Robotics 1

Trajectory planning in Cartesian space

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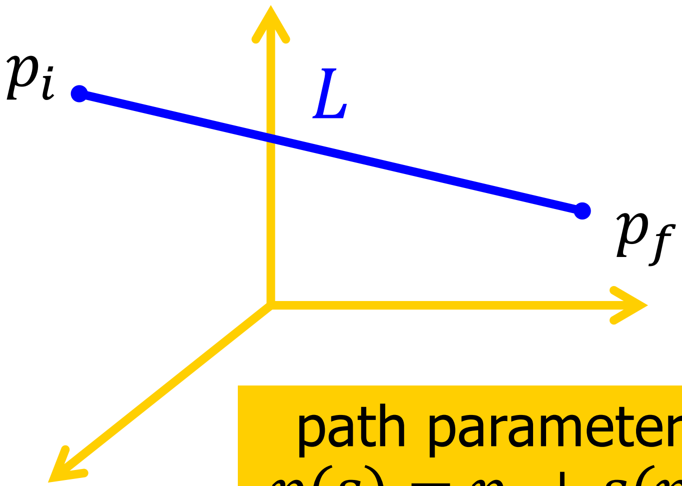
Trajectories in Cartesian space

- in general, the trajectory planning methods proposed in the joint space can be applied also in the Cartesian space
  - consider independently each component of the task vector (i.e., a position or an angle of a minimal representation of orientation)
- however, when planning a trajectory for the three orientation angles, the resulting global motion cannot be intuitively visualized in advance
- if possible, we still prefer to plan Cartesian trajectories separately for position and orientation
- the number of knots to be interpolated in the Cartesian space is typically low (e.g., 2 knots for a PTP motion, 3 if a "via point" is added) ⇒ use simple interpolating paths, such as straight lines, arc of circles, …

Planning a linear Cartesian path
(position only)



GIVEN
$p_i, p_f \in \mathbb{R}^3$; $v_i, v_f \in \mathbb{R}$ (typically = 0);
bounds $v_{max}, a_{max} \in \mathbb{R}^+$

$$L = \|p_f - p_i\|$$

$$\frac{p_f - p_i}{\|p_f - p_i\|} =$$ unit vector of directional cosines of the line

path parameterization
$$p(s) = p_i + s(p_f - p_i)$$

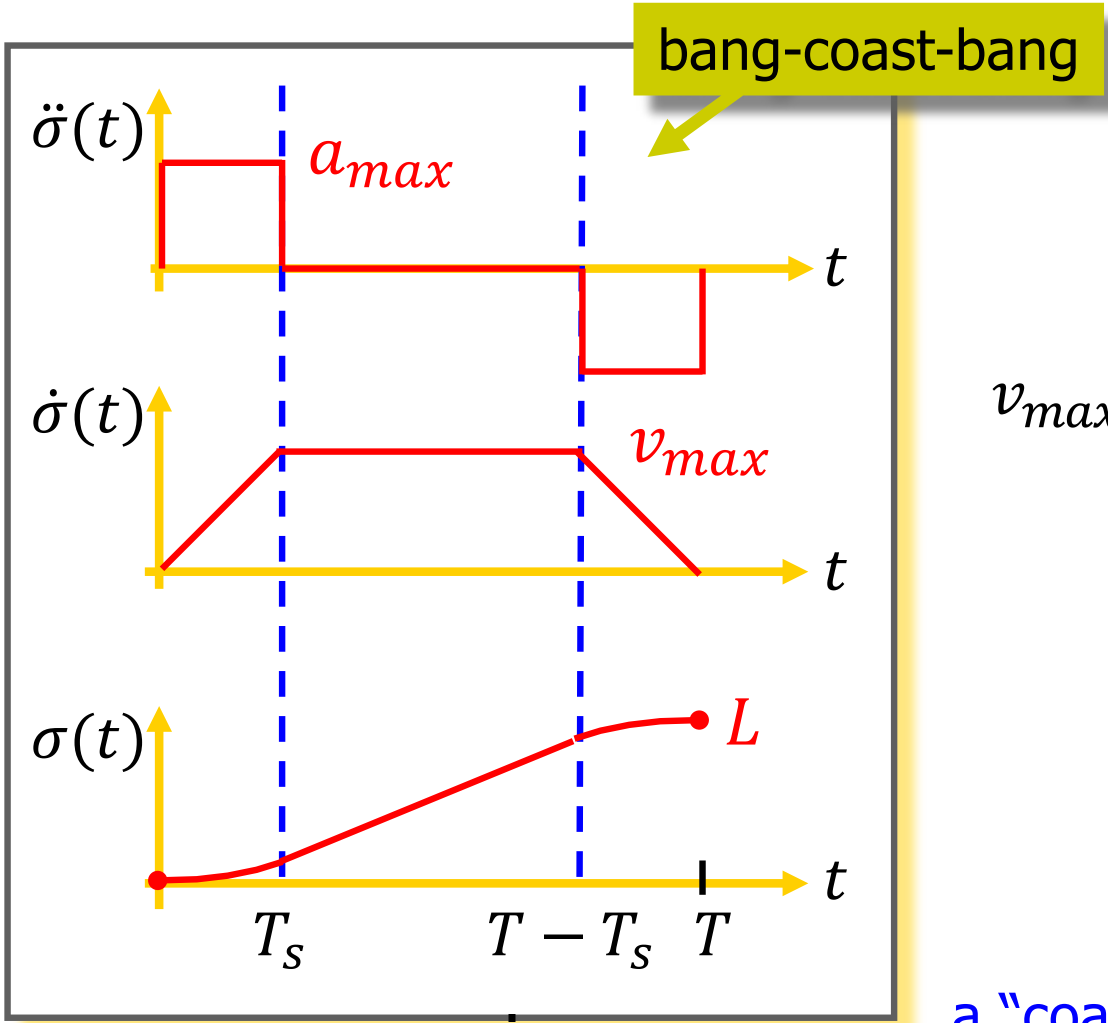
$s \in [0,1]$

may also use $s = \sigma / L$, where $\sigma \in [0, L]$ is the arc length (gives the current length of the path)

$$\dot{p}(s) = \frac{dp}{ds}\dot{s} = (p_f - p_i)\,\dot{s} = \frac{p_f - p_i}{L}\,\dot{\sigma}$$

$$\ddot{p}(s) = \cancel{\frac{d^2 p}{ds^2}}\dot{s}^2 + \frac{dp}{ds}\ddot{s} = (p_f - p_i)\,\ddot{s} = \frac{p_f - p_i}{L}\,\ddot{\sigma}$$

Timing law with trapezoidal speed - 1



given*: L, v_{max}, a_{max}
find: T_s, T

$$v_{max}\,(T - T_s) = L$$

= area of the speed profile

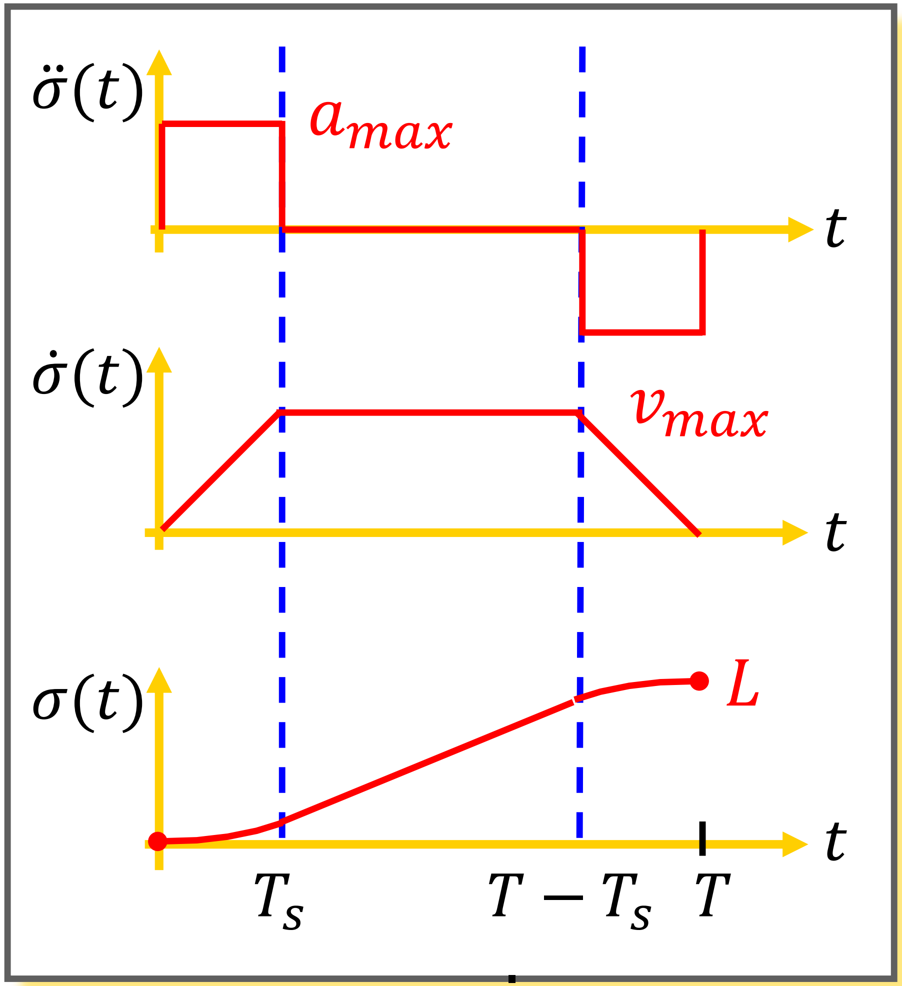
$$T_s = \frac{v_{max}}{a_{max}}$$

$$T = \frac{L a_{max} + v_{max}^2}{a_{max} v_{max}}$$

a "coast" phase exists iff $L > v_{max}^2 / a_{max}$

* = other input data combinations are possible (see textbook)

Timing law with trapezoidal speed - 2

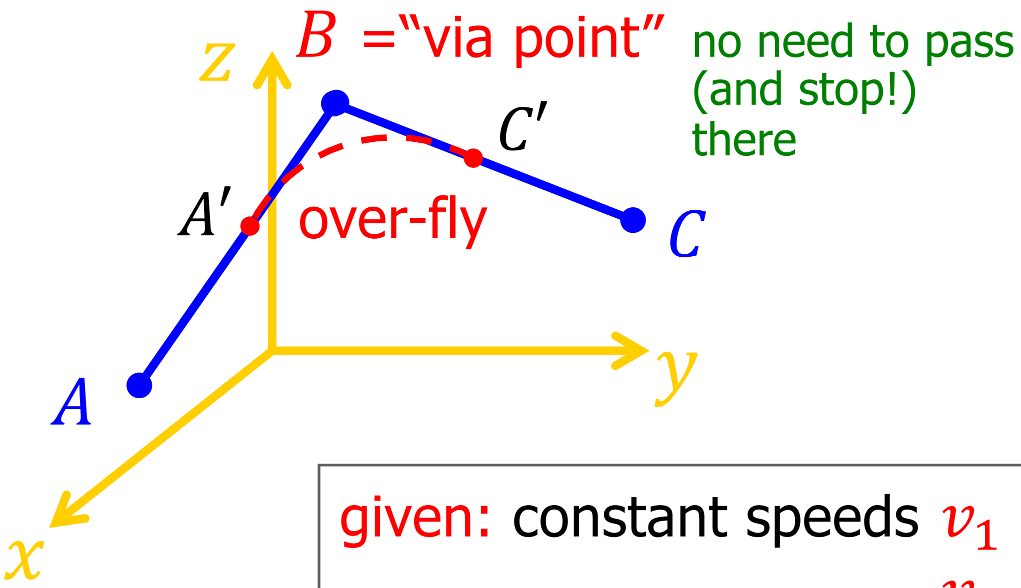


$$\sigma(t) = \begin{cases} \dfrac{a_{max} t^2}{2}, & t \in [0, T_s] \\ v_{max} t - \dfrac{v_{max}^2}{2\, a_{max}}, & t \in [T_s, T - T_s] \\ -\dfrac{a_{max}(t-T)^2}{2} + v_{max} T - \dfrac{v_{max}^2}{a_{max}}, & t \in [T - T_s, T] \end{cases}$$

discontinuous acceleration profile!
if needed, use for instance a
a rest-to-rest quintic polynomial timing

can be used also in the joint space!

