_1 = 0$ (otherwise arbitrary). We remark that the optimal configuration changes for a different set of link lengths. Only when all link lengths are equal (say, to $\ell$), the optimal configuration would not depend on $\ell$ (while the associated value $H^*$ would have a scaling factor $\ell^2$).

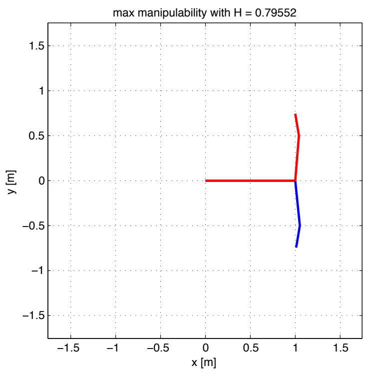


Figure 6: Optimal configurations for the manipulability index $H$ (shown at $q_1 = 0$)

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