Robotics I

February 11, 2010

Consider a planar 2R manipulator having link lengths $\ell_1 = 0.6$, $\ell_2 = 0.5$ [m]. The joint angles θ_1, θ_2 are defined using the DH convention. The joint ranges are unlimited. The base of the manipulator is placed at the origin of the given $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ frame.

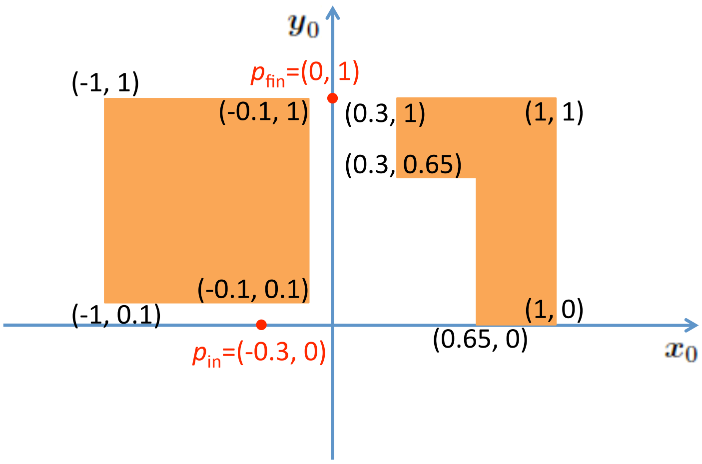

Figure 1: Workspace for the assigned task

With reference to Fig. 1, plan a continuous parametric path so that the end-effector is transferred between the Cartesian points

$$\boldsymbol{p}_{\rm in} = \begin{pmatrix} -0.3 \\ 0 \end{pmatrix} \qquad \mapsto \qquad \boldsymbol{p}_{\rm fin} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad [\mathrm{m}],$$

and the following conditions are satisfied:

- the path is made of polynomial functions of the lowest possible degree;
- the path tangent is continuous with respect to the path parameter;
- the manipulator avoids collision with the two obstacles shown in orange.

Provide a solution and check graphically (e.g., using Matlab) the collision avoidance. If a singularity is encountered in the proposed solution, indicate how this situation is handled. Moreover, explain how a timing law should be assigned so that the resulting trajectory has a satisfactory behavior.

[150 minutes; open books]

Solution

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The manipulator workspace is a circular ring with inner circumference of radius $|\ell_1 - \ell_2| = 0.1$ and outer circumference of radius $\ell_1 + \ell_2 = 1.1$. Therefore, the two given Cartesian points are within the manipulator workspace. Using the inverse kinematics function of the 2R robot yields the two solutions

$$\boldsymbol{\theta}_{\text{in}}^{\text{left}} = \begin{pmatrix} -2.1598 \\ -2.6193 \end{pmatrix} \qquad \boldsymbol{\theta}_{\text{in}}^{\text{right}} = \begin{pmatrix} 2.1598 \\ 2.6193 \end{pmatrix} \quad \text{[rad]}$$

for the initial Cartesian point $\boldsymbol{p}_{\text{in}}$, and the two solutions

$$\boldsymbol{\theta}_{\text{fin}}^{\text{left}} = \begin{pmatrix} 1.9606 \\ -0.8632 \end{pmatrix} \qquad \boldsymbol{\theta}_{\text{fin}}^{\text{right}} = \begin{pmatrix} 1.1810 \\ 0.8632 \end{pmatrix} \quad \text{[rad]}$$

for the final Cartesian point $\boldsymbol{p}_{\text{fin}}$. To avoid collision at the initial and final point, we need to choose $\boldsymbol{\theta}_{\text{in}}^{\text{left}}$ and $\boldsymbol{\theta}_{\text{fin}}^{\text{right}}$, respectively. Since these inverse solutions are of two different kinds, it is clear that the arm will need to pass through a singular configuration (stretched or folded) during motion.