Concatenation of linear paths



$$\frac{B-A}{\|B-A\|} = K_{AB}$$

$$\frac{C-B}{\|C-B\|} = K_{BC}$$

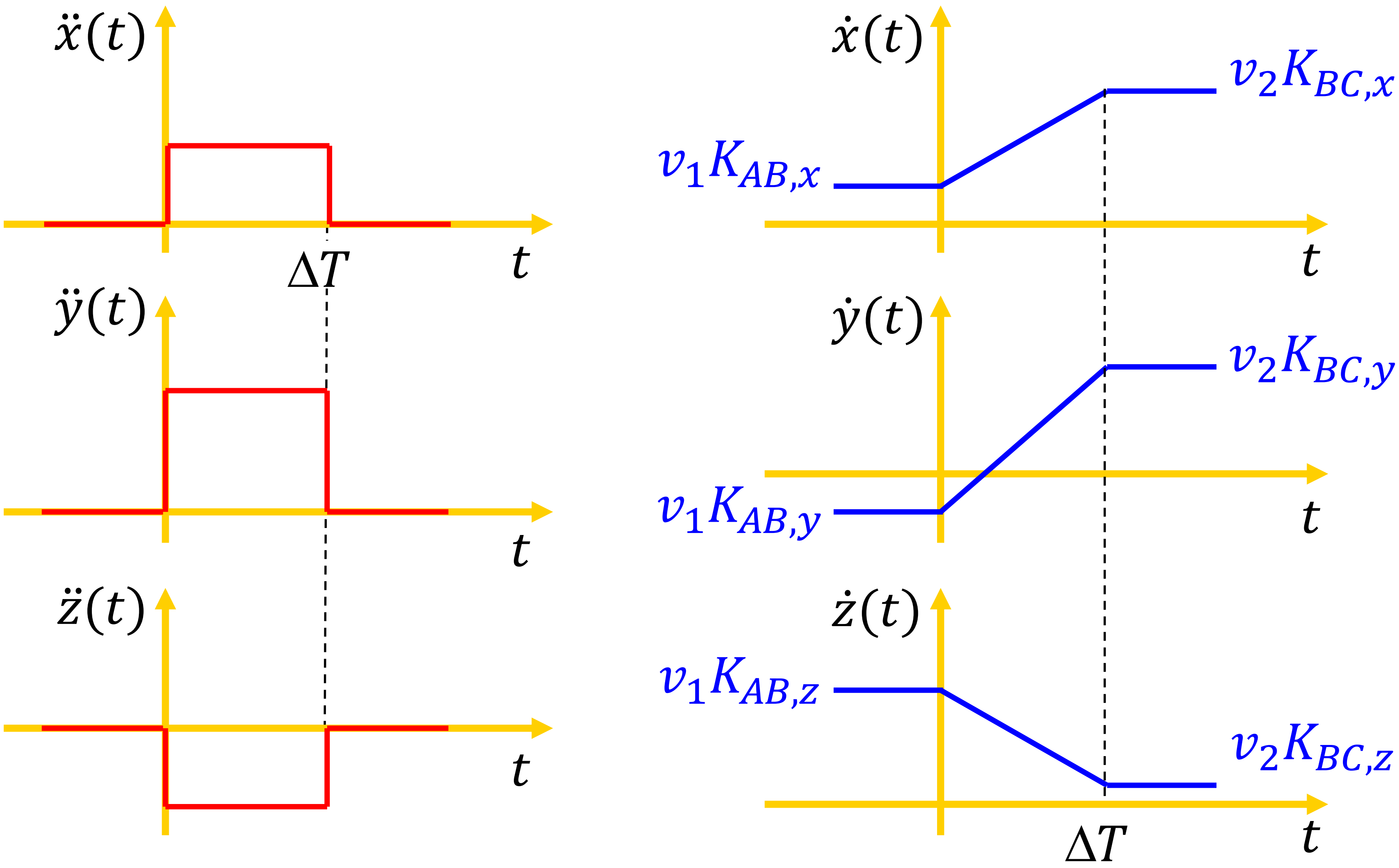
unit vectors of directional cosines

given: constant speeds v_1 on linear path AB
v_2 on linear path BC
desired transition: with constant acceleration for a time ΔT

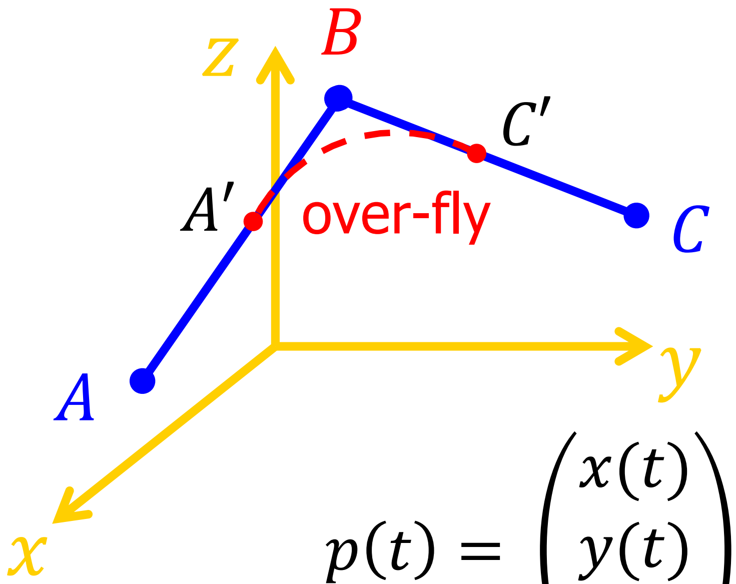
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad t \in [0, \Delta T] \quad \text{(transition starts at } t = 0\text{)}$$

note: during over-fly, the path remains always in the plane specified by the two lines intersecting at B (in essence, it is a planar problem)

Time profiles on components



Timing law during transition



$$\frac{B-A}{\|B-A\|} = K_{AB}$$

$$\frac{C-B}{\|C-B\|} = K_{BC}$$

unit vectors of directional cosines

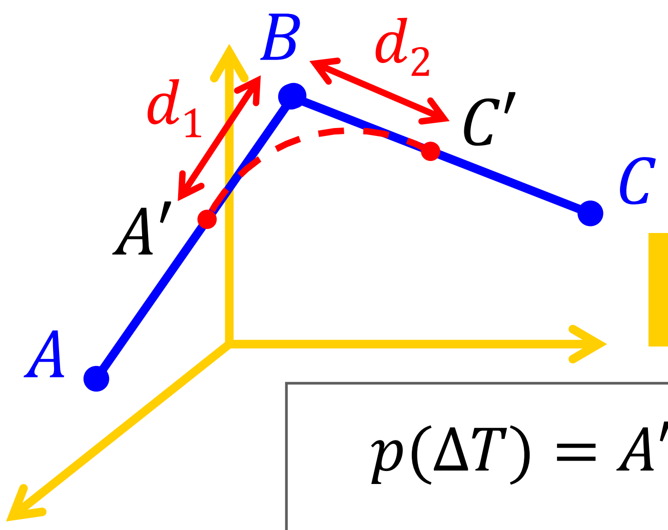
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad t \in [0, \Delta T] \quad \text{(transition starts at } t = 0\text{)}$$

$$\ddot{p}(t) = (v_2 K_{BC} - v_1 K_{AB})/\Delta T \quad \xrightarrow{\int} \quad \dot{p}(t) = v_1 K_{AB} + (v_2 K_{BC} - v_1 K_{AB})t/\Delta T$$

$$\xrightarrow{\int} \quad p(t) = A' + v_1 K_{AB} t + (v_2 K_{BC} - v_1 K_{AB}) t^2/(2\Delta T)$$

thus, we obtain a parabolic blending
(see textbook for this same approach in the joint space)

Solution
(various options)



$$B - A' = d_1 K_{AB}$$
$$C' - B = d_2 K_{BC} \quad (1)$$

$$p(t) = A' + v_1 K_{AB} t + (v_2 K_{BC} - v_1 K_{AB}) t^2/(2\Delta T)$$

$$p(\Delta T) = A' + (\Delta T/2)(v_1\, KAB + v_2 K_{BC}) = C'$$

$\Rightarrow$ $$-B + A' + (\Delta T/2)\,(v_1 K_{AB} + v_2 K_{BC}) = C' - B$$

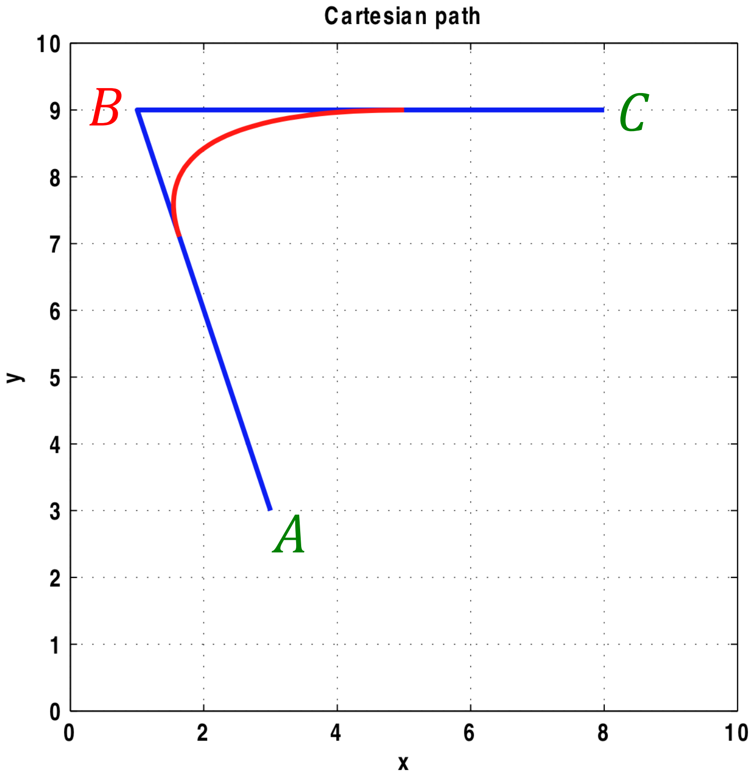
(1) $\Rightarrow$ $$d_1 K_{AB} + d_2 K_{BC} = (\Delta T/2)(v_1 K_{AB} + v_2 K_{BC})$$

