Robotics I

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The direct kinematics of a 4-dof robot of the SCARA type (2R-1P-1R) is

$$
\begin{aligned}
x &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\
y &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \\
z &= q_3 \\
\alpha &= 0 \\
\beta &= 0 \\
\gamma &= q_1 + q_2 + q_4,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{p} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ is the position of the end-effector and $\boldsymbol{\phi} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}^T$ is its orientation, expressed in XYZ angles (roll-pitch-yaw) with respect to the fixed base frame. The length of links 1 and 2 is $l_1 = l_2 = 1$ [m].

In the configuration $\boldsymbol{q}_A = \begin{pmatrix} 0 & \pi/2 & 1 & -\pi/2 \end{pmatrix}^T$, check if the following end-effector generalized (linear/angular) velocity vector

$$
\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}
\tag{2}
$$

is feasible or not. Provide then the expression and the numerical value of the joint velocity vector $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that guarantees the exact execution of the desired end-effector velocity (if the task (2) is feasible), or at least the minimization of the Cartesian/task velocity error norm in a least squares sense (if (2) is unfeasible).

Repeat the above analysis and the synthesis of a joint velocity vector with the same properties when the robot is in the configuration $\boldsymbol{q}_B = \begin{pmatrix} \pi/4 & 0 & 1 & 0 \end{pmatrix}^T$.

[120 minutes; open books]

Solution

Denote the 6-dimensional task vector associated to the end-effector pose of the robot as

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{\phi} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

where $\boldsymbol{f}(\boldsymbol{q})$ is the direct kinematics expressed by the right-hand side of (1). The 6×4 analytic Jacobian matrix $\boldsymbol{J}_a$ relating $\dot{\boldsymbol{q}}$ to $\dot{\boldsymbol{r}}$

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_a(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and the 6×4 geometric Jacobian matrix $\boldsymbol{J}_g$ relating $\dot{\boldsymbol{q}}$ to the linear and angular velocity of the end-effector

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \boldsymbol{J}_g(\boldsymbol{q})\dot{\boldsymbol{q}}$$

will coincide in this case ($\boldsymbol{J}_a = \boldsymbol{J}_g = \boldsymbol{J}$), since the only admissible rotations of the end-effector frame of this robot occur around a single constant axis (the Z axis). In particular, it will always be $\omega_x = \dot{\alpha} = 0$, $\omega_y = \dot{\beta} = 0$, and $\omega_z = \dot{\gamma}$.

Therefore, we compute (for convenience, analytically from (1))

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)) & -l_2 \sin(q_1 + q_2) & 0 & 0 \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Evaluating this matrix at the configuration $\boldsymbol{q}_A = \begin{pmatrix} 0 & \pi/2 & 1 & -\pi/2 \end{pmatrix}^T$, and with the given link lengths, provides

$$\boldsymbol{J}_A = \boldsymbol{J}(\boldsymbol{q}_A) = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

which has full row rank equal to 4 (det $\boldsymbol{J}_A^T \boldsymbol{J}_A = 1 \neq 0$). The desired linear and angular velocity (2), due to its structure with zeros in the fourth and fifth components, will certainly be feasible and a joint velocity solution is computed as

$$
\begin{aligned}
\dot{\boldsymbol{q}}_A &= \boldsymbol{J}_A^{\#} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} = \left(\boldsymbol{J}_A^T \boldsymbol{J}_A \right)^{-1} \boldsymbol{J}_A^T \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ [\text{rad/s}].
\end{aligned}
\tag{3}
$$

We note that in this case $\dot{\boldsymbol{q}}_A$ is also the unique joint velocity realizing the desired task velocity.

At the configuration $\boldsymbol{q}_B = \begin{pmatrix} \pi/4 & 0 & 1 & 0 \end{pmatrix}^T$, we have

$$
\boldsymbol{J}_B = \boldsymbol{J}(\boldsymbol{q}_B) = \begin{pmatrix} -\sqrt{2} & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2} & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},
$$

which has clearly rank equal to 3. Still the desired linear and angular velocity lies in the range of matrix $\boldsymbol{J}_B$ and thus it is feasible. The only difference with respect to the previous case is that a joint velocity solution should be computed using the numerical algorithm for the pseudoinverse of $\boldsymbol{J}_B$ (i.e., using the SVD of $\boldsymbol{J}_B$, or by simply calling the `pinv` function in Matlab). We obtain, with some numerical rounding,

$$
\begin{aligned}
\dot{\boldsymbol{q}}_B &= \boldsymbol{J}_B^{\#} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} \\
&= \begin{pmatrix} -0.3536 & 0.3536 & 0 & 0 & 0 & -0.1667 \\ 0 & 0 & 0 & 0 & 0 & 0.3333 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3536 & -0.3536 & 0 & 0 & 0 & 0.8333 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.5404 \\ -0.3333 \\ 1 \\ -0.1262 \end{pmatrix} \ [\text{rad/s}].
\end{aligned}
\tag{4}
$$

