as $x^k \to \bar{x}$ and $y_n^k \to y_n''$. Therefore, as $k \to \infty$ (i.e., $r^k \to 0$),

$$r^k \phi\left(g_n(y_n^k, x^k), h_n(y_n^k)\right) \to 0,$$

that is,

$$P_n^{r^k}(y_n^k, x^k) \to f_n(y_n'', \bar{x}),$$

which implies the existence of a positive integer K_2 such that

$$|P_n^{r^k}(y_n^k, x^k) - f_n(y_n'', \bar{x})| < \epsilon, \qquad \text{for all } k > K_2 \quad (25)$$

for ϵ in (23). Besides, from the continuity of f_n at any (y_n, x), we have the existence of a positive integer K_3 such that

$$|f_n(\bar{y}_n, \bar{x}) - f_n\left(\check{y}_n^{r^k}(x^k), x^k\right)| < \epsilon, \qquad \text{for all } k > K_3. \quad (26)$$

Set $K = \max(K_1, K_2, K_3)$. Then, using (25), (23), and (26), in turn, we have the following relations for all $k > K$.

$$P_n^{r^k}(y_n^k, x^k) < f_n(y_n'', \bar{x}) + \epsilon = f_n(\bar{y}_n, \bar{x}) - \epsilon < f_n\left(\check{y}_n^{r^k}(x^k), x^k\right) \quad (27)$$

Since $\phi > 0$,

$$f_n\left(\check{y}_n^{r^k}(x^k), x^k\right) < P_n^{r^k}\left(\check{y}_n^{r^k}(x^k), x^k\right). \quad (28)$$

Equations (27) and (28) yield

$$P_n^{r^k}(y_n^k, x^k) < P_n^{r^k}\left(\check{y}_n^{r^k}(x^k), x^k\right), \qquad \text{for all } k > K.$$

This relation and (24) contradict that $\check{y}_n^{r^k}(x^k)$ is optimal for (3) in response to x^k and r^k. Therefore, any accumulation point is optimal for (4) with $\bar{x}$.

Existence and Optimality of the Limit Point: Since the optimal solution $\check{y}_n(\bar{x})$ to (4) with $\bar{x}$ is unique under the assumption d), the accumulation point of $\{\check{y}_n^{r^k}(x^k)\}$ is also unique. Therefore, the accumulation point becomes a limit point of $\{\check{y}_n^{r^k}(x^k)\}$. Thus, we can conclude that $\check{y}_n^{r^k}(x^k)$ converges to $\check{y}_n(\bar{x})$.

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Kinematic Control Equations for Simple Manipulators

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***Abstract*—The basis for all advanced manipulator control is a relationship between the Cartesian coordinates of the end-effector and the manipulator joint coordinates. A direct method for assigning link coordinate systems and obtaining the end-effector position in terms of joint coordinates is reviewed. Techniques for obtaining the solution to these equations for kinematically simple manipulators, which includes all commercially available manipulators, are presented.**

INTRODUCTION

A serial link manipulator consists of a sequence of mechanical links connected together by actuated joints. Such a structure forms a kinematic chain and may be analyzed by methods developed by Denavit and Hartenberg [10]. The results of this analysis are the matrix equations expressing manipulator end-effector Cartesian position and orientation in terms of the joint coordinates. These equations may be obtained for any manipulator independent of the number of links or degrees of freedom.

In this correspondence we first review the method of obtaining these equations extending the procedure of assigning coordinate frames to include simple manipulators which have many zero length links and intersecting joint axes. While we may obtain these kinematic equations for any manipulator it is their solution which is of interest. Given a desired Cartesian position and orientation of the manipulator's end-effector what are the necessary joint coordinates? While there is only one end-effector position corresponding to a given set of joint coordinates, there are a number of configurations of the manipulator's links all of which place the end-effector in the same position and orientation. Normally only one solution corresponding to a given kinematic configuration is desired (e.g., elbow up or down, etc.), rather than the entire set of solutions. Frequently the solution is to be embedded in a real-time servo loop and only a very minimum number of mathematical operations may be performed.

When the manipulator geometry is simple and well understood a trigonometric solution may often be obtained [1]–[3], [8], [9]. However, six-degree-of-freedom manipulators are sufficiently complex that the direct trigonometric method is too difficult to apply. We present a method of obtaining a solution to the kinematic equations based on the Hartenberg–Denavit matrices from which the solution is obtained explicitly in the case of simple manipulators. The existence of an explicit solution to the kinematic equations for any manipulator is of great importance in evaluating the manipulator's suitability for computer control. Iterative solution techniques can involve an order of magnitude and more computation than an explicit solution. Pieper [5] in his thesis considers a series of simple manipulators for which a closed-form solution is obtainable. It is to these "simple" manipulators that the solution method presented in this correspondence is applicable. We have solved the kinematic equations for all commercially available manipulators and find that the equations can be readily obtained in a matter of hours.