$\Rightarrow$ $$d_1 = v_1 \Delta T/2 \qquad d_2 = v_2 \Delta T/2$$

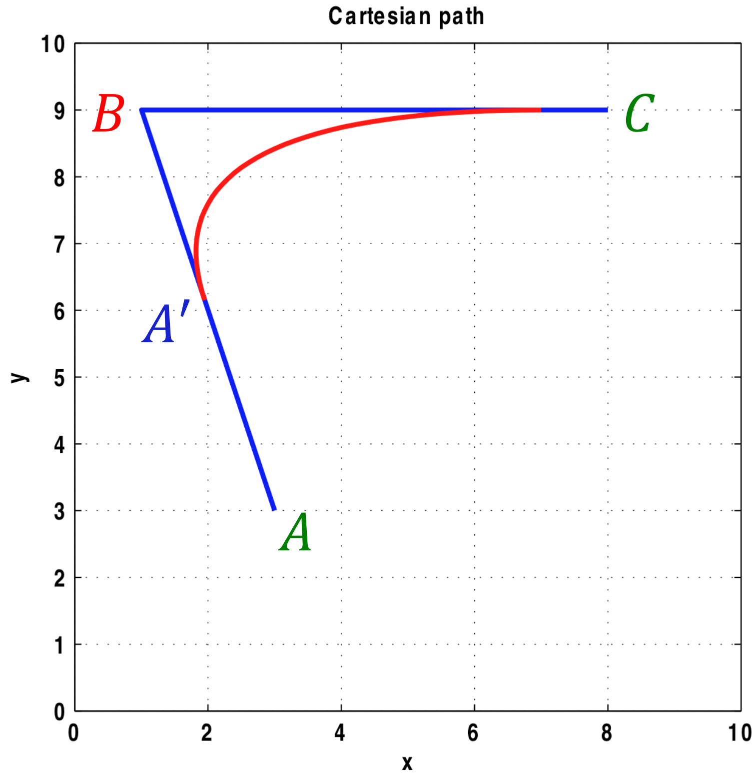
by choosing, e.g., d_1 (namely A') $\rightarrow$ $\Delta T = 2 d_1/v_1 \Rightarrow d_2 = d_1 v_2/v_1$

A numerical example

- transition: $A = (3,3)$ to $C = (8,9)$ via $B = (1,9)$, with speed from $v_1 = 1$ to $v_2 = 2$
- exploiting two options for solution (resulting in different paths!)
  - assign transition time: $\Delta T = 4$ (we re-center it here for $t \in [-\Delta T/2, \Delta T/2]$)
  - assign distance from B for departing: $d_1 = 3$ (assign d_2 for landing is handled similarly)



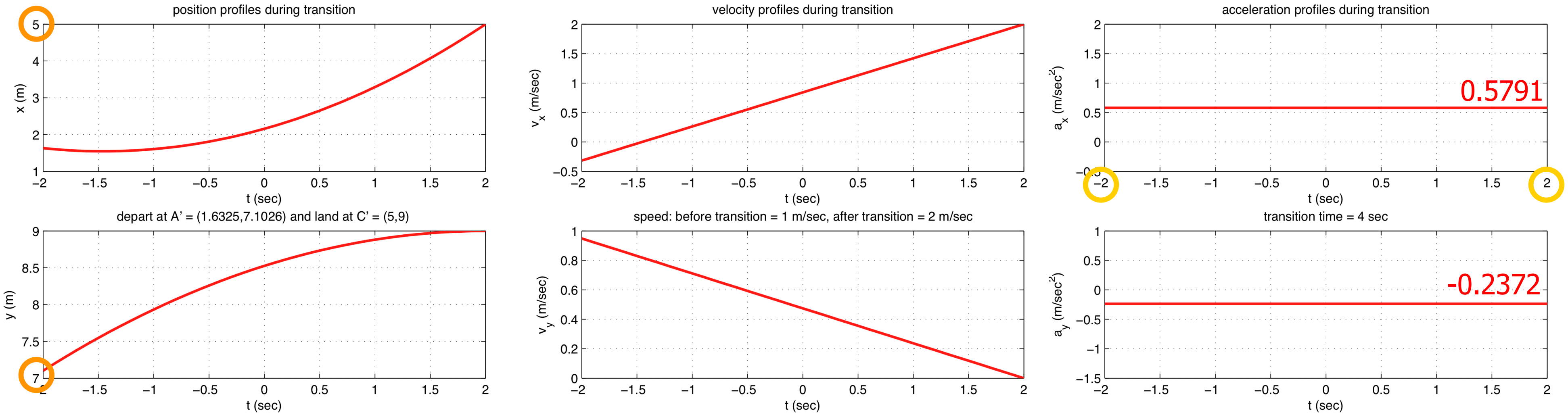
$\Delta T = 4$



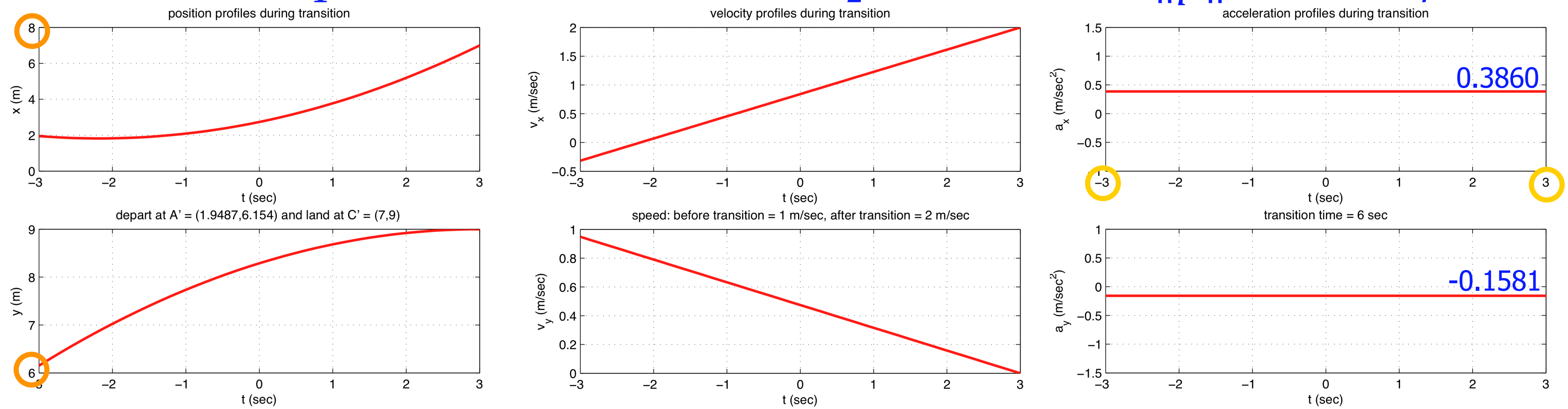
$d_1 = 3$

A numerical example (cont'd)

first option: $\Delta T = 4$ (resulting in $d_1 = 2, d_2 = 4$) $\Rightarrow \|\ddot{p}\| = 0.39$ m/s²

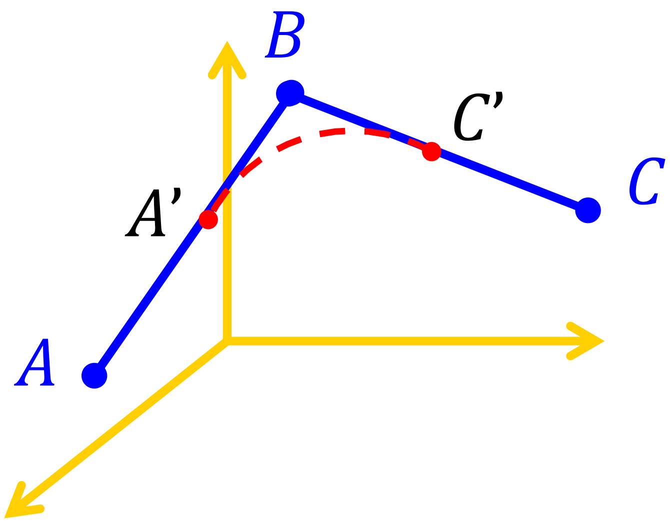


second option: $d_1 = 3$ (resulting in $\Delta T = 6, d_2 = 6$) $\Rightarrow \|\ddot{p}\| = 0.17$ m/s²



actually: similar velocity/acceleration profiles, but with a different time scale!!

Alternative solution
(imposing acceleration)



$$\ddot{p}(t) = (v_2 K_{BC} - v_1 K_{AB})/\Delta T$$

$v_1 = v_2 = v_{max}$ (for simplicity)