Therefore, the easiest way to address the problem is to define a path in the joint space, possibly using one (or more) via points. The path can then cross singular configurations without control problems at run time during path/trajectory execution (no Jacobian inversion is needed). Indeed, the problem of avoiding collisions remains.

The straightforward interpolation of the initial and final configurations by a single linear joint path (the polynomial of lowest possible degree) is not feasible. To see this, introduce a parameter $s \in [0,1]$ for describing the path $\boldsymbol{\theta} = \boldsymbol{q}(s) = \begin{pmatrix} q_1(s) & q_2(s) \end{pmatrix}^T$. The interpolating linear path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \left(\boldsymbol{\theta}_{\text{fin}}^{\text{right}} - \boldsymbol{\theta}_{\text{in}}^{\text{left}}\right) s + \boldsymbol{\theta}_{\text{in}}^{\text{left}}, \qquad s \in [0,1]. \tag{1}$$

The arm will pass through the stretched singularity for $s = s^*$ such that $\theta_2 = q_2(s^*) = 0$. This happens at

$$s^* = 0.7521 \qquad \Rightarrow \qquad \theta_1 = q_1(s^*) = 0.3529 \quad \text{[rad]}.$$

It is easy to see that a collision occurs with the obstacle on the right, e.g., for the manipulator configuration $\boldsymbol{\theta} = \begin{pmatrix} 0.3529 & 0 \end{pmatrix}^T$. This is confirmed graphically by a simple Matlab code, implementing joint path generation and manipulator direct kinematics, and plotting results as in Fig. 2. Initial (green) and final (red) configuration, and end-effector path (dotted) are also shown.

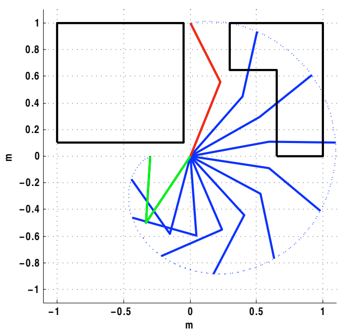

Figure 2: Stroboscopic view of manipulator motion for a linear joint path, resulting in a collision

In order to avoid this situation, we may replace in (1) the second component of $\boldsymbol{\theta}_{\text{fin}}^{\text{right}}$ by its value modulo 2π, i.e.,

$$\boldsymbol{\theta}_{\text{fin}}^{\text{right}^-} = \begin{pmatrix} 1.1810 \\ 0.8632 - 2\pi \end{pmatrix} = \begin{pmatrix} 1.1810 \\ -5.4200 \end{pmatrix} \quad \text{[rad]}.$$

The linear path of the second joint will now necessarily cross the value $\theta_2 = -\pi$, i.e., the arm will pass through the folded singularity. Unfortunately, this is not yet sufficient to avoid collision (see Fig. 3). For instance, the tip of the robot is in collision at $s^* = 0.75$, being the arm in the configuration $\boldsymbol{\theta} = \begin{pmatrix} 0.3458 & -4.7198 \end{pmatrix}^T$ (or, $\theta_1 \approx 20°$ and $\theta_2 = -270°$).

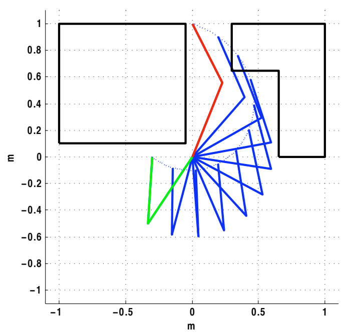

Figure 3: Stroboscopic view of manipulator motion for another linear joint path, resulting again in a collision

As a result of this analysis, we need to introduce an intermediate joint configuration, say at $s = 0.5$, which is conveniently chosen as the folded arm configuration in correspondence to the point where the passage between the obstacles begins, i.e.,

$$\boldsymbol{\theta}_{\text{mid}} = \begin{pmatrix} 0 \\ -\pi \end{pmatrix} \quad \text{[rad]} \qquad \Leftrightarrow \qquad \boldsymbol{p}_{\text{mid}} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} \quad \text{[m]}.$$