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Coordinate Frames

A serial link manipulator consists of a sequence of links connected together by actuated joints. For an n-degree-of-freedom manipulator, there will be n links and n joints. The base of the manipulator is link 0 and is not considered one of the six links. Link 1 is connected to the base link by joint 1. There is no joint at the end of the final link. The only significance of links is that they maintain a fixed relationship between the manipulator joints at each end of the link (7). Any link can be characterized by two dimensions: the common normal distance a_n, and α_n the angle between the axes in a plane perpendicular to a_n. It is customary to call a_n "the length" and α_n "the twist" of the link (see Fig. 1). Generally, two links are connected at each joint axis (see Fig. 2). The axis will have two normals connected to it, one for each link. The relative position of two such connected links is given by d_n, the distance between the normals along the joint n axis, and θ_n the angle between the normals measured in a plane normal to the axis. d_n and θ_n are called 'the distance" and "the angle" between the links, respectively.

In order to describe the relationship between links, we will assign coordinate frames to each link. We will first consider revolute joints in which θ_n is the joint variable. The origin of the coordinate frame of link n is set to be at the intersection of the common normal between joints n and $n+1$ and the axis of joint $n+1$. In the case of intersecting joint axes, the origin is at the point of intersection of the joint axes. If the axes are parallel, the origin is chosen to make the joint distance zero for the next link whose coordinate origin is defined. The z axis for link n shall be aligned with the axis of joint $n+1$. The x axis will be aligned with any common normal which exists and is directed along the normal from joint n to joint $n+1$. In the case of intersecting joints, the direction of the x axis is parallel or antiparallel to the vector cross product $z_{n-1} \times z_n$. Notice this condition is also satisfied for the x axis directed along the normal between joints n and $n+1$. For the nth revolute joint when x_{n-1} and x_n are parallel and have the same direction, θ_n is at its zero position.

In the case of a prismatic joint the distance d_n is the joint variable. The direction of the joint axis is the direction in which the joint moves. Although the direction of the axis is defined, unlike a revolute joint, its position in space is not defined (see Fig. 3). In the case of a prismatic joint the length a_n has no meaning and is set to zero. The origin of the coordinate frame for a prismatic joint is coincident with the next defined link origin. The z axis of the prismatic link is aligned with the axis of joint $n+1$. The x_n axis is parallel or antiparallel to the vector cross product of the direction of the prismatic joint and z_n. For a prismatic joint, we will define its zero position, with $d_i = 0$, to be when x_{n-1} and x_n intersect. With the manipulator in its zero position, the positive sense of rotation for revolute joints or displacement for prismatic joints can be decided and the sense of the direction of the z axes determined.

The origin of the base link (zero) will be coincident with the origin of link 1. If it is desired to define a different reference coordinate system then the relationship between the reference and base coordinate systems can be described by a fixed homogeneous transformation [6]. At the end of the manipulator the final displacement d_6 or rotation θ_6 occurs with respect to z_5. The origin of the coordinate system for link 6 is chosen to be coincident with that of the link 5 coordinate system. If a tool or end-effector is used whose origin and axes do not coincide with the coordinate system of link 6, the tool can be related by a fixed homogeneous transformation to link 6.

Having assigned coordinate frames to all links according to the preceding scheme, we can establish the relationship between successive frames $n-1$, n by the following rotations and translations.

Rotate about z_{n-1}, an angle θ_n.
Translate along z_{n-1}, a distance d_n.

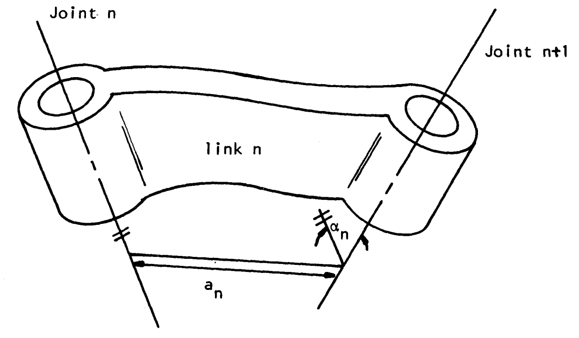

Fig. 1. Length a, and twist α, of a link.

Translate along rotated $x_{n-1} = x_n$, a length a_n.
Rotate about x_n, the twist angle α_n.

This may be expressed as the product of four homogeneous transformations relating the coordinate frame of link n to the coordinate frame of link $n-1$. This relationship is called an A matrix:

$$A_n = \begin{bmatrix} C\theta & -S\theta C\alpha & S\theta S\alpha & aC\theta \\ S\theta & C\theta C\alpha & -C\theta S\alpha & aS\theta \\ 0 & S\alpha & C\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

where S and C refer to sine and cosine, respectively. For a prismatic joint the A matrix reduces to

$$A_n = \begin{bmatrix} C\theta & -S\theta C\alpha & S\theta S\alpha & 0 \\ S\theta & C\theta C\alpha & -C\theta S\alpha & 0 \\ 0 & S\alpha & C\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Once the link coordinate frames have been assigned to the manipulator the various constant link parameters can be tabulated: d, a, and α for a link following a revolute joint and, θ and α for a link following a prismatic joint. Based on these parameters, the constant sine and cosine values of α may be evaluated and the values for the six A_i transformation matrices determined.