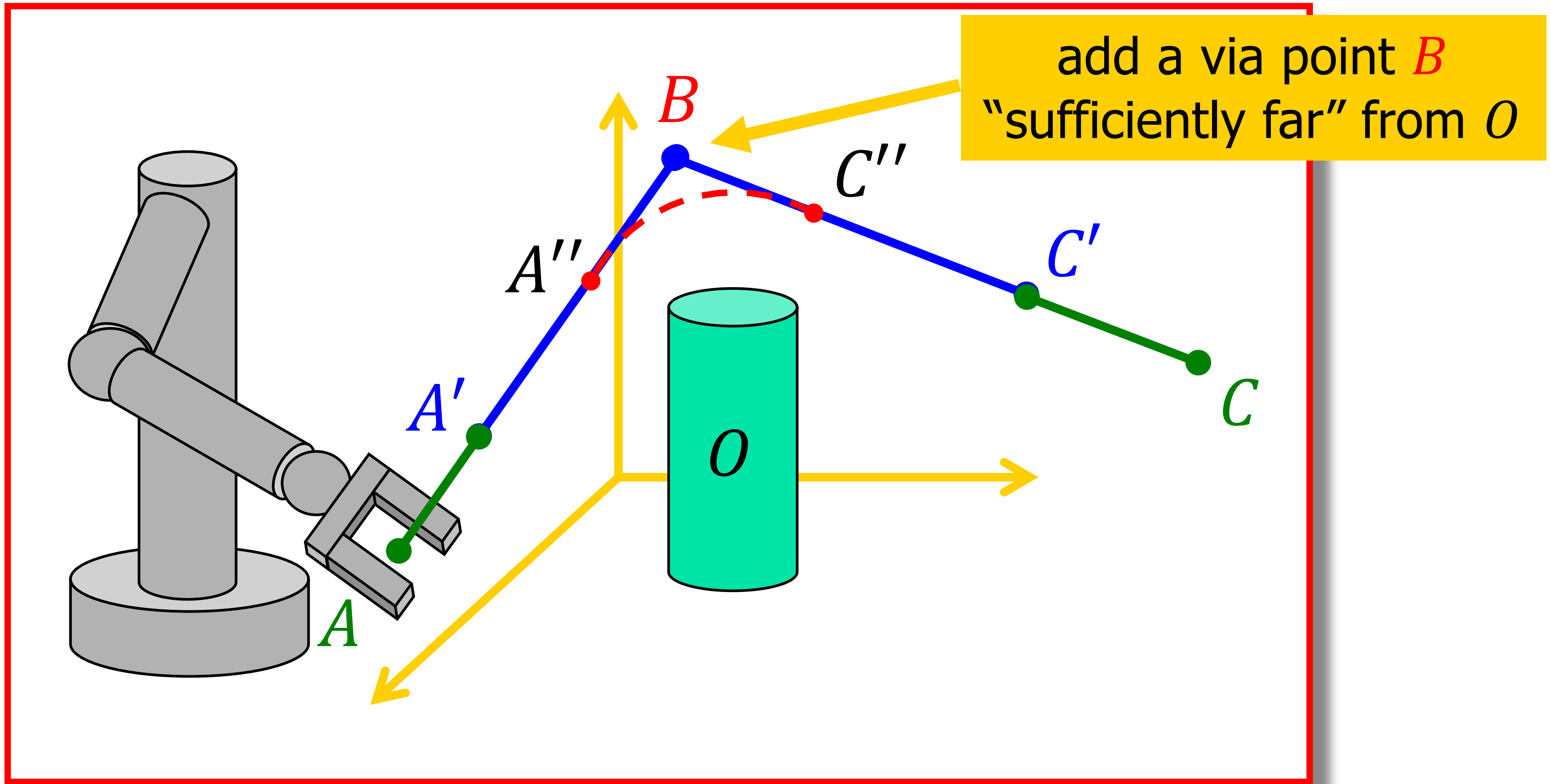
$\|\ddot{p}(t)\| = a_{max}$

$$\Delta T = (v_{max}/a_{max})\, \|K_{BC} - K_{AB}\|$$
$$= (v_{max}/a_{max}) \sqrt{2(1 - K_{BC,x}K_{AB,x} - K_{BC,y}K_{AB,y} - K_{BC,z}K_{AB,z})}$$

then, $d_1 = d_2 = v_{max}\,\Delta T/2$

Application example

plan a Cartesian trajectory from A to C (rest-to-rest)
that avoids the obstacle O, with $a \leq a_{max}$ and $v \leq v_{max}$



on $\overline{AA'} \rightarrow a_{max}$; on $\overline{A'B}$ and $\overline{BC'} \rightarrow v_{max}$; on $\overline{C'C} \rightarrow -a_{max}$;
+ over-fly between A'' e C'' (e.g., with a_{max} in norm)

Other Cartesian paths

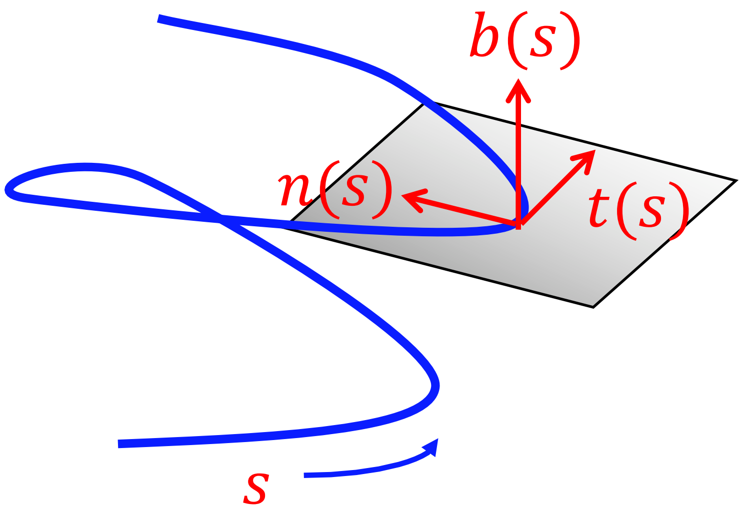
- circular path through 3 points in 3D (often built-in feature)
- linear path for the end-effector with constant orientation
- in robots with spherical wrist: planning may be decomposed into a path for wrist center and one for E-E orientation, with a common timing law
- though more complex in general, it is often convenient to parameterize the Cartesian geometric path $p(s)$ in terms of its arc length (e.g., with $s = R\theta$ for circular paths), so that the following hold:
  - velocity $\dot{p} = dp/dt = (dp/ds)(ds/dt) = p'\dot{s}$
    - p' = unit vector ($\|\cdot\| = 1$) tangent to the path ⇒ tangent direction $t(s)$
    - $\dot{s} \geq 0$ is the absolute value of the tangential velocity (= speed)
  - acceleration $\ddot{p} = (d^2p/ds^2)(ds/dt)^2 + (dp/ds)(d^2s/dt^2) = p''\dot{s}^2 + p'\ddot{s}$
    - $\|p''\|$ = curvature $\kappa(s)$ ($= 1$/radius of curvature)
    - $p''\dot{s}^2$ = centripetal acceleration ⇒ normal direction $n(s) \perp$ to the path, on the osculating plane; the binormal direction is $b(s) = t(s) \times n(s)$
    - $\ddot{s}$ = scalar value (with any sign) of the tangential acceleration

Definition of Frenet frame

- for a smooth and non-degenerate curve $p(s) \in \mathbb{R}^3$, parameterized by s (not necessarily its arc length), one can define a reference frame as shown

$$p' = dp/ds \quad p'' = d^2p/ds^2$$

derivatives w.r.t. the parameter s



unit tangent vector

$$t(s) = p'(s)/\|p'(s)\|$$

unit normal vector ($\in$ osculating plane)

$$n(s) = t'(s)/\|t'(s)\|$$
$$= p'(s)\times(p''(s)\times p'(s))/(\|p'(s)\| \cdot \|p''(s)\times p'(s)\|)$$

unit binormal vector

$$b(s) = t(s) \times n(s)$$
$$= p'(s)\times p''(s)/\|p'(s)\times p''(s)\|$$

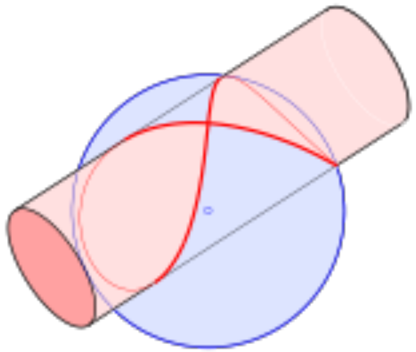
- general expressions of path curvature and torsion (at a path point $p(s)$)

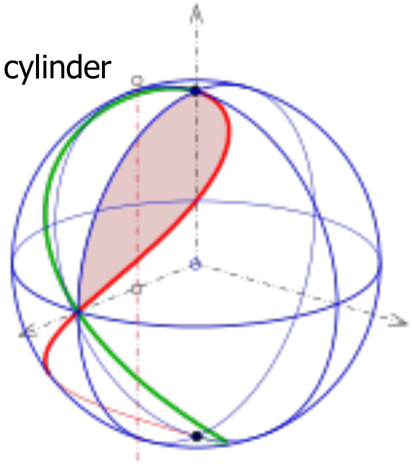
$$\kappa(s) = \|p'(s) \times p''(s)\|/\|p'(s)\|^3$$
$$\tau(s) = [p'(s) \cdot (p''(s) \times p'''(s))]/\|p'(s) \times p''(s))\|^2$$

Examples of paths with Frenet frame

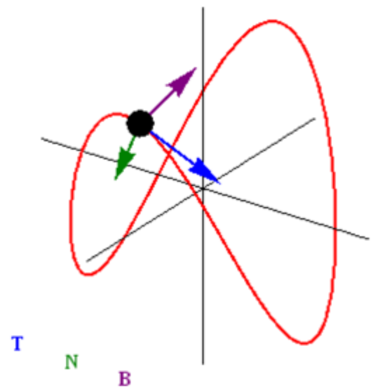
Viviani curve

= intersection of a sphere with a tangent cylinder





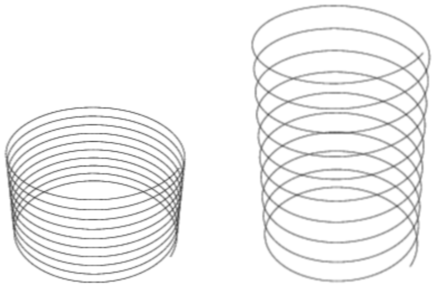
$$x = r\cos^2 s$$
$$y = r\cos s\sin s$$
$$z = r\sin s$$
$$s \in [-\pi/2, \pi/2]$$
$$x = r\cos^2 s$$
$$y = -r\cos s\sin s$$
$$z = -r\sin s$$



By Ag2gaeh - https://commons.wikimedia.org/w/index.php?curid=81698760

By Gonfer https://commons.wikimedia.org/w/index.php?curid=18558097

Helix curve (right handed)



$$x = r\cos s$$
$$y = r\sin s$$
$$z = h\,s$$
$$s \in [0, 2\pi]$$

⇒

$$\kappa = \frac{r}{r^2 + h^2}$$
$$\tau = \frac{h}{r^2 + h^2}$$

By Goldencako - https://commons.wikimedia.org/w/index.php?curid=7519084

Exercise

given the path $p(s) = \begin{pmatrix} 6s+2 \\ 5s^2 \\ -8s \end{pmatrix}, \quad s \in [0,1]$ ⇒