The choice of a negative value $\theta_{\text{mid},2} = -\pi$ (rather than π) is based on the same previous argument: by continuity of motion, the second link will always rotate in the clockwise direction, reaching the folded singularity at the specified location $\boldsymbol{p}_{\text{mid}}$, and then unfolding itself so as to avoid collision with the obstacle on the right. The boundary conditions for the interpolating joint path are then:

$$\boldsymbol{q}(0) = \boldsymbol{\theta}_{\text{in}}^{\text{left}} = \begin{pmatrix} -2.1598 \\ -2.6193 \end{pmatrix}, \quad \boldsymbol{q}(\tfrac{1}{2}) = \boldsymbol{\theta}_{\text{mid}} = \begin{pmatrix} 0 \\ -\pi \end{pmatrix}, \quad \boldsymbol{q}(1) = \boldsymbol{\theta}_{\text{fin}}^{\text{right}^-} = \begin{pmatrix} 1.1810 \\ -5.4200 \end{pmatrix}. \tag{2}$$

Further, we need to impose now also continuity of the joint path tangent at the mid point, i.e.,

$$\left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}^-} = \left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}^+}. \tag{3}$$

Therefore, we can select (for each joint) a quadratic and a linear function of s (or viceversa) on the two tracts of the path, allowing a total of five coefficients for satisfying the five boundary conditions. Such a mixed polynomial path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \begin{cases} \boldsymbol{a}s^2 + \boldsymbol{b}s + \boldsymbol{c}, & \text{for } s \in [0, \frac{1}{2}] \\ \boldsymbol{d}s + \boldsymbol{e}, & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

where $\boldsymbol{a}, \ldots, \boldsymbol{e}$ are suitable two-dimensional vectors of coefficients. Imposing the boundary conditions (2–3), and dropping for compactness the superscripts 'left' and 'right$^-$', yields:

$$\boldsymbol{q}(s) = \begin{cases} \left(4\boldsymbol{\theta}_{\text{fin}} - 8\boldsymbol{\theta}_{\text{mid}} + 4\boldsymbol{\theta}_{\text{in}}\right) s^2 + \left(6\boldsymbol{\theta}_{\text{mid}} - 4\boldsymbol{\theta}_{\text{in}} - 2\boldsymbol{\theta}_{\text{fin}}\right) s + \boldsymbol{\theta}_{\text{in}}, & \text{for } s \in [0, \frac{1}{2}] \\ 2\left(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}}\right) s + 2\boldsymbol{\theta}_{\text{mid}} - \boldsymbol{\theta}_{\text{fin}}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{4}$$

The planned joint path and the path tangent are given in Figs. 4 and 5, respectively. A stroboscopic view of the resulting manipulator motion is shown in Fig. 6. It can be seen that there are no collisions.

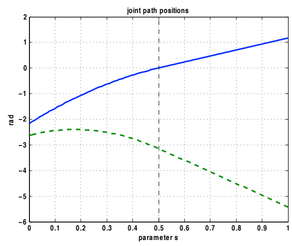

Figure 4: Quadratic/linear path in the joint space: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

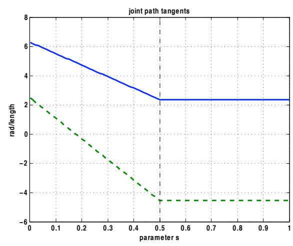

Figure 5: Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)

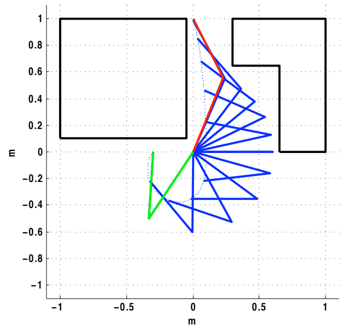

Figure 6: Stroboscopic view of manipulator motion for the quadratic/linear joint path

To convert this path into a trajectory, one should associate a timing law $s = s(t)$ for $t \in [0, T]$. Any timing law can be chosen (bang-bang in acceleration, with trapezoidal speed profile, cubic time polynomial, ...), depending on additional task requests and robot performance limits. The only care is that a *single* timing law should be chosen for both joints. Otherwise, the joint motion would be uncoordinated in time and the executed Cartesian robot path would *not* be the planned one, with a possible danger of collisions.

