Robotics 1

Inverse differential kinematics
Statics and force transformations

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Inversion of differential kinematics

- find the joint velocity vector that realizes a desired end-effector “generalized” velocity (linear and angular)

\[ \mathbf{v} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \]

J square and non-singular

\[ \dot{\mathbf{q}} = \mathbf{J}^{-1}(\mathbf{q}) \mathbf{v} \]

- problems
  - near a singularity of the Jacobian matrix (high \( \dot{\mathbf{q}} \))
  - for redundant robots (no standard “inverse” of a rectangular matrix)

in these cases, “more robust” inversion methods are needed
Incremental solution
to inverse kinematics problems

- joint velocity inversion can be used also to solve on-line and incrementally a “sequence” of inverse kinematics problems
- each problem differs by a small amount $dr$ from previous one

\[
\begin{align*}
    r &= f_r(q) \\
    dr &= \frac{\partial f_r(q)}{\partial q} \ dq = J_r(q) \ dq
\end{align*}
\]

direct kinematics

differential kinematics

\[
\begin{align*}
    r + dr &= f_r(q) \\
    dq &= J_r^{-1}(q) \ dr \\
    q &= f_r^{-1}(r + dr) \\
    q &= q + dq
\end{align*}
\]

first, increment the desired task variables

then, solve the inverse kinematics problem

then, increment the original joint variables
Behavior near a singularity

\[ \dot{q} = J^{-1}(q) \, v \]

- Problems arise only when commanding joint motion by inversion of a given Cartesian motion task.
- Here, a linear Cartesian trajectory for a planar 2R robot.
- There is a sudden increase of the displacement/velocity of the first joint near \( \theta_2 = -\pi \) (end-effector close to the origin), despite the required Cartesian displacement is small.
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \nu \]

regular case

A line from right to left, at \( \alpha = 170^\circ \) angle with x-axis, executed at constant speed \( \nu = 0.6 \text{ m/s} \) for \( T = 6 \text{ s} \).
Simulation results
planar 2R robot in straight line Cartesian motion

path at $\alpha=170^\circ$

regular case

error due only to numerical integration ($10^{-10}$)
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \mathbf{v} \]

close to singular case

a line from right to left, at $\alpha=178^\circ$ angle with x-axis, executed at constant speed $v=0.6 \text{ m/s}$ for $T=6 \text{ s}$
Simulation results
planar 2R robot in straight line Cartesian motion

path at $\alpha=178^\circ$

close to singular case

still very small, but increased numerical integration error ($2 \cdot 10^{-9}$)

large peak of $\dot{q}_1$
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \mathbf{v} \]

close to singular case
with joint velocity saturation at $V_i = 300^\circ/s$

a line from right to left, at $\alpha = 178^\circ$ angle with x-axis,
executed at constant speed $v = 0.6$ m/s for $T = 6$ s
Simulation results
planar 2R robot in straight line Cartesian motion

path at $\alpha=178^\circ$

close to singular case

saturated value of $q_1$

actual position error!! (6 cm)
Damped Least Squares method

\[
\min_{\dot{q}} H = \frac{\lambda}{2} \| \dot{q} \|^2 + \frac{1}{2} \| J \dot{q} - v \|^2, \quad \lambda \geq 0
\]

\[
\dot{q} = (\lambda I_n + J^T J)^{-1} J^T v = J^T (\lambda I_m + J J^T)^{-1} v
\]

- inversion of differential kinematics as an optimization problem
- function \( H = \text{weighted} \) sum of two objectives (minimum error norm on achieved end-effector velocity and minimum norm of joint velocity)
- \( \lambda = 0 \) when “far enough” from a singularity
- with \( \lambda > 0 \), there is a (vector) error \( \varepsilon = v - J\dot{q} \) in executing the desired end-effector velocity \( v \) (check that \( \varepsilon = \lambda \left( \lambda I_m + J J^T \right)^{-1} v \)), but the joint velocities are always reduced ("damped")
- \( J_{DLS} \) can be used for both \( m = n \) and \( m < n \) cases
Simulation results
planar 2R robot in straight line Cartesian motion