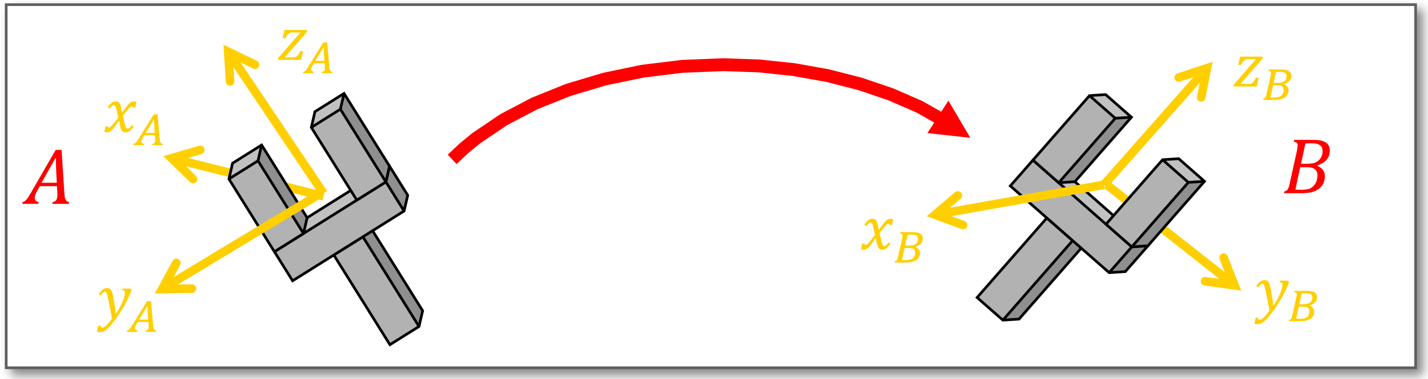
a) define the Frenet frame $\{t(s), n(s), b(s)\}$

b) compute the curvature $\kappa(s)$ and the torsion $\tau(s)$

Optimal trajectories

- for Cartesian robots (e.g., PPP joints)
  1. the straight line joining two position points in the Cartesian space is one path that can be executed in minimum time under velocity/acceleration constraints (but other such paths exist, if (joint) motion is not coordinated)
  2. the optimal timing law is of the bang-coast-bang type in acceleration (in this special case, also in terms of motor torques)
- for articulated robots (with at least one R joint)
  - 1. e 2. are no longer true in general in the Cartesian space, but time-optimality still holds in the joint space when assuming bounds on joint velocity/acceleration
    - straight line paths in the joint space do not correspond to straight line paths in the Cartesian space, and vice-versa
  - bounds on joint acceleration are conservative (though kinematically tractable) w.r.t. actual bounds on motor torques, which involve the full robot dynamics
    - when changing robot configuration/state, different torque values are needed to impose the same joint accelerations …

Planning orientation trajectories



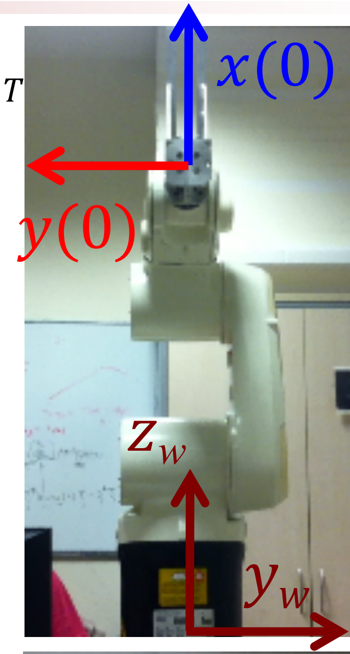
- using minimal representations of orientation (e.g., ZXZ Euler angles ϕ, θ, ψ), we can plan a trajectory for each component independently
  - e.g., a linear path in space ϕ, θ, ψ, with a cubic timing law

    $\Rightarrow$ but poor prediction/understanding of the resulting intermediate orientations
- alternative method based on the axis/angle representation
  - determine the (neutral) axis r and the angle θ_{AB}: $R(r, \theta_{AB}) = R_A^T R_B$ (rotation matrix changing the orientation from A to B $\Rightarrow$ inverse axis-angle problem)
  - plan a timing law $\theta(t)$ for the (scalar) angle interpolating $\theta = 0$ with $\theta = \theta_{AB}$ in time T (with possible constraints/boundary conditions on its time derivatives)
  - $\forall t \in [0, T], R_A R(r, \theta(t))$ specifies the actual end-effector orientation at time t

A complete position/orientation Cartesian trajectory

- initial given configuration $q(0) = (0 \quad \pi/2 \quad 0 \quad 0 \quad 0 \quad 0)^T$
- initial end-effector position $p(0) = (0.540 \quad 0 \quad 1.515)^T$
- initial orientation

$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

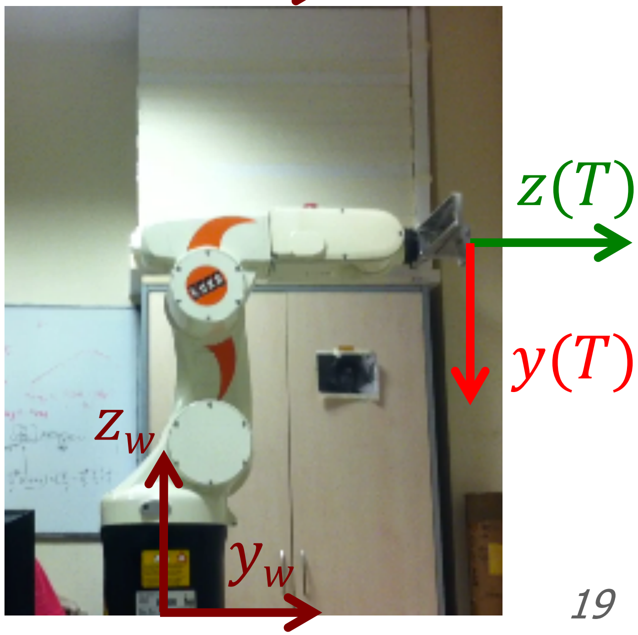


linear path for position ⇓ **axis-angle** method for orientation

- final end-effector position $p(T) = (0 \quad 0.540 \quad 1.515)^T$
- final orientation

$$R(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

- the final configuration is NOT specified a priori



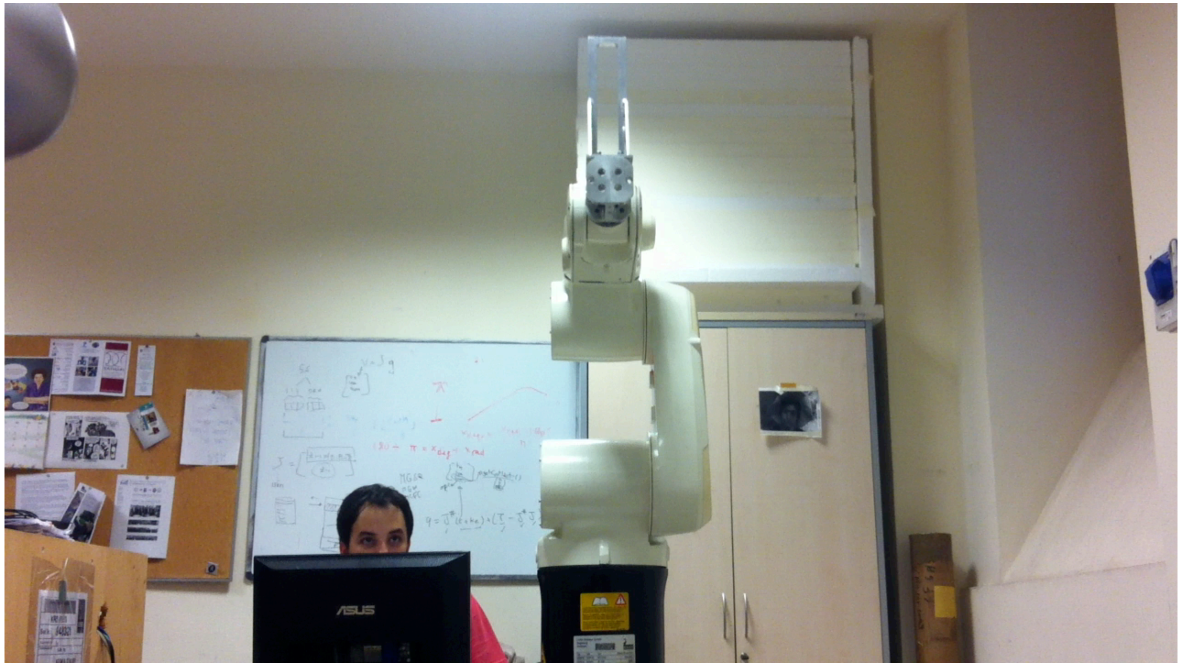
Axis-angle orientation trajectory

video

$$L = \|p_{\text{final}} - p_{\text{init}}\| = 0.763 \text{ [m]}$$

$$\omega = r\dot{\theta} \;\rightarrow\; \|\omega\| = |\dot{\theta}|$$

$$\dot{\omega} = r\ddot{\theta} \;\rightarrow\; \|\dot{\omega}\| = |\ddot{\theta}|$$



coordinated
Cartesian motion
with bounds

$$v_{max} = 0.4 \text{ [m/s]}$$
$$a_{max} = 0.1 \text{ [m/s}^2\text{]}$$
$$\omega_{max} = \pi/4 \text{ [rad/s]}$$
$$\dot{\omega}_{max} = \pi/8 \text{ [rad/s}^2\text{]}$$

triangular
speed profile $\dot{s}(t)$
with minimum
time $T = 5.52$ s
(imposed by the bounds
on linear motion)

$$p(s) = p_{\text{init}} + s(p_{\text{final}} - p_{\text{init}}) = (0.540 \quad 0 \quad 1.515)^T + s(-0.540 \quad 0.540 \quad 0)^T, \qquad s \in [0,1]$$

$$\boldsymbol{s = s(t),\ t \in [0, T]}$$

$$R_{\text{init}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = R_{\text{init}}^T$$

$$R_{\text{final}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$R_{\text{init}}^T R_{\text{final}} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = Rot(r, \theta_{\text{if}})$$

$$r = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \theta_{\text{if}} = \frac{2\pi}{3} \text{ [rad]} (= 120°)$$

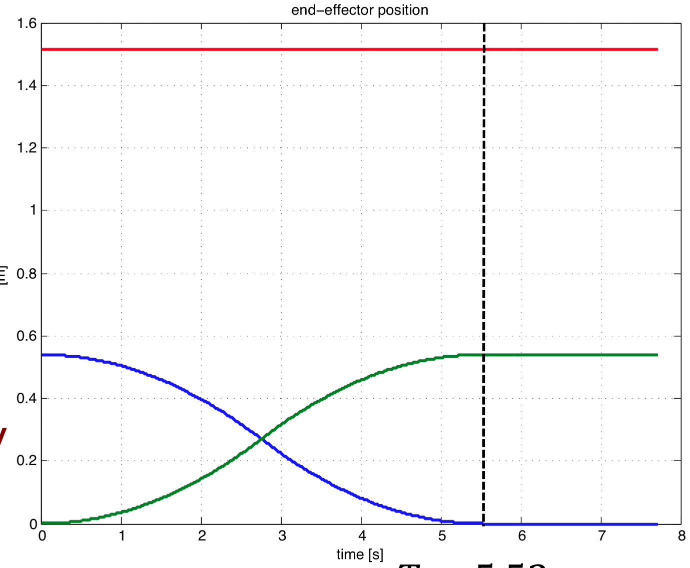
$$R(s) = R_{\text{init}} Rot(r, \theta(s))$$