Additional considerations. In the following, we present complementary material to the given solution. In particular, we consider a more balanced path planning solution using two quadratic functions of s, with a total of six coefficients for each joint. It can be expected that this provides a further degree of freedom for shaping the resulting joint path. An additional constraint should be specified in this case, in order to 'square' the interpolation problem. This is obtained by imposing a specific value $\boldsymbol{\theta}'_{mid}$ for the joint path tangent at the mid point, i.e.,

$$\left.\frac{d\boldsymbol{q}(s)}{ds}\right|_{s=\frac{1}{2}} = \boldsymbol{\theta}'_{mid}. \tag{5}$$

Indeed, various choices can be made for this (vector) value. The fully quadratic path $\boldsymbol{q}(s)$ is defined as

$$\boldsymbol{q}(s) = \begin{cases} \boldsymbol{a}s^2 + \boldsymbol{b}s + \boldsymbol{c}, & \text{for } s \in [0, \frac{1}{2}] \\ \boldsymbol{d}s^2 + \boldsymbol{e}s + \boldsymbol{f}, & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

where $\boldsymbol{a}, \dots, \boldsymbol{f}$ are suitable two-dimensional vectors of coefficients. Imposing the boundary conditions (2–5) yields now:

$$\boldsymbol{q}(s) = \begin{cases} \left(4(\boldsymbol{\theta}_{\rm in} - \boldsymbol{\theta}_{\rm mid}) + 2\boldsymbol{\theta}'_{mid}\right) s^2 + \left(4(\boldsymbol{\theta}_{\rm mid} - \boldsymbol{\theta}_{\rm in}) - \boldsymbol{\theta}'_{mid}\right) s + \boldsymbol{\theta}_{\rm in}, & \text{for } s \in [0, \frac{1}{2}] \\ \left(4(\boldsymbol{\theta}_{\rm fin} - \boldsymbol{\theta}_{\rm mid}) - 2\boldsymbol{\theta}'_{mid}\right) s^2 + \left(4(\boldsymbol{\theta}_{\rm mid} - \boldsymbol{\theta}_{\rm fin}) + 3\boldsymbol{\theta}'_{mid}\right) s + \boldsymbol{\theta}_{\rm fin} - \boldsymbol{\theta}'_{mid}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases} \tag{6}$$

Accordingly, its first derivative w.r.t. s (the path tangent in the joint space) is:

$$\frac{d\boldsymbol{q}(s)}{ds} = \begin{cases} \left(8(\boldsymbol{\theta}_{\rm in} - \boldsymbol{\theta}_{\rm mid}) + 4\boldsymbol{\theta}'_{mid}\right) s + 4(\boldsymbol{\theta}_{\rm mid} - \boldsymbol{\theta}_{\rm in}) - \boldsymbol{\theta}'_{mid}, & \text{for } s \in [0, \frac{1}{2}] \\ \left(8(\boldsymbol{\theta}_{\rm fin} - \boldsymbol{\theta}_{\rm mid}) - 4\boldsymbol{\theta}'_{mid}\right) s + 4(\boldsymbol{\theta}_{\rm mid} - \boldsymbol{\theta}_{\rm fin}) + 3\boldsymbol{\theta}'_{mid}, & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