A comparison of inverse and damped inverse Jacobian methods
even closer to singular case

\[ \dot{q} = J^{-1}(q) \, v \]

\[ \dot{q} = J_{DLS}(q) \, v \]

A line from right to left, at \( \alpha = 179.5^\circ \) angle with x-axis,
executed at constant speed \( v = 0.6 \) m/s for \( T = 6 \) s
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \mathbf{v} \]

path at \( \alpha = 179.5^\circ \)

\[ \dot{q} = J_{DLS}(q) \mathbf{v} \]

here, a very fast reconfiguration of first joint ...

\[ \dot{q} = J_{DLS}(q) \mathbf{v} \]

a completely different inverse solution, around/after crossing the region close to the folded singularity
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \, v \]

\[ \dot{q} = J_{DLS}(q) \, v \]

extremely large peak velocity of first joint!!

smooth joint motion with limited joint velocities!
Simulation results
planar 2R robot in straight line Cartesian motion

\[ \dot{q} = J^{-1}(q) \mathbf{v} \]

\[ \dot{q} = J_{DLS}(q) \mathbf{v} \]

Increased numerical integration error (\(3 \cdot 10^{-8}\))

Error (25 mm) when crossing the singularity, later recovered by feedback action \((\mathbf{v} \Rightarrow \mathbf{v} + K \mathbf{e})\)

Minimum singular value of \(JJ^T\) and \(\lambda I + JJ^T\)

They differ only when damping factor is non-zero

Robots 1
Pseudoinverse method

A constrained optimization (minimum norm) problem

\[ \min_{\dot{q}} H = \frac{1}{2} \| \dot{q} \|^2 \text{ such that } J\dot{q} - \nu = 0 \]

\[ \min_{\dot{q}\in S} H = \frac{1}{2} \| \dot{q} \|^2 \]

\[ S = \{ \dot{q} \in \mathbb{R}^n : \| J\dot{q} - \nu \| \text{ is minimum} \} \]

Solution

\[ \dot{q} = J^\#\nu \]

Pseudoinverse of J

- If \( \nu \in \mathcal{R}(J) \), the constraint is satisfied (\( \nu \) is feasible)
- Else \( J\dot{q} = \nu^\perp \), where \( \nu^\perp \) minimizes the error \( \| J\dot{q} - \nu \| \)

Orthogonal projection of \( \nu \) on \( \mathcal{R}(J) \)
Properties of the pseudoinverse

it is the **unique** matrix that satisfies the **four** relationships

- \( JJ^\#J = J \quad J^\#JJ^\# = J^\# \)

- \((J^\#J)^T = J^\#J \quad (JJ^\#)^T = JJ^\# \)

- if rank \( \rho = m = n \): \( J^\# = J^{-1} \)

- if \( \rho = m < n \): \( J^\# = J^T (JJ^T)^{-1} \)

**it always** exists and is computed in general numerically using the SVD = Singular Value Decomposition of \( J \)
(e.g., with the MATLAB function **pinv**)
Numerical example

Jacobian of 2R arm with $l_1 = l_2 = 1$ and $q_2 = 0$ (rank $\rho = 1$)

$$J = \begin{bmatrix} -2s_1 & -s_1 \\ 2c_1 & c_1 \end{bmatrix} \quad J^\# = \frac{1}{5} \begin{bmatrix} -2s_1 & 2c_1 \\ -s_1 & c_1 \end{bmatrix}$$

$$\dot{q} = J^\# v$$

is the minimum norm joint velocity vector that realizes $v^\perp$
General solution for $m < n$

All solutions (an infinite number) of the inverse differential kinematics problem can be written as

$$
\dot{q} = J^\# \nu + (I - J^\# J) \xi
$$

"projection" matrix in the null space of $J$

This is also the solution to a slightly modified constrained optimization problem ("biased" toward the joint velocity $\xi$, chosen to avoid obstacles, joint limits, etc.)