$$\theta(s) = s\theta_{\text{if}}, \qquad s \in [0,1]$$

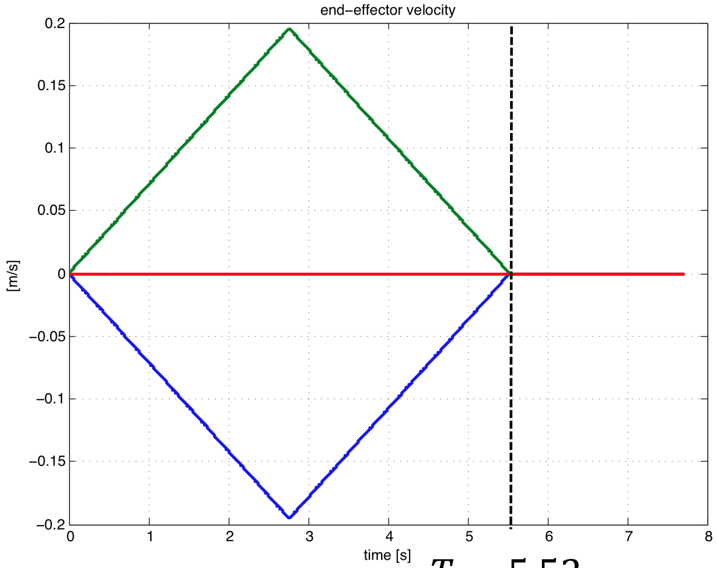
Axis-angle orientation trajectory

x = —— (blue)
y = —— (green)
z = —— (red)

planned motion of Cartesian position and velocity



$T = 5.52$

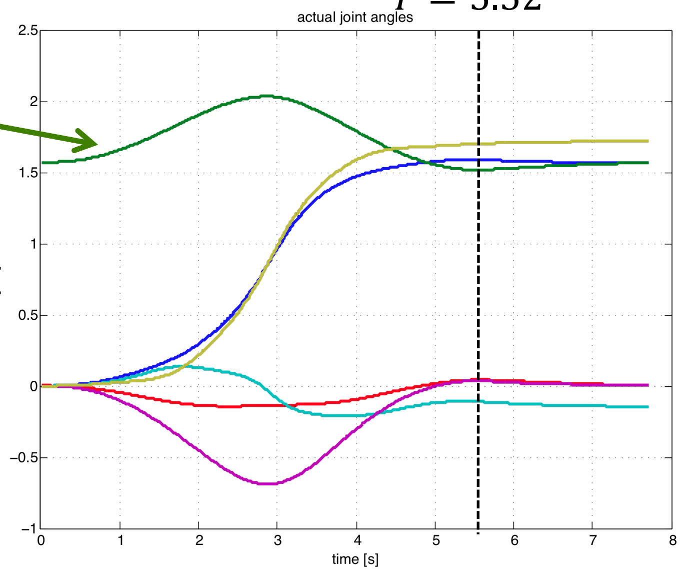


$T = 5.52$

triangular profile for linear speed $T = 5.52$ s

joint 2

actual joint motion



$T = 5.52$

- the robot joint velocity was commanded by inversion of the **geometric** Jacobian
- a **user** program, via KUKA RSI interface at $T_c = 12$ ms sampling time (two-way communication)
- robot motion execution is ≈ what was planned, but only thanks to an external **kinematic control** loop (at **task** level)

Comparison of orientation trajectories
Euler angles vs. axis-angle method

- initial configuration $q(0) = (0 \quad \pi/2 \quad \pi/2 \quad 0 \quad -\pi/2 \quad 0)^T$
- initial end-effector position $p(0) = (0.115 \quad 0 \quad 1.720)^T$
- initial orientation

$$R(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

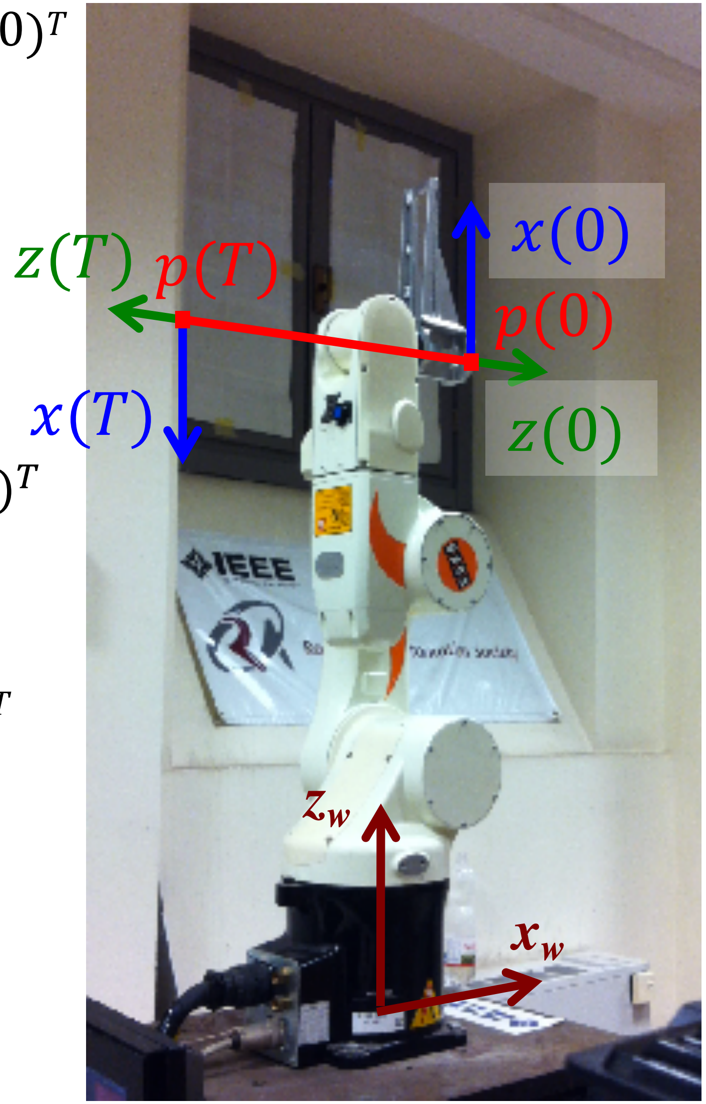
- initial Euler ZYZ (α, β, γ) angles $\phi_{ZYZ}(0) = (0 \quad \pi/2 \quad \pi)^T$

via a **linear path** (for position)

- final end-effector position $p(T) = (-0.172 \quad 0 \quad 1.720)^T$
- final orientation

$$R(T) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- final Euler ZYZ angles $\phi_{ZYZ}(T) = (-\pi \quad \pi/2 \quad 0)^T$



Comparison of orientation trajectories
Euler angles vs. axis-angle method

$$R_{\text{init}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

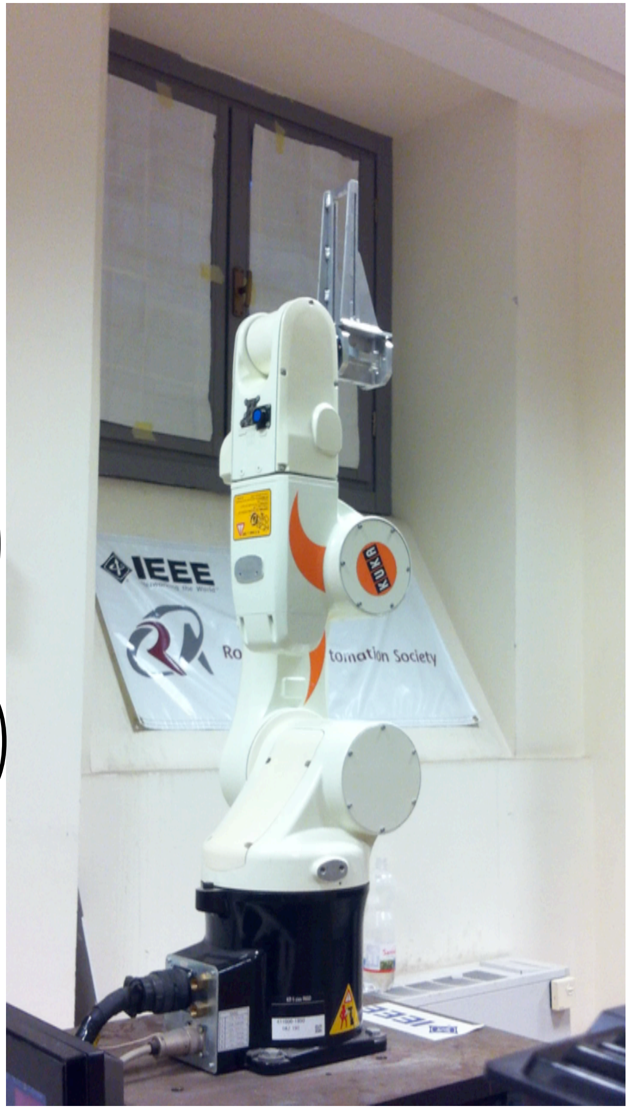
$$\Longrightarrow \phi_{ZYZ,\text{init}} = \begin{pmatrix} 0 \\ \pi/2 \\ \pi \end{pmatrix}$$

$$R_{\text{final}} = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \phi_{ZYZ,\text{final}} = \begin{pmatrix} -\pi \\ \pi/2 \\ 0 \end{pmatrix}$$

(singularity at $\beta = 0$ avoided!)

video

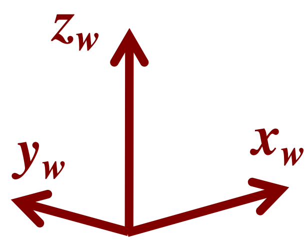


using ZYZ Euler angles

using axis-angle method

$$R_{\text{init}}^T R_{\text{final}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Longrightarrow r = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \theta = \pi$$