A possible choice for the vector value $\boldsymbol{\theta}'_{mid}$ is obtained by imposing at $\boldsymbol{p}_{\rm mid}$ a tangent to the Cartesian path in the direction of $\boldsymbol{y}_0$ and of unit norm, i.e.,

$$\left.\frac{d\boldsymbol{p}(s)}{ds}\right|_{s=\frac{1}{2}} = \left.\frac{d\boldsymbol{p}}{d\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{mid}} \left.\frac{d\boldsymbol{q}}{ds}\right|_{s=\frac{1}{2}} = \boldsymbol{J}(\boldsymbol{\theta}_{mid})\,\boldsymbol{\theta}'_{mid} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{7}$$

where $\boldsymbol{p} = \boldsymbol{kin}(\boldsymbol{\theta})$ is the direct kinematics of the manipulator. This choice is indeed feasible, despite the manipulator is in a (folded) singular configuration. In fact, the robot Jacobian

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{pmatrix} -\ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1+\theta_2) & -\ell_2 \sin(\theta_1+\theta_2) \\ \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) & \ell_2 \cos(\theta_1+\theta_2) \end{pmatrix}$$

takes the value

$$\boldsymbol{J}(\boldsymbol{\theta}_{mid}) = \begin{pmatrix} 0 & 0 \\ \ell_1 - \ell_2 & -\ell_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0.1 & -0.5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{R}(\boldsymbol{J}(\boldsymbol{\theta}_{mid})).$$

The (minimum norm) solution for $\boldsymbol{\theta}'_{mid}$ is obtained using pseudoinversion (or, equivalently, by pseudoinversion of the second row/scalar equation only) in (7), i.e.,

$$\boldsymbol{\theta}'_{mid} = \boldsymbol{J}^{\#}(\boldsymbol{\theta}_{mid}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^{\#} \cdot 1 = \frac{1}{0.26} \begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix}. \tag{8}$$

The resulting joint path, path tangent, and path curvature are given in Figs. 7–8. Note that the curvature has a discontinuity at the midpoint. A stroboscopic view of the manipulator motion is shown in Fig. 9. Also in this case, it can be seen that there are no collisions.

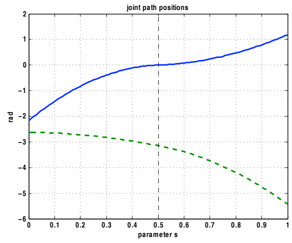

Figure 7: Fully quadratic path in the joint space: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

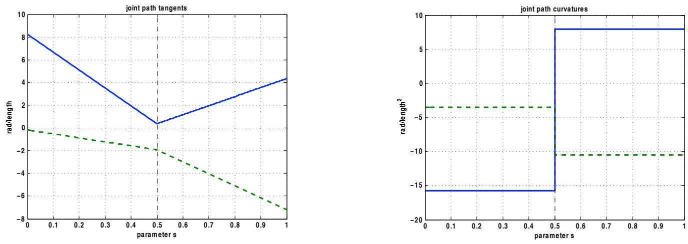

Figure 8: [Left] Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)
. [Right] Path curvature: $d^2q_1(s)/ds^2$ (solid, blue), $d^2q_2(s)/ds^2$ (dashed, green)

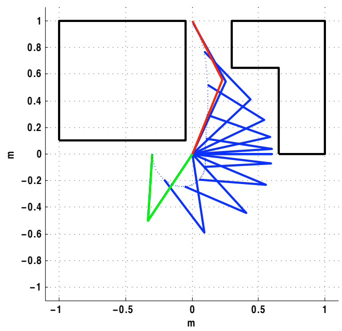

Figure 9: Stroboscopic view of manipulator motion for a fully quadratic joint path