Kinematic Equations

Having assigned coordinate frames to a manipulator it is possible to obtain the Cartesian position and orientation of the manipulator end-effector when given the joint coordinates.

The description of the end of the manipulator, link coordinate frame 6, with respect to link coordinate frame $n-1$ is given by U_n where

$$U_n = A_n * A_{n+1} * \cdots * A_6. \tag{3}$$

The end of the manipulator with respect to the base, known as T_6, is given by U_1:

$$T_6 = U_1 = A_1 * A_2 * A_3 * A_4 * A_5 * A_6. \tag{4}$$

If the manipulator is related to a reference coordinate frame by a transformation Z and has a tool attached to its end described by E, we have the description of the end of the tool with respect to the reference coordinate system described by X as follows (4):

$$X = Z * T_6 * E. \tag{5}$$

In Fig. 4 the PUMA arm (Unimate 600 Robot) is shown with coordinate frames assigned to the links. The parameters are shown in Table I.

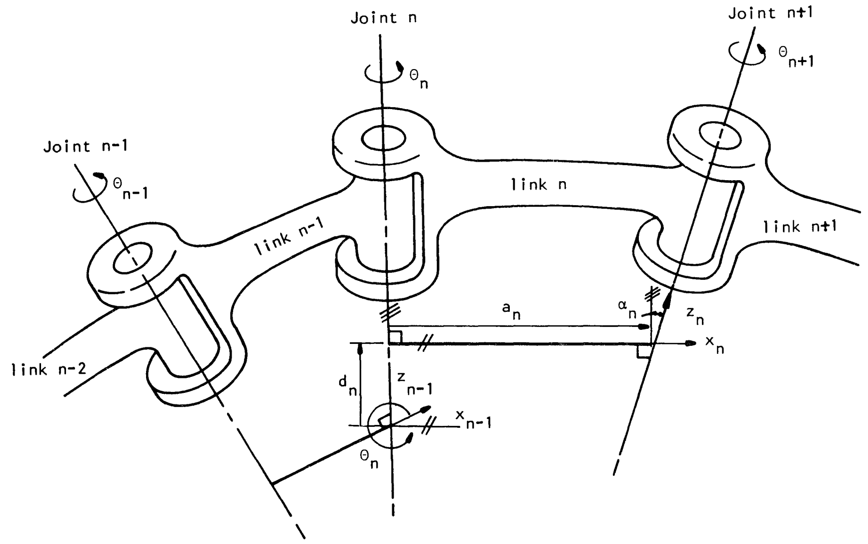

Fig. 2. Link parameters θ, d, a, α.

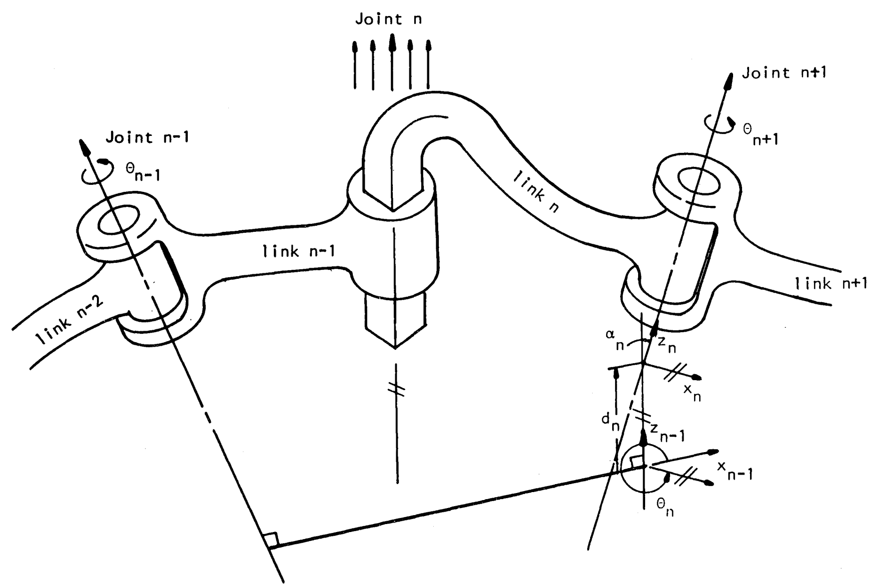

Fig. 3. Link parameters d, α for prismatic joint.