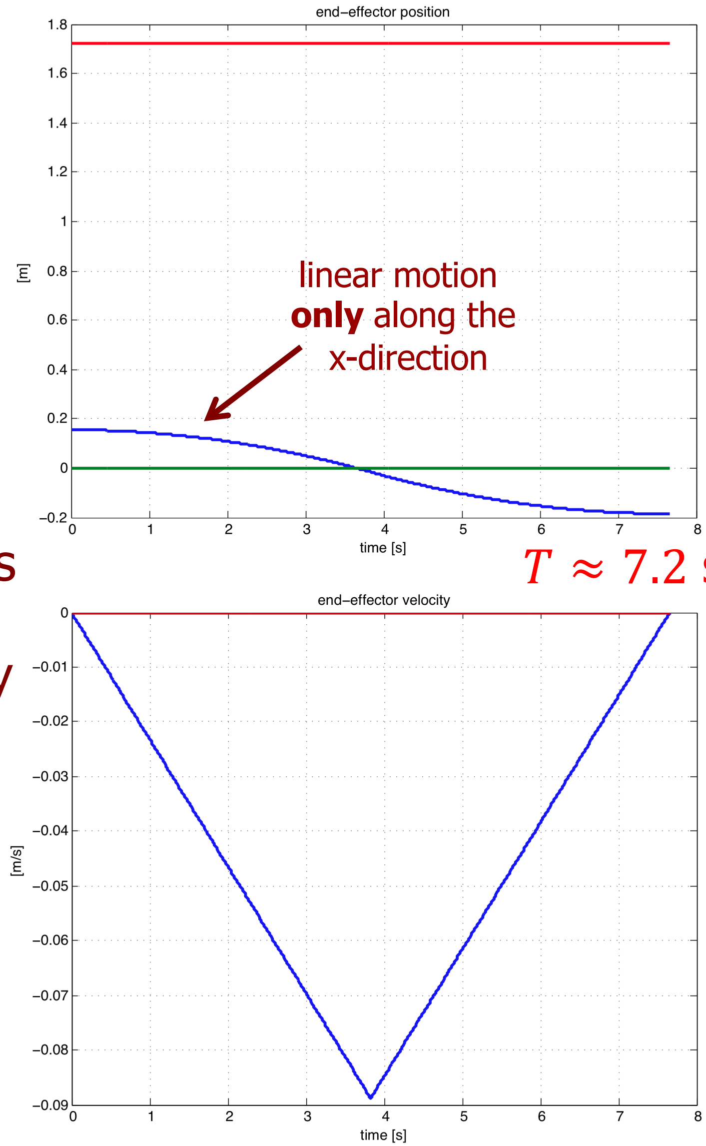


video

Comparison of orientation trajectories
Euler angles vs. axis-angle method

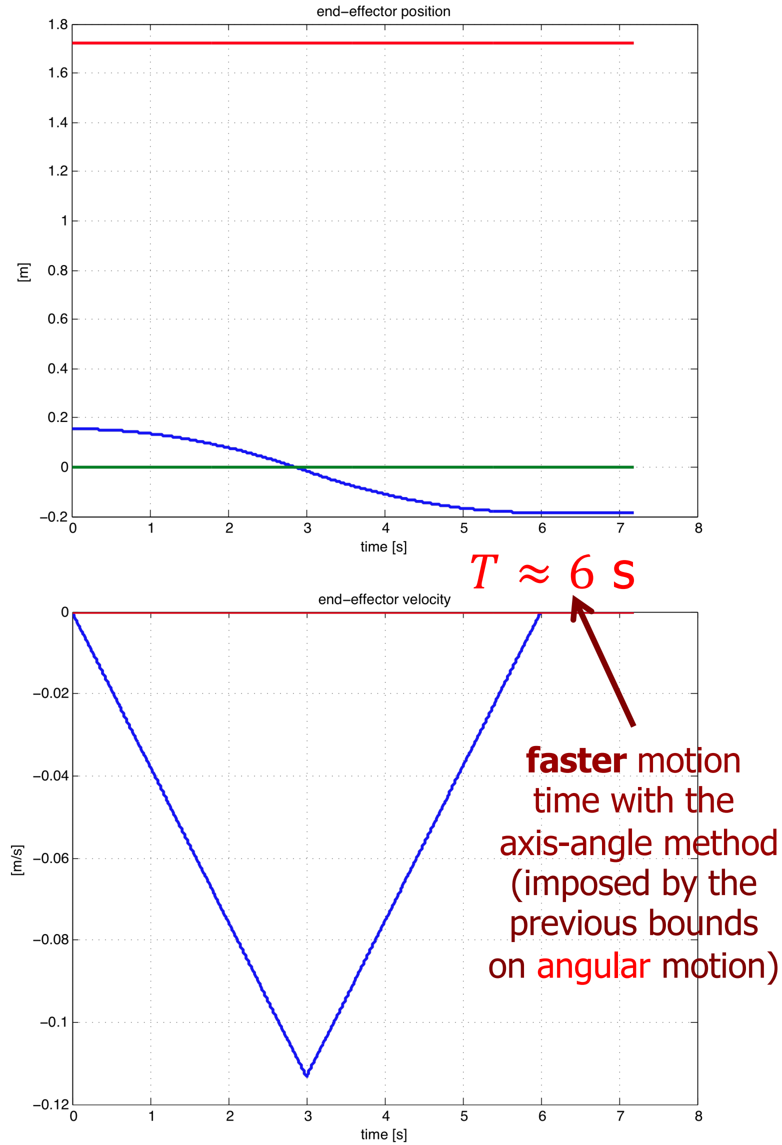
$x =$ —— (blue)
$y =$ —— (green)
$z =$ —— (red)

planned Cartesian components of position and velocity



$T \approx 7.2$ s

using ZYZ Euler angles



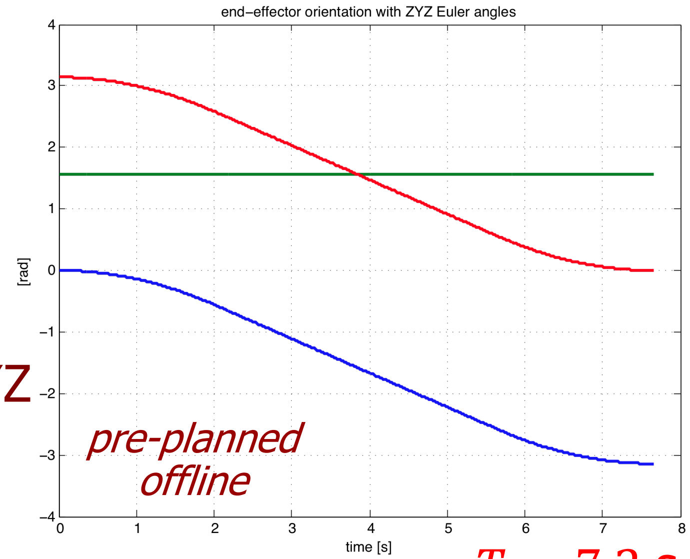
$T \approx 6$ s

using axis-angle method

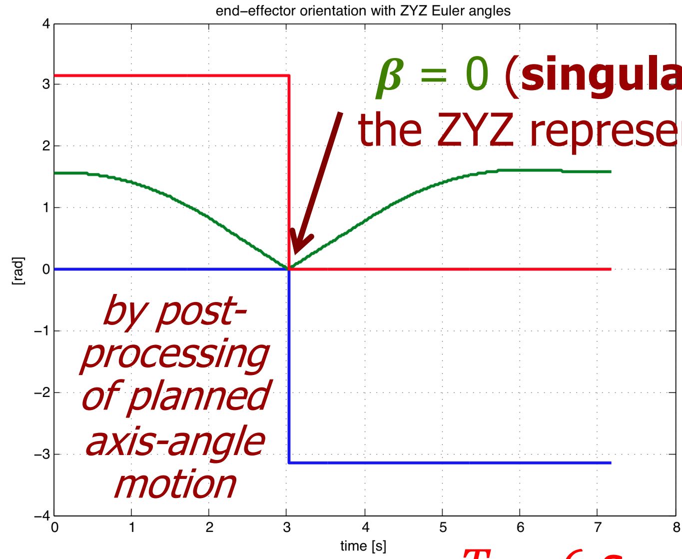
Comparison of orientation trajectories
Euler angles vs. axis-angle method

α = (blue line)
β = (green line)
γ = (red line)

orientation in terms of ZYZ Euler angles

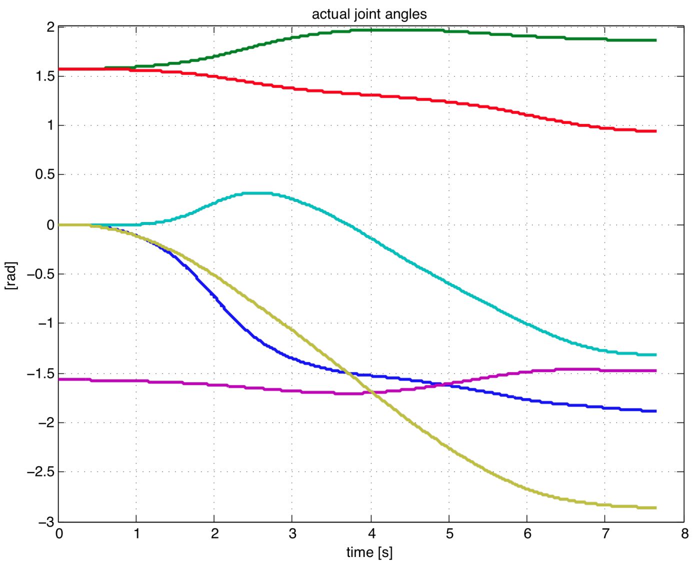


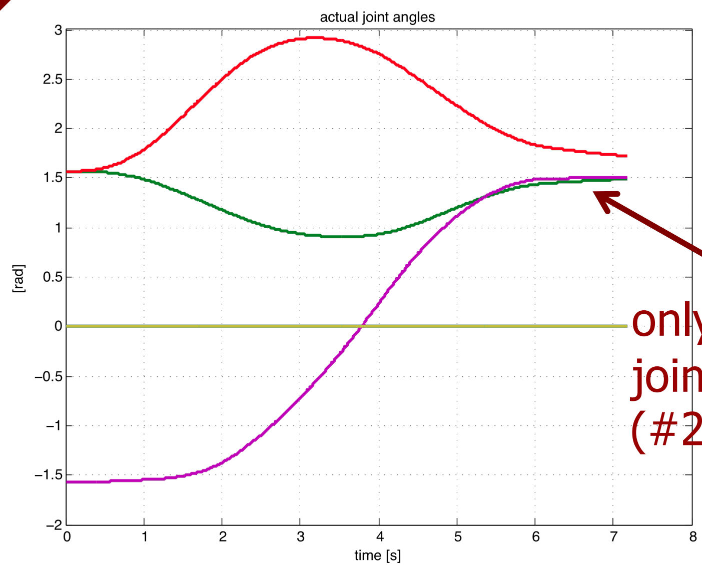
$T \approx 7.2$ s



$T \approx 6$ s

actual joint motion





only **three** joints move (#2,#3,#5)