We show also the outcome of other possible choices for $\boldsymbol{\theta}'_{mid}$. One can avoid to select a specific value, and resolve the problem by imposing also curvature continuity of the joint path at the mid point. The second derivative w.r.t. s of the interpolating path function (6) is

$$\frac{d^2\boldsymbol{q}(s)}{ds^2} = \begin{cases} 8(\boldsymbol{\theta}_{\text{in}} - \boldsymbol{\theta}_{\text{mid}}) + 4\boldsymbol{\theta}'_{mid}, & \text{for } s \in [0, \frac{1}{2}] \\ 8(\boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{mid}}) - 4\boldsymbol{\theta}'_{mid} & \text{for } s \in [\frac{1}{2}, 1], \end{cases}$$

i.e., piecewise constant. Imposing equality at $s = 0.5$ leads to:

$$\boldsymbol{\theta}'_{mid} = \boldsymbol{\theta}_{\text{fin}} - \boldsymbol{\theta}_{\text{in}} = \begin{pmatrix} 3.3408 \\ -2.8007 \end{pmatrix}.$$

The resulting joint path, path tangent, and path curvature are shown in Figs. 10–11, where the curvature is now continuous (constant). However, the obtained motion is unfeasible due to the collision with the obstacle on the left, close to the reaching of the final point (Fig. 12). This is a result of the additional smoothness imposed (at least in the chosen class of interpolating functions).

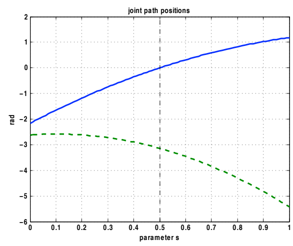

Figure 10: Quadratic path in the joint space with continuous curvature: $q_1(s)$ (solid, blue) and $q_2(s)$ (dashed, green)

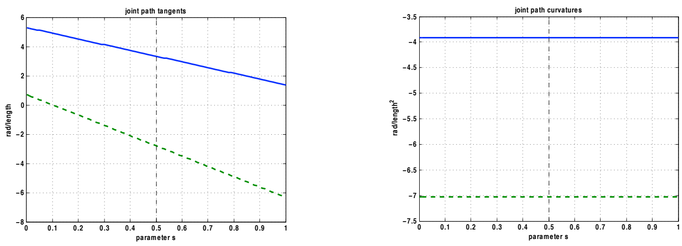

Figure 11: [Left] Path tangent: $dq_1(s)/ds$ (solid, blue), $dq_2(s)/ds$ (dashed, green)
. [Right] Path curvature: $d^2q_1(s)/ds^2$ (solid, blue), $d^2q_2(s)/ds^2$ (dashed, green)

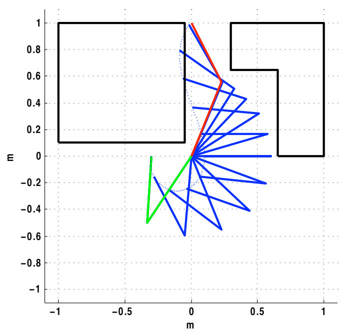

Figure 12: Stroboscopic view of manipulator motion for the quadratic path with continuous curvature, resulting in a collision

As a matter of fact, the problem is due to the large 'swing' imposed to the robot when passing through the mid point. The criticality of the choice of $\boldsymbol{\theta}'_{mid}$ in the considered class of interpolating functions becomes even more dramatically clear when setting for instance

$$\boldsymbol{\theta}'_{mid} = \frac{100}{0.26} \begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix},$$

i.e., a value that is *hundred* times larger than the minimum norm solution given in (8). The planned robot motion goes wild in this case, as shown in Fig. 13. On the other hand, the solution obtained for $\boldsymbol{\theta}'_{mid} = \mathbf{0}$ is quite similar to the one in Fig. 9.

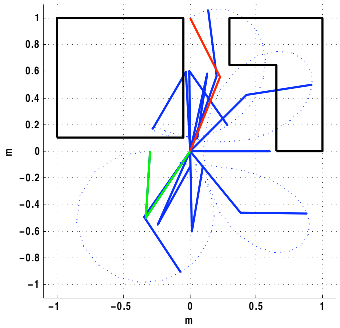

Figure 13: Stroboscopic view of manipulator motion for a quadratic path, with very large $\boldsymbol{\theta}'_{mid}$

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