The A matrices for the PUMA arm are as follows:

$$A_1=\begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$$A_2=\begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$A_3=\begin{bmatrix} C_3 & 0 & S_3 & a_3C_3 \\ S_3 & 0 & -C_3 & a_3S_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$A_4=\begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$A_5=\begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$A_6=\begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

where S_i refers to $\sin(\theta_i)$ and C_i refers to $\cos(\theta_i)$. The product of

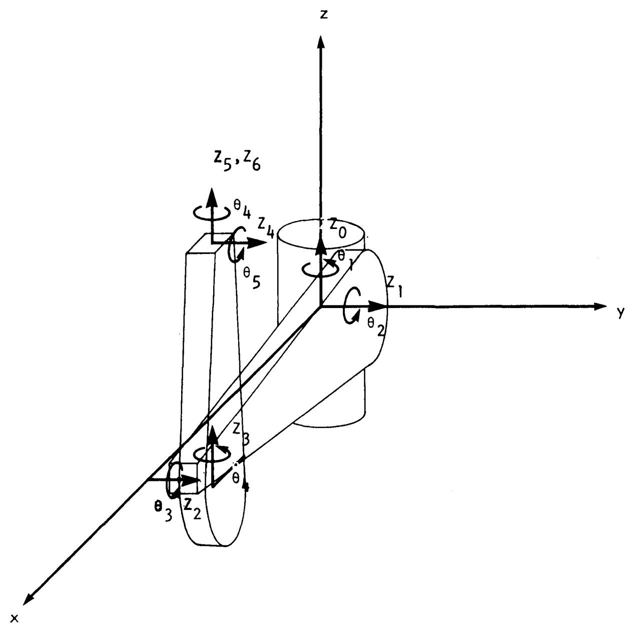

Fig. 4. PUMA manipulator.

TABLE I
LINK PARAMETERS FOR PUMA ARM

Joint	$\alpha°$	$\theta°$	d	a	Range
1	$-90°$	θ_1	0	0	θ_1: $+/-160°$
2	0	θ_2	0	a_2	θ_2: $+45 \rightarrow -225°$
3	90°	θ_3	d_3	a_3	θ_3: $225° \rightarrow -45°$
4	$-90°$	θ_4	d_4	0	θ_4: $+/-170°$
5	90°	θ_5	0	0	θ_5: $+/-135°$
6	0	θ_6	0	0	θ_6: $+/-170°$
$a_2 = 17.000$	$a_3 = 0.75$				
$d_3 = 4.937$	$d_4 = 17.000$				

the A matrices, starting at link 6 and working back to the base, for the PUMA arm are

$$U_6 = A_6 \tag{12}$$

$$U_5 = A_5 U_6 = \begin{bmatrix} C_5C_6 & -C_5S_6 & S_5 & 0 \\ S_5C_6 & -S_5S_6 & -C_5 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$U_4 = A_4 U_5 = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & 0 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & 0 \\ -S_5C_6 & S_5S_6 & C_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

where S_{23} refers to $\sin(\theta_2 + \theta_3)$ and C_{23} refers to $\cos(\theta_2 + \theta_3)$,

$$U_1 = A_1 U_5 = T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

where

$$n_x = C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - S_1[S_4C_5C_6 + C_4S_6] \tag{18}$$

$$n_y = S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + C_1[S_4C_5C_6 + C_4S_6] \tag{19}$$

$$n_z = -S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6 \tag{20}$$

$$o_x = C_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] - S_1[-S_4C_5S_6 + C_4C_6] \tag{21}$$

$$o_y = S_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] + C_1[-S_4C_5S_6 + C_4C_6] \tag{22}$$

$$o_z = S_{23}(C_4C_5S_6 + S_4C_6) + C_{23}S_5S_6 \tag{23}$$

$$a_x = C_1(C_{23}C_4S_5 + S_{23}C_5) - S_1S_4S_5 \tag{24}$$

$$a_y = S_1(C_{23}C_4S_5 + S_{23}C_5) + C_1S_4S_5 \tag{25}$$

$$a_z = -S_{23}C_4S_5 + C_{23}C_5 \tag{26}$$

$$p_x = C_1(d_4S_{23} + a_3C_{23} + a_2C_2) - S_1d_3 \tag{27}$$

$$p_y = S_1(d_4S_{23} + a_3C_{23} + a_2C_2) + C_1d_3 \tag{28}$$

$$p_z = -(-d_4C_{23} + a_3S_{23} + a_2S_2). \tag{29}$$

In order to compute the right hand three columns of T_6, we require 12 transcendental function calls, 34 multiplies, and 16 additions. The first column of T_6 can be obtained as the vector cross product of the second and third columns.

If the joint coordinates are given, the position and orientation of the hand are obtained by evaluating these equations to obtain T_6. The position and orientation of a tool with respect to a base coordinate frame can now be obtained from (5).

SOLUTION

In order to control the manipulator, we are interested in the reverse problem, that is, given X in (5), what are the corresponding joint coordinates?