The solution $\dot{\boldsymbol{q}}_B$ is not unique in this case, but it has automatically the minimum norm property among all joint velocities realizing the task. For illustration, another joint velocity satisfying the task is

$$
\dot{\boldsymbol{q}}_B' = \begin{pmatrix} -0.1322 \\ -1.1498 \\ 1 \\ 0.2820 \end{pmatrix} \ [\text{rad/s}],
$$

which is obtained by adding to $\dot{\boldsymbol{q}}_B$ a vector $\boldsymbol{n}_B$ in the one-dimensional null space of $\boldsymbol{J}_B$:

$$
\dot{\boldsymbol{q}}_B' = \dot{\boldsymbol{q}}_B + \boldsymbol{n}_B, \qquad \boldsymbol{n}_B = \begin{pmatrix} 0.4082 \\ -0.8165 \\ 0 \\ 0.4082 \end{pmatrix}, \qquad \text{being } \boldsymbol{J}_B \boldsymbol{n}_B = \boldsymbol{0}.
$$

3

Checking the norm of these two joint velocity solutions gives

$$\|\dot{\boldsymbol{q}}_B\| = 1.1913 < 1.554 = \|\dot{\boldsymbol{q}}'_B\|,$$

as expected.

In both cases A and B (as well as in B'), the task velocity error is zero, since the desired linear and angular velocity (2) is exactly realized. Indeed, one can verify that

$$\boldsymbol{J}_A\,\dot{\boldsymbol{q}}_A = \boldsymbol{J}_B\,\dot{\boldsymbol{q}}_B = \boldsymbol{J}_B\,\dot{\boldsymbol{q}}'_B = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

In case B, there exist other task velocity vectors (still with structural zeros in the fourth and fifth components) that are not feasible. The joint velocity obtained by using the pseudoinverse of $\boldsymbol{J}_B$ will still provide a task velocity error of minimum norm, although this error would then be different from zero. The reader is invited to provide an example.

We finally note that, due to the presence of structural zeros in the fourth and fifth rows of $\boldsymbol{J}$, all previous results could have been obtained by reducing the problem to the 4-dimensional task space $\bar{\boldsymbol{r}} = \begin{pmatrix} x & y & z & \gamma \end{pmatrix}^T$ and working with the square 4×4 matrix $\bar{\boldsymbol{J}}$ obtained by deleting rows 4 and 5 from $\boldsymbol{J}$. For case A, we would obtain the same solution $\dot{\boldsymbol{q}}_A$ given in (3) by computing $\dot{\boldsymbol{q}}_A = \bar{\boldsymbol{J}}^{-1}\bar{\boldsymbol{r}}$, while for case B the same solution $\dot{\boldsymbol{q}}_B$ given in (4) would be obtained as $\dot{\boldsymbol{q}}_B = \bar{\boldsymbol{J}}^{\#}\bar{\boldsymbol{r}}$.

In fact we can state in general that, given a $m \times n$ matrix $\boldsymbol{J}$ (with $m > n$) of the form

$$\boldsymbol{J} = \boldsymbol{T}\begin{pmatrix} \bar{\boldsymbol{J}} \\ \boldsymbol{O} \end{pmatrix},$$

where $\boldsymbol{T}$ is a row permutation (and thus, an orthonormal) matrix and $\bar{\boldsymbol{J}}$ is a square $m \times m$ matrix, the following expression holds for the pseudoinverse:

$$\boldsymbol{J}^{\#} = \bar{\boldsymbol{J}}^{\#}\begin{pmatrix} \boldsymbol{I}_{m \times m} & \boldsymbol{O} \end{pmatrix}\boldsymbol{T}^T.$$

Morever, when $\det \bar{\boldsymbol{J}} \neq 0$, it is $\bar{\boldsymbol{J}}^{\#} = \bar{\boldsymbol{J}}^{-1}$. In our application, this relation means that we can simply work with the reduced task space and the associated reduced Jacobian, discarding the zero rows.

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