$$
\min_{\dot{q}} H = \frac{1}{2} \| \dot{q} - \xi \|^2 \quad \text{such that} \quad J \dot{q} - \nu = 0 \iff \min_{\dot{q} \in S} H = \frac{1}{2} \| \dot{q} - \xi \|^2 \\
S = \{ \dot{q} \in \mathbb{R}^n : \| J \dot{q} - \nu \| \text{ is minimum} \}
$$

Verification of which actual task velocity is going to be obtained

$$v_{\text{actual}} = J \dot{q} = J \left( J^\# \nu + (I - J^\# J) \xi \right) = JJ^\# \nu + (J - JJ^\# J) \xi = JJ^\# (Jw) = (JJ^\# J)w = Jw = v$$

If $\nu \in \mathcal{N}(J) \Rightarrow \nu = Jw$, for some $w$
Geometric interpretation

a simple case with n=2, m=1
at a given configuration:

\[ J \dot{q} = \begin{bmatrix} j_1 & j_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = v \in \mathcal{R} \]

task equality constraint

\[ J \dot{q} = v \]

associated homogeneous equation

\[ J \dot{q} = 0 \]

space of joint velocities (at a configuration \( q \))

minimum norm solution

\[ J^\# v \]

solution with minimum norm

\[ (I - J^\# J)\xi \]

orthogonal projection of \( \xi \) on \( \mathcal{N}(J) \)

linear subspace

\[ \mathcal{N}(J) = \{ \dot{q} \in \mathcal{R}^2 : J \dot{q} = 0 \} \]

“biasing” joint velocity (in general, not a solution)

all possible solutions lie on this line ...
Higher-order differential inversion

- inversion of motion from task to joint space can be performed also at a higher differential level

- **acceleration**-level: given $q, \dot{q}$
  \[
  \ddot{q} = J_r^{-1}(q) \left( \ddot{r} - \dot{J}_r(q)\dot{q} \right)
  \]

- **jerk**-level: given $q, \dot{q}, \ddot{q}$
  \[
  \dddot{q} = J_r^{-1}(q) \left( \dddot{r} - \dot{J}_r(q)\ddot{q} - 2\dddot{J}_r(q)\dot{q} \right)
  \]

- the (inverse) of the Jacobian is always the **leading** term

- **smoother** joint motions are expected (at least, due to the existence of higher-order time derivatives $\dddot{r}, \dddot{r}, ...$)

*Robotics 1*
Generalized forces and torques

- \( \tau \) = forces/torques exerted by the motors at the robot joints
- \( F \) = equivalent forces/torques exerted at the robot end-effector
- \( F_e \) = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: \( F_e = -F \)
  
  *reaction from environment is equal and opposite to the robot action on it*
Transformation of forces – Statics

in a given configuration

- what is the transformation between $F$ at robot end-effector and $\tau$ at joints?

in static equilibrium conditions (i.e., no motion):

- what $F$ will be exerted on environment by a $\tau$ applied at the robot joints?
- what $\tau$ at the joints will balance a $F_e (= -F)$ exerted by the environment?

all equivalent formulations
Virtual displacements and works

Infinitesimal (or "virtual", i.e., satisfying all possible constraints imposed on the system) displacements at an equilibrium.

\[
\begin{pmatrix}
\frac{dp}{\omega \ dt}
\end{pmatrix} = J \ dq
\]

- Without kinetic energy variation (zero acceleration)
- Without dissipative effects (zero velocity)

The "virtual work" is the work done by all forces/torques acting on the system for a given virtual displacement.
Principle of virtual work