We may first obtain T_6 from (5) as

$$T_6 = Z^{-1} * X * E^{-1} \tag{30}$$

and then the traditional approach is to solve the matrix equation

$$T_6 = A_1 * A_2 * A_3 * A_4 * A_5 * A_6 \tag{31}$$

$$U_3 = A_3 U_4 = \begin{bmatrix} C_3(C_4C_5C_6 - S_4S_6) - S_3S_5C_6 & -C_3(C_4C_5S_6 + S_4C_6) + S_3S_5S_6 & C_3C_4S_5 + S_3C_5 & d_4S_3 + a_3C_3 \\ S_3(C_4C_5C_6 - S_4S_6) + C_3S_5C_6 & -S_3(C_4C_5S_6 + S_4C_6) - C_3S_5S_6 & S_3C_4S_5 - C_3C_5 & -d_4C_3 + a_3S_3 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$U_2 = A_2 U_3 =$$

$$\begin{bmatrix} C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6 & -C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6 & C_{23}C_4S_5 + S_{23}C_5 & d_4S_{23} + a_3C_{23}a_2C_2 \\ S_{23}(C_4C_5C_6 - S_4S_6) + C_{23}S_5C_6 & -S_{23}(C_4C_5S_6 + S_4C_6) - C_{23}S_5S_6 & S_{23}C_4S_5 - C_{23}C_5 & -d_4C_{23} + a_3S_{23} + a_2S_2 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

where T_6 is given numeric values. With numeric values assigned to the elements of T_6, the required values of θ_1, θ_2, θ_3, θ_4, θ_5, and θ_6 can be obtained by simultaneously solving (18)–(29). This approach is difficult for the following reasons: the equations are transcendental; we will need both the sine and cosine in order to determine angles uniquely and accurately; the manipulator exhibits more than one solution for a given position; and we have twelve equations in six unknowns.

There are, however, six other matrix equations obtained by successively premultiplying (31) by the A matrix inverses:

$$A_1^{-1} * T_6 = U_2 \tag{32}$$

$$A_2^{-1} * A_1^{-1} * T_6 = U_3 \tag{33}$$

$$A_3^{-1} * A_2^{-1} * A_1^{-1} * T_6 = U_4 \tag{34}$$

$$A_4^{-1} * A_3^{-1} * A_2^{-1} * A_1^{-1} * T_6 = U_5 \tag{35}$$

$$A_5^{-1} * A_4^{-1} * A_3^{-1} * A_2^{-1} * A_1^{-1} * T_6 = U_6. \tag{36}$$

The matrix elements of the left sides of these equations are functions of the elements of T_6 and of the first $n-1$ joint variables. The matrix elements of the right hand sides are either zero, constants, or functions of the nth to 6th joint variables. As matrix equality implies element by element equality we obtain 12 equations from each matrix equation, that is, one equation for each of the components of the four vectors $\boldsymbol{n}$, $\boldsymbol{o}$, $\boldsymbol{a}$ and $\boldsymbol{p}$. Equating elements of these matrix equations frequently results in equations yielding joint variables explicitly. We will illustrate the various forms of these equations by developing the equations for the PUMA arm.

If we premultipy (31) by A_1^{-1} we obtain

$$A_1^{-1} * T_6 = A_2 * A_3 * A_4 * A_5 * A_6 \tag{37}$$

$$A_1^{-1} * T_6 = U_2. \tag{38}$$

The left side of (38) is given by

$$A_1^{-1} * T_6 = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

The inverse of a homogeneous transformation is simple to obtain (see Appendix I) and the product of these two matrices is

$$A_1^{-1} * T_6 = \begin{bmatrix} f_{11}(n) & f_{11}(o) & f_{11}(a) & f_{11}(p) \\ f_{12}(n) & f_{12}(o) & f_{12}(a) & f_{12}(p) \\ f_{13}(n) & f_{13}(o) & f_{13}(a) & f_{13}(p) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

where

$$f_{11} = C_1 x + S_1 y \tag{41}$$

$$f_{12} = -z \tag{42}$$

$$f_{13} = -S_1 x + C_1 y \tag{43}$$

and x, y, and z refer to components of the vectors given as arguments to f_{11}, f_{12}, and f_{13}, for example

$$f_{11}(n) = C_1 n_x + S_1 n_y. \tag{44}$$

The right side of (38) is obtained from (16) and is given by

$$U_2 = \begin{bmatrix} C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6 & -C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6 & C_{23}C_4S_5 + S_{23}C_5 & d_4S_{23} + a_3C_{23} + a_2C_2 \\ S_{23}(C_4C_5C_6 - S_4S_6) + C_{23}S_5C_6 & -S_{23}(C_4C_5S_6 + S_4C_6) - C_{23}S_5S_6 & S_{23}C_4S_5 - C_{23}C_5 & -d_4C_{23} + a_3S_{23} + a_2S_2 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