using ZYZ Euler angles

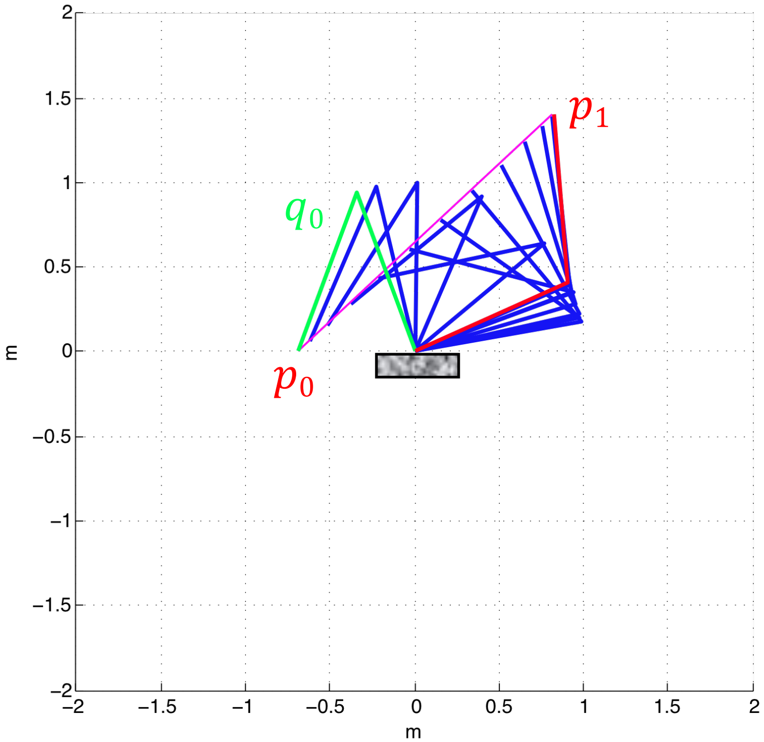
using axis-angle method

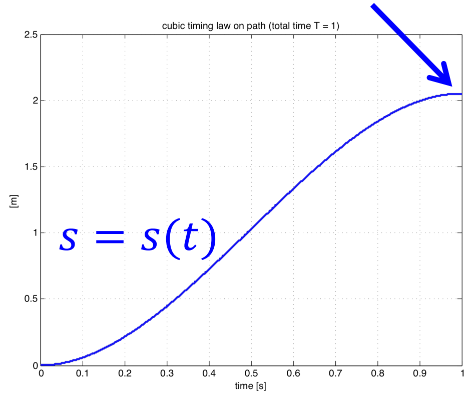
Uniform time scaling

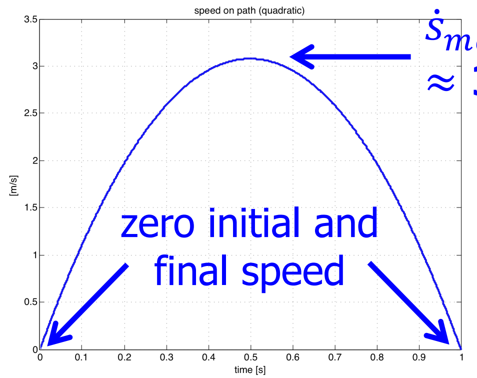
- for a given path $p(s)$ (in joint or Cartesian space) and timing law $s(\tau)$ ($\tau = t/T$, T="motion time"), we need to check if existing bounds v_{max} on (joint) velocity and/or a_{max} on (joint) acceleration are violated or not
  - … unless such constraints have already been taken into account during the trajectory planning, e.g., by using a bang-coast-bang acceleration timing law
- velocity scales **linearly** with motion time
  - $dp/dt = (dp/ds)\,(ds/d\tau) \cdot 1/T$
- acceleration scales **quadratically** with motion time
  - $d^2p/dt^2 = \left((d^2p/ds^2)(ds/d\tau)^2 + (dp/ds)(d^2s/d\tau^2)\right) \cdot 1/T^2$
- if motion is unfeasible, scale (increase) time $T \rightarrow kT$ ($k > 1$), based on the "most violated" constraint (max of the ratios $|v|/v_{max}$ and $|a|/a_{max}$)
- if motion is "too slow" w.r.t. the robot capabilities, decrease T ($k < 1$)
  - in both cases, after scaling, there will be (at least) one instant of saturation (for at least one variable)
  - no need to re-compute motion profiles from scratch!

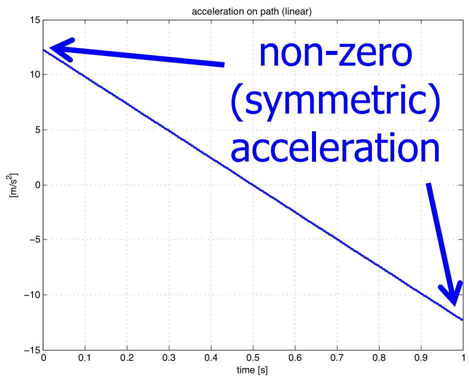
Numerical example - 1

- 2R planar robot with links of unitary length (1 [m])
- linear Cartesian path $p(s)$: $q_0 = (110°, 140°) \Rightarrow p_0 = f(q_0) = (-0.684, 0)$ $\Rightarrow p_1 = (0.816, 1.4)$ [m], with rest-to-rest cubic timing law $s(t)$, $T = 1$ [s]
- joint space bounds: max (absolute) velocity $v_{max,1} = 2, v_{max,2} = 2.5$ [rad/s], max (absolute) acceleration $a_{max,1} = 5, a_{max,2} = 7$ [rad/s²]





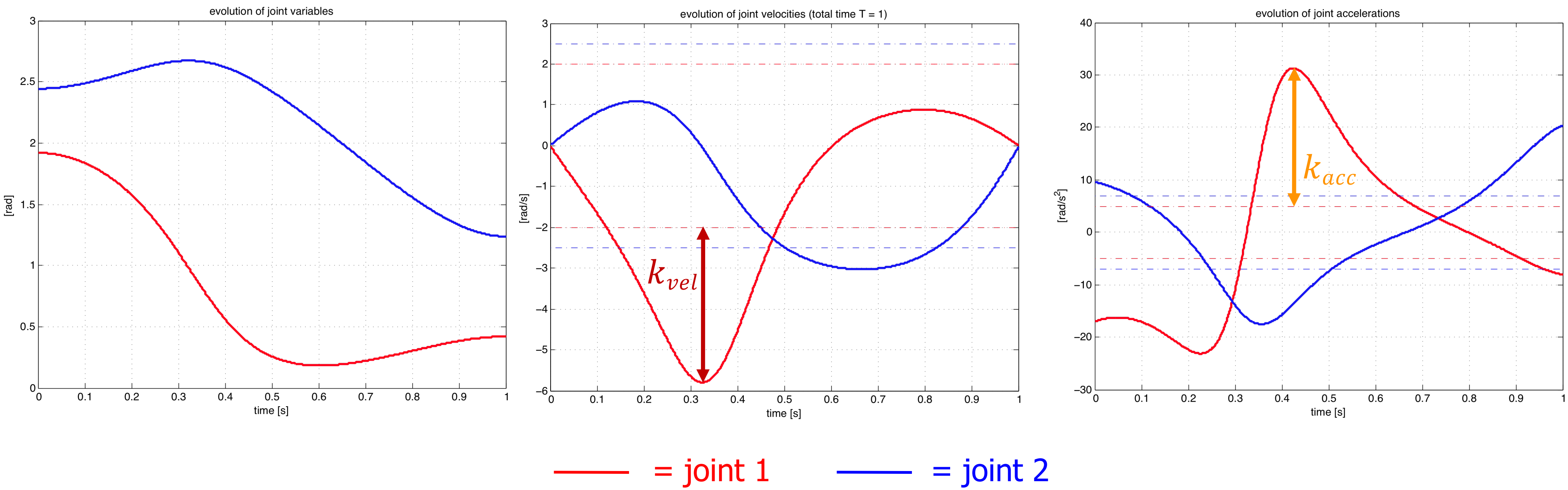




Numerical example - 2

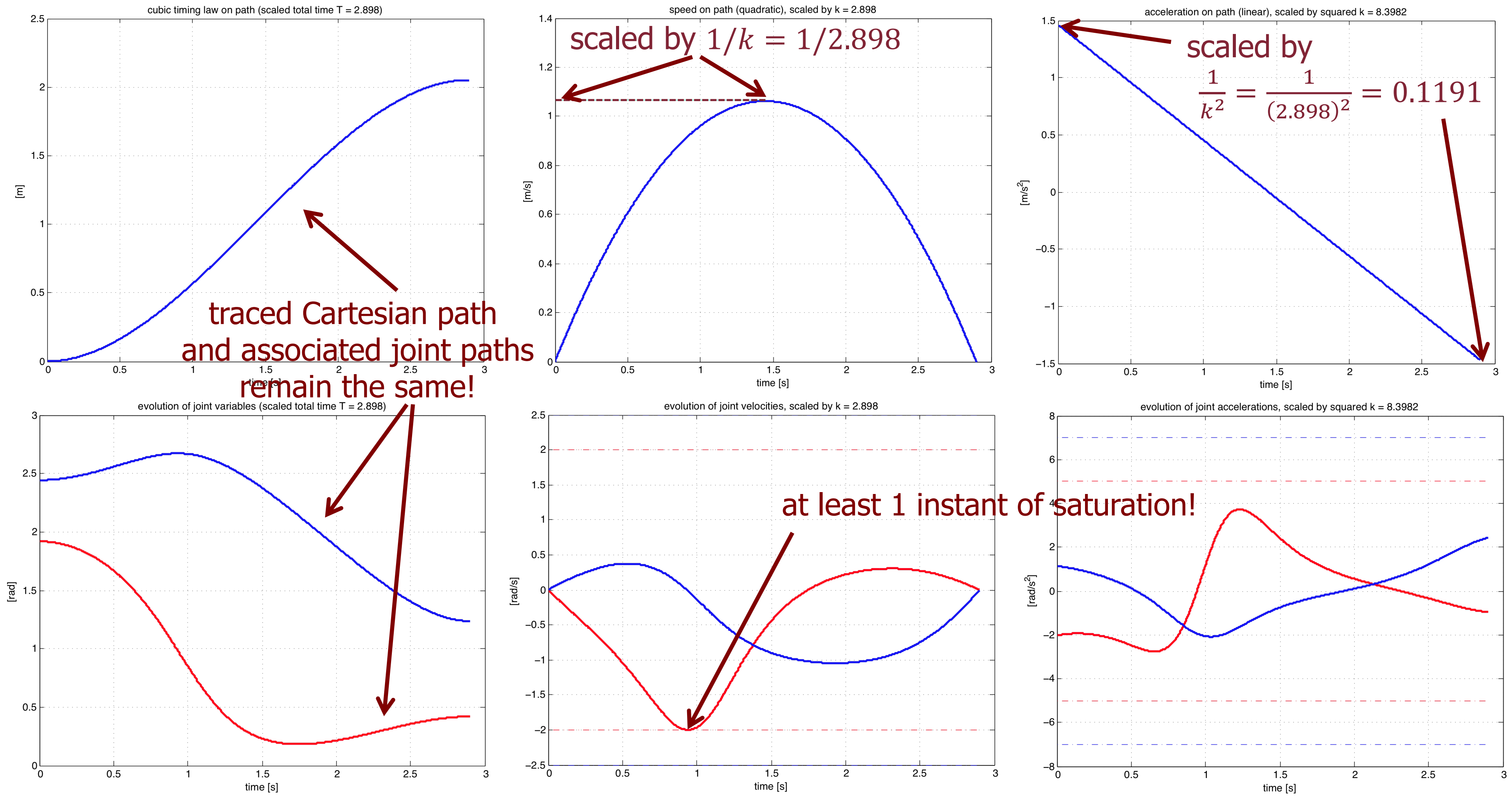
- violation of both joint velocity and acceleration bounds with $T = 1$ [s]
  - max relative violation of joint velocities: $k_{vel} = 2.898 = \max\left\{1, |\dot{q}_1|/v_{max,1}, |\dot{q}_2|/v_{max,2}\right\}$
  - .... and of joint accelerations: $k_{acc} = 6.2567 = \max\left\{1, |\ddot{q}_1|/a_{max,1}, |\ddot{q}_2|/a_{max,2}\right\}$
- minimum uniform time scaling of Cartesian trajectory to recover feasibility

$$k = \max\left\{1, k_{vel}, \sqrt{k_{acc}}\right\} = 2.898 \quad \Rightarrow \quad T_{scaled} = kT = 2.898 > T$$



Numerical example - 3

- scaled trajectory with $T_{scaled} = 2.898$ [s]
  - speed [acceleration] on path and joint velocities [accelerations] scale linearly [quadratically]



—— = joint 1 —— = joint 2