The sum of the “virtual works” done by all forces/torques acting on the system = 0

\[ \tau^T dq - \mathbf{F}^T \left[ \begin{array}{c} dp \\ \omega dt \end{array} \right] = \tau^T dq - \mathbf{F}^T \mathbf{J} dq = 0 \quad \forall dq \]

\[ \tau = \mathbf{J}^T(q) \mathbf{F} \]
Duality between velocity and force

velocity \( \dot{q} \) (or displacement \( dq \)) in the joint space

\[ J(q) \]

generalized velocity \( \mathbf{v} \) (or e-e displacement in the Cartesian space)

\[ \begin{bmatrix} dp \\ \omega dt \end{bmatrix} \]

forces/torques \( \mathbf{\tau} \) at the joints

\[ J^T(q) \]

generalized forces \( \mathbf{F} \) at the Cartesian e-e

the singular configurations for the velocity map are the same as those for the force map

\[ \rho(J) = \rho(J^T) \]
Dual subspaces of velocity and force
summary of definitions

\[ \mathcal{R}(J) = \{ v \in \mathbb{R}^m : \exists q \in \mathbb{R}^n, Jq = v \} \]
\[ \mathcal{N}(J^T) = \{ F \in \mathbb{R}^m : J^T F = 0 \} \]
\[ \mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^m \]

\[ \mathcal{R}(J^T) = \{ \tau \in \mathbb{R}^n : \exists F \in \mathbb{R}^m, J^T F = \tau \} \]
\[ \mathcal{N}(J) = \{ \dot{q} \in \mathbb{R}^n : J \dot{q} = 0 \} \]
\[ \mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^n \]
Velocity and force singularities
list of possible cases

\[ \rho = \text{rank}(J) = \text{rank}(J^T) \leq \min(m,n) \]

1. \( \rho = m \)
   \[ \exists \dot{q} \neq 0 : J\dot{q} = 0 \]
   \[ \mathcal{N}(J) = \{0\} \]
   \[ \mathcal{N}(J^T) = \{0\} \]

2. \( \rho < m \)
   \[ \exists \dot{q} \neq 0 : J\dot{q} = 0 \]
   \[ \exists F \neq 0 : J^TF = 0 \]

1. \( \det J \neq 0 \)
   \[ \mathcal{N}(J) = \{0\} \]
   \[ \mathcal{N}(J^T) = \{0\} \]

2. \( \det J = 0 \)
   \[ \exists \dot{q} \neq 0 : J\dot{q} = 0 \]
   \[ \exists F \neq 0 : J^TF = 0 \]

1. \( \rho = n \)
   \[ \exists F \neq 0 : J^TF = 0 \]

2. \( \rho < n \)
   \[ \exists \dot{q} \neq 0 : J\dot{q} = 0 \]
   \[ \exists F \neq 0 : J^TF = 0 \]
Example of singularity analysis

planar 2R arm with generic link lengths $l_1$ and $l_2$

$$J(q) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$\text{det } J(q) = l_1 l_2 s_2$$

singularity at $q_2 = 0$ (arm straight)

$\mathcal{R}(J) = \alpha \begin{bmatrix} -s_1 \\ c_1 \end{bmatrix}$

$\mathcal{N}(J^T) = \alpha \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$

$\mathcal{R}(J^T) = \beta \begin{bmatrix} l_1 + l_2 \\ l_2 \end{bmatrix}$

$\mathcal{N}(J) = \beta \begin{bmatrix} l_2 \\ -(l_1 + l_2) \end{bmatrix}$

singularity at $q_2 = \pi$ (arm folded)

$\mathcal{R}(J)$ and $\mathcal{N}(J^T)$ as above

$\mathcal{R}(J^T) = \beta \begin{bmatrix} l_2 - l_1 \\ l_2 \end{bmatrix}$ (for $l_1 = l_2$, $\beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

$\mathcal{N}(J) = \beta \begin{bmatrix} l_2 \\ -(l_2 - l_1) \end{bmatrix}$ (for $l_1 = l_2$, $\beta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$)
Velocity manipulability