All the elements on the right side of (45) are functions of θ_2, θ_{23}, d_4, θ_4, θ_5, and θ_6 except for element 34. We may equate the 34 elements to obtain

$$f_{13}(p) = d_3 \tag{46}$$

or

$$-S_1 p_x + C_1 p_y = -d_3. \tag{47}$$

In order to solve equations of this form we make the following trigonometric substitutions:

$$p_x = r\cos\phi \tag{48}$$

$$p_y = r\sin\phi \tag{49}$$

where

$$r = +\left(p_x^2 + p_y^2\right)^{1/2} \tag{50}$$

$$\phi = \tan^{-1}\left(\frac{p_y}{p_x}\right). \tag{51}$$

As either the numerator or denominator of (51) can be zero we will use the arctangent function of two arguments to obtain values of ϕ. This arctangent function uses the sign of the numerator and denominator to determine the correct quadrant for the resulting angle and is defined over the range $-\pi \leqslant \phi < \pi$. Substituting for p_x and p_y in (47) we obtain

$$S\phi C\theta_1 - C\phi S\theta_1 = d_3/r \tag{52}$$

with

$$0 < d_3/r \leqslant 1.$$

Equation (52) reduces to

$$S(\phi - \theta_1) = d_3/r \tag{53}$$

with

$$0 < \phi - \theta_1 < \pi.$$

We may obtain the cosine as

$$C(\phi - \theta_1) = \pm\sqrt{1 - (d_3/r)^2} \tag{54}$$

where the minus sign corresponds to a left-hand shoulder configuration of the manipulator and the plus sign corresponds to a right-hand shoulder configuration. Finally,

$$\theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right) - \tan^{-1}\frac{d_3}{\pm\sqrt{r^2 - d_3^2}}. \tag{55}$$

Having determined θ_1 the left side of (38) is now defined. Whenever we have the left side of one of (32)–(36) defined, we examine the right side for elements which are a function of individual joint coordinates. In the case of the PUMA arm, as with any arm with two or more joint axes parallel, the T_6 matrix is expressed in terms of sums or differences of the angles relating to the parallel axes. In order to solve the kinematic equations, the sum or difference of the angles must be determined before the angles themselves can be found. In addition the solution for these sums of angles involves the sum of the squares of two equations. Such is the case in order to solve for θ_2 and θ_3. The 14 and 24 elements of (38) are

$$d_4S_{23} + a_3C_{23} + a_2C_2 = C_1p_x + S_1p_y \tag{56}$$

$$-d_4C_{23} + a_3S_{23} + a_2S_2 = -p_z \tag{57}$$

where

$$C_1 p_x + S_1 p_y = f_{11p} \tag{58}$$

$$-p_z = f_{12p}. \tag{59}$$

Squaring, adding, and simplifying:

$$f_{11p}^2 + f_{12p}^2 - d_4^4 - a_3^2 - a_2^2 = 2a_2 d_4 S_3 + 2a_2 a_3 C_3. \tag{60}$$

Since the left side is known and the only variables are S_3 and C_3, this equation is of the form of (47). It can be solved to yield

$$\theta_3 = \arctan\frac{a_3}{-d_4} - \arctan\frac{d}{\pm\sqrt{e-d^2}} \tag{61}$$

where

$$d = f_{11p}^2 + f_{12p}^2 - d_4^2 - a_3^2 - a_2^2 \tag{62}$$

$$e = 4a_2^2 a_3^2 + 4a_2^2 d_4^2 \qquad \text{(constant)}. \tag{63}$$

Evaluating the elements of (33) we obtain

$$\begin{bmatrix} f_{21}(n) & f_{21}(o) & f_{21}(a) & f_{21}(p) - a_2 \\ f_{22}(n) & f_{22}(o) & f_{22}(a) & f_{22}(p) \\ f_{23}(n) & f_{23}(o) & f_{23}(a) & f_{23}(p) \\ 0 & 0 & 0 & 1 \end{bmatrix} = U_3 \tag{64}$$

where

$$f_{21} = C_2(C_1 x + S_1 y) - S_2 z \tag{65}$$

$$f_{22} = -S_2(C_1 x + S_1 y) - C_2 z \tag{66}$$

$$f_{23} = -S_1 x + C_1 y. \tag{67}$$

Since this yields nothing, we evaluate (34) as

$$\begin{bmatrix} f_{31}(n) & f_{31}(o) & f_{31}(a) & f_{31}(p) - a_2 C_3 - a_3 \\ f_{32}(n) & f_{32}(o) & f_{32}(a) & f_{32}(p) + d_3 \\ f_{33}(n) & f_{33}(o) & f_{33}(a) & f_{33}(p) - a_2 S_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U_4 \tag{68}$$

$$f_{31} = C_{23}(C_1 x + S_1 y) - S_{23} z \tag{69}$$

$$f_{32} = -S_1 x + C_1 y \tag{70}$$

$$f_{33} = S_{23}(C_1 x + S_1 y) + C_{23} z \tag{71}$$

equating the 14 and 34 terms we obtain

$$C_{23} f_{11p} - S_{23} p_z = a_2 C_3 + a_3 \tag{72}$$

$$S_{23} f_{11p} + C_{23} p_z = d_4 + a_2 S_3. \tag{73}$$

Since C_{23} and S_{23} are the only variables, we can solve the above equations simultaneously to yield:

$$S_{23} = \frac{w_2 f_{11p} - w_1 p_2}{f_{11p}^2 + p_z^2} \tag{74}$$

$$C_{23} = \frac{w_1 f_{11p} + w_2 p_z}{f_{11p}^2 + p_z^2} \tag{75}$$

where

$$w_1 = a_2 C_3 + a_3 \tag{76}$$

$$w_2 = d_4 + a_2 S_3 \tag{77}$$

therefore

$$\theta_{23} = \arctan\frac{w_2 f_{11p} - w_1 p_z}{w_1 f_{11p} + w_2 p_z} \tag{78}$$

and

$$\theta_2 = \theta_{23} - \theta_3. \tag{79}$$

With the left side of (68) now defined, we check the right side for functions of single variables. The 13 and 23 elements give us equations for the sine and cosine of θ_4 if $\sin(\theta_5)$ is not zero. When $\sin\theta_5 = 0$, $\theta_5 = 0$ and the manipulator becomes degenerate with both the axes of joint 4 and joint 6 aligned. In this state it is only the sum of θ_4 and θ_5 which is significant. If θ_5 is zero we are free to choose any value for θ_4. The current value is frequently assigned:

$$C_4 S_5 = C_{23}(C_1 a_x + S_1 a_y) - S_{23} a_z \tag{80}$$

$$S_4 S_5 = -S_1 a_x + C_1 a_y \tag{81}$$

and

$$\theta_4 = \tan^{-1}\frac{-S_1 a_x + C_1 a_y}{C_{23}(C_1 a_x + S_1 a_y) - S_{23} a_z} \tag{82}$$

if

$$\theta_5 > 0$$

and

$$Q_4 = \theta_4 + 180^\circ \quad \text{if } \theta_5 < 0. \tag{83}$$

Evaluating the elements of (35) we obtain

$$\begin{bmatrix} f_{41}(n) & f_{41}(o) & f_{41}(a) & 0 \\ f_{42}(n) & f_{42}(o) & f_{42}(a) & 0 \\ f_{43}(n) & f_{43}(o) & f_{43}(a) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_5 C_6 & -C_5 S_6 & S_5 & 0 \\ S_5 C_6 & -S_5 S_6 & -C_5 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{84}$$

where

$$f_{41} = C_4[C_{23}(C_1 x + S_1 y) - S_{23} z] + S_4[-S_1 x + C_1 y] \tag{85}$$

$$f_{42} = -S_{23}(C_1 x + S_1 y) - C_{23} z \tag{86}$$

$$f_{43} = -S_4[C_{23}(C_1 x + S_1 y) - S_{23} z] + C_4[-S_1 x + C_1 y]. \tag{87}$$

From the right side of (84), we can then obtain equations for S_5, C_5, S_6 and C_6 by inspection. When both sine and cosine are defined we obtain a unique value for the joint angle. We obtain a value for θ_5 by equating the 13 and 23 elements of (84):

$$S_5 = C_4[C_{23}(C_1 a_x + S_1 a_y) - S_{23} a_z] + S_4[-S_1 a_x + C_1 a_y] \tag{88}$$

$$C_5 = S_{23}(C_1 a_x + S_1 a_y) + C_{23} a_z \tag{89}$$

and obtain θ_5 as

$$\theta_5 = \tan^{-1}\frac{C_4[C_{23}(C_1 a_x + S_1 a_y) - S_{23} a_z] + S_4[-S_1 a_x + C_1 a_y]}{S_{23}(C_1 a_x + S_1 a_y) + C_{23} a_z}. \tag{90}$$

While we have equations for both S_6 and C_6, the equation for S_6 is in terms of elements of the first column which involves the use of the $\boldsymbol{n}$ vector of T_6. The $\boldsymbol{n}$ vector of T_6 is not usually made available as it represents redundant information. It can always be computed by the vector cross product of the $\boldsymbol{o}$ and $\boldsymbol{a}$ vectors. By evaluating the elements of (36) we can obtain equations for S_6 and C_6 as a function of the $\boldsymbol{o}$ vector:

$$\begin{bmatrix} f_{51}(n) & f_{51}(o) & 0 & 0 \\ f_{52}(n) & f_{52}(o) & 0 & 0 \\ f_{53}(n) & f_{53}(o) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{91}$$

where

$$f_{51}=C_5\{C_4[C_{23}(C_1x+S_1y)-S_{23}z]+S_4[-S_1x+C_1y]\}$$
$$+S_5\{-S_{23}(C_1x+S_1y)-C_{23}z\} \quad (92)$$
$$f_{52}=-S_4[C_{23}(C_1x+S_1y)-S_{23}z]+C_4[-S_1x+C_1y] \quad (93)$$
$$f_{53}=S_5\{C_4[C_{23}(C_1x+S_1y)-S_{23}z]+S_4[-S_1x+C_1y]\}$$
$$+C_5\{S_{23}(C_1x+S_1y)+C_{23}z\}. \quad (94)$$