- In a given configuration, we wish to evaluate how “effective” is the mechanical transformation between joint velocities and end-effector velocities.
  - “How easily” can the end-effector be moved in the various directions of the task space.
  - Equivalently, “how far” is the robot from a singular condition.
- We consider all end-effector velocities that can be obtained by choosing joint velocity vectors of unit norm.

\[
\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad v^T J\#^T J\# v = 1
\]

**Task velocity manipulability ellipsoid**

\[
(JJ^T)^{-1} \quad \text{if } \rho = m
\]

Note: The “core” matrix of the ellipsoid equation \(v^T A^{-1} v = 1\) is the matrix \(A\).
Manipulability ellipsoid in velocity

length of principal (semi-)axes: singular values of $J$ (in its SVD)
\[ \sigma_i \{ J \} = \sqrt{\lambda_i \{ J J^T \}} \geq 0 \]

in a singularity, the ellipsoid loses a dimension (for $m=2$, it becomes a segment)

direction of principal axes: (orthogonal) eigenvectors associated to $\lambda_i$

proportional to the volume of the ellipsoid (for $m=2$, to its area)

planar 2R arm with unitary links
Manipulability measure

planar 2R arm with unitary links: Jacobian J is square

\[ \sqrt{\det(JJ^T)} = \sqrt{\det J \cdot \det J^T} = |\det J| = \prod_{i=1}^{2} \sigma_i \]

- Manipulability measure
- Best posture for manipulation (similar to a human arm!)
- Full isotropy is never obtained in this case, since it always \( \sigma_1 \neq \sigma_2 \)
Force manipulability

- in a given configuration, evaluate how “effective” is the transformation between joint torques and end-effector forces
  - “how easily” can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces/torques are balanced by the robot without the need of any joint torque
- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

\[ \tau^T \tau = 1 \quad \Rightarrow \quad F^T J J^T F = 1 \]

same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths

task force manipulability ellipsoid
Velocity and force manipulability

dual comparison of actuation vs. control

planar 2R arm with unitary links

note: velocity and force ellipsoids have here a different scale for a better view

Cartesian actuation task (high joint-to-task transformation ratio):
preferred velocity (or force) directions are those where the ellipsoid stretches

Cartesian control task (low transformation ratio = high resolution):
preferred velocity (or force) directions are those where the ellipsoid shrinks
Velocity and force transformations

- same reasoning made for relating end-effector to joint forces/torques (virtual work principle + static equilibrium) used also transforming forces and torques applied at different places of a rigid body and/or expressed in different reference frames

Transformation among generalized velocities

\[
\begin{bmatrix}
    A \mathbf{v} \\
    A \mathbf{\omega}
\end{bmatrix} =
\begin{bmatrix}
    A \mathbf{R} & -A \mathbf{R} S( B \mathbf{r}_{BA}) \\
    0 & A \mathbf{R} & B \mathbf{\omega}
\end{bmatrix} =
J_{BA} \begin{bmatrix}
    B \mathbf{v} \\
    B \mathbf{\omega}
\end{bmatrix}
\]

Transformation among generalized forces

\[
\begin{bmatrix}
    B \mathbf{f} \\
    B \mathbf{m}
\end{bmatrix} =
J_{BA}^{T} \begin{bmatrix}
    A \mathbf{f} \\
    A \mathbf{m}
\end{bmatrix} =
\begin{bmatrix}
    B \mathbf{R} & 0 \\
    -S( B \mathbf{r}_{BA}) B \mathbf{R} & B \mathbf{R} & A \mathbf{f}
\end{bmatrix}
\]

Note: for skew-symmetric matrices, \(-S^{T}(r) = S(r)\)
Example: 6D force/torque sensor

frame of measure for the forces/torques (attached to the wrist sensor)

frame of interest for evaluating forces/torques in a task with environment contact
Example: Gear reduction at joints

transmission element with motion reduction ratio N:1

motor

\[ \dot{\theta}_m = N \dot{\theta} \]
\[ u = Nu_m \]

one of the simplest applications of the principle of virtual work...

here, \( J = J^T = N \) (a scalar!)