By equating the 12 and 22 elements we obtain expressions for S_6 and C_6:

$$S_6=-C_5\{C_4[C_{23}(C_1o_x+S_1o_y)-S_{23}o_z]+S_4[-S_1o_x$$
$$+C_1o_y]\}+S_5\{S_{23}(C_1o_x+S_1o_y)+C_{23}o_z\} \quad (95)$$
$$C_6=-S_4[C_{23}(C_1o_x+S_1o_y)-S_{23}o_z]+C_4[-S_1o_x+C_1o_y]. \quad (96)$$

We obtain an equation for θ_6 as:

$$\theta_6=\tan^{-1}\frac{-C_5\{C_4[C_{23}(C_1o_x+S_1o_y)-S_{23}o_z]+S_4[-S_1o_x+C_1o_y]\}+S_5\{S_{23}(C_1o_x+S_1o_y)+C_{23}o_z\}}{-S_4[C_{23}(C_1o_x+S_1o_y)-S_{23}o_z]+C_4[-S_1o_x+C_1o_y]}. \quad (97)$$

Even in the case where θ_4 is undefined because the manipulator configuration is degenerate, once a value is assigned to θ_4 the correct values for θ_5 and θ_6 are determined by these equations. This solution corresponds to 16 transcendental function calls, 38 multiplies, and 25 additions.

Extension to Other Manipulators

This solution technique, demonstrated with the PUMA manipulator, is valid for kinematically simple manipulators, including all commercially available manipulators for which solutions have been obtained. There are, however, some manipulators whose configurations mandate a slightly different approach to the solution. In the case of a manipulator with an offset at the hand, the problem was inverted and the solution to the kinematic problem to position the base at T_6^{-1} was solved.

There are two common pitfalls in obtaining solutions which should be avoided. One of these is division by the sine or cosine of an angle. The other is not maximizing the use of common expressions. For example, after solving for θ_4 from (82), a possible method to determine θ_5 would be to equate the 2,3 and 3,3 elements of (68). In order to do this, the 2,3 element (S_4S_5) would have to be divided by S_4. This leads to inaccuracy when S_4 is near or equal to zero. By extending the method one more step and premultiplying by A_3^{-1} both problems were avoided.

Summary

We have reviewed the method of assigning coordinate frames to the links of a manipulator. In terms of these coordinate frames the kinematic equations can be developed in a straightforward manner. These equations can be obtained for any manipulator. If the manipulator is kinematically "simple," the solution to the kinematic equations can be obtained in a very straightforward, error-free manner.

Appendix I

Given a homogeneous transformation represented by four vectors $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{n}$, and $\boldsymbol{p}$

$$T=\begin{bmatrix} l_x & m_x & n_x & p_x \\ l_y & m_y & n_y & p_y \\ l_z & m_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (98)$$

Its inverse is given by

$$T^{-1}=\begin{bmatrix} l_x & l_y & l_z & -\boldsymbol{p}\cdot\boldsymbol{l} \\ m_x & m_y & m_z & -\boldsymbol{p}\cdot\boldsymbol{m} \\ n_x & n_y & n_z & -\boldsymbol{p}\cdot\boldsymbol{n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (99)$$

where the terms of the right-hand column are obtained using vector dot product. That (99) represents the inverse is easily verified by forming the matrix product and checking that the result is an identity matrix:

$$T^{-1}*T=\begin{bmatrix} l_x & l_y & l_z & -\boldsymbol{p}\cdot\boldsymbol{l} \\ m_x & m_y & m_z & -\boldsymbol{p}\cdot\boldsymbol{m} \\ n_x & n_y & n_z & -\boldsymbol{p}\cdot\boldsymbol{n} \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} l_x & m_x & n_x & p_x \\ l_y & m_y & n_y & p_y \\ l_x & m_z & n_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (100)$$

$$T^{-1}*T=\begin{bmatrix} \boldsymbol{l}\cdot\boldsymbol{l} & \boldsymbol{l}\cdot\boldsymbol{m} & \boldsymbol{l}\cdot\boldsymbol{m} & 0 \\ \boldsymbol{m}\cdot\boldsymbol{l} & \boldsymbol{m}\cdot\boldsymbol{m} & \boldsymbol{m}\cdot\boldsymbol{n} & 0 \\ \boldsymbol{n}\cdot\boldsymbol{l} & \boldsymbol{n}\cdot\boldsymbol{m} & \boldsymbol{n}\cdot\boldsymbol{n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (101)$$

As the three vectors $\boldsymbol{l}$, $\boldsymbol{m}$, and $\boldsymbol{n}$ are orthogonal we have

$$\boldsymbol{l}\cdot\boldsymbol{l}=\boldsymbol{m}\cdot\boldsymbol{m}=\boldsymbol{n}\cdot\boldsymbol{n}=1 \quad (102)$$

and

$$\boldsymbol{l}\cdot\boldsymbol{m}=\boldsymbol{l}\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\boldsymbol{m}=0 \quad (103)$$

and thus (101) reduces to an identity matrix.

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