Robotics 1

Differential kinematics

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Differential kinematics

- “relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)”

- instantaneous velocity mappings can be obtained through time derivation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the link between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation
Angular velocity of a rigid body

“rigidity” constraint on distances among points:
\[ \| r_{ij} \| = \text{constant} \]
\[ v_{Pi} - v_{Pj} \text{ orthogonal to } r_{ij} \]

1. \( v_{P2} - v_{P1} = \omega_1 \times r_{12} \)
2. \( v_{P3} - v_{P1} = \omega_1 \times r_{13} \)
3. \( v_{P3} - v_{P2} = \omega_2 \times r_{23} \)

\[ 2 - 1 = 3 \]
\[ \omega_1 = \omega_2 = \omega \]

the angular velocity \( \omega \) is associated to the whole body (not to a point)
if \( \exists P_1, P_2 \text{ with } v_{P1} = v_{P2} = 0 \): pure rotation (circular motion of all \( P_j \notin \text{line } P_1P_2 \))
\( \omega = 0 \): pure translation (all points have the same velocity \( v_{P} \))
Linear and angular velocity of the robot end-effector

- $\mathbf{v}$ and $\mathbf{\omega}$ are “vectors”, namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - these contributions will be those of the single the joint velocities
- on the other hand, $\phi$ (and $\dot{\phi}$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

in general, $\mathbf{\omega} \neq \dot{\phi}$
Finite and infinitesimal translations

- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal $dx, dy, dz$ translations (linear displacements) always commute

same final position
Finite rotations do not commute

example

mathematical fact: $\omega$ is NOT an exact differential form
(the integral of $\omega$ over time depends on the integration path!)

different final orientations!

note: finite rotations still commute when made around the same fixed axis
Infinitesimal rotations commute!

- **Infinitesimal rotations** \( d\phi_X, d\phi_Y, d\phi_Z \) around \( x, y, z \) axes

\[
R_X(\phi_X) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi_X & -\sin \phi_X \\
0 & \sin \phi_X & \cos \phi_X
\end{bmatrix}
\]

\[
R_Y(\phi_Y) = \begin{bmatrix}
\cos \phi_Y & 0 & \sin \phi_Y \\
0 & 1 & 0 \\
-\sin \phi_Y & 0 & \cos \phi_Y
\end{bmatrix}
\]

\[
R_Z(\phi_Z) = \begin{bmatrix}
\cos \phi_Z & -\sin \phi_Z & 0 \\
\sin \phi_Z & \cos \phi_Z & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
R_X(d\phi_X) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -d\phi_X \\
0 & d\phi_X & 1
\end{bmatrix}
\]

\[
R_Y(d\phi_Y) = \begin{bmatrix}
1 & 0 & d\phi_Y \\
0 & 1 & 0 \\
-d\phi_Y & 0 & 1
\end{bmatrix}
\]

\[
R_Z(d\phi_Z) = \begin{bmatrix}
1 & -d\phi_Z & 0 \\
d\phi_Z & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- \( R(d\phi) = R(d\phi_X, d\phi_Y, d\phi_Z) \) in any order

\[
= I + S(d\phi)
\]

neglecting second- and third-order (infinitesimal) terms
Time derivative of a rotation matrix

- Let \( R = R(t) \) be a rotation matrix, given as a function of time.
- Since \( I = R(t)R^T(t) \), taking the time derivative of both sides yields:
  \[
  0 = \frac{d[R(t)R^T(t)]}{dt} = \frac{dR(t)}{dt} R^T(t) + R(t) \frac{dR^T(t)}{dt} \\
  = \frac{dR(t)}{dt} R^T(t) + [\frac{dR(t)}{dt} R^T(t)]^T 
  \]
  Thus \( \frac{dR(t)}{dt} R^T(t) = S(t) \) is a skew-symmetric matrix.

- Let \( p(t) = R(t)p' \) a vector (with constant norm) rotated over time.
- Comparing:
  \[
  \frac{dp(t)}{dt} = \frac{dR(t)}{dt} p' = S(t)R(t) p' = S(t) p(t) \\
  \frac{dp(t)}{dt} = \omega(t) \times p(t) = S(\omega(t)) p(t) 
  \]
  We get \( S = S(\omega) \)

\[
\dot{R} = S(\omega) R \\
S(\omega) = \dot{R} R^T
\]
Example

Time derivative of an elementary rotation matrix

\[
R_x(\phi(t)) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi(t) & -\sin \phi(t) \\
0 & \sin \phi(t) & \cos \phi(t)
\end{bmatrix}
\]

\[
\dot{R}_x(\phi) R^T_x(\phi) = \dot{\phi} \begin{bmatrix}
0 & 0 & 0 \\
0 & -\sin \phi & -\cos \phi \\
0 & \cos \phi & -\sin \phi
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -\dot{\phi} \\
0 & \dot{\phi} & 0
\end{bmatrix} = S(\omega)
\]

\[
\omega = \begin{bmatrix}
\dot{\phi} \\
0 \\
0
\end{bmatrix}
\]
Time derivative of RPY angles and $\omega$

\[ R_{\text{RPY}}(\alpha_x, \beta_y, \gamma_z) = R_{ZYX''}(\gamma_z, \beta_y, \alpha_x) \]

the three contributions $\dot{\gamma}z$, $\dot{\beta}y'$, $\dot{\alpha}x''$ to $\omega$ are simply summed as vectors

\[ \omega = \begin{bmatrix} c\beta & c\gamma & -s\gamma & 0 \\ c\beta & s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 & 0 \\ x'' & y' & z & 0 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \]

\[ \det T_{\text{RPY}}(\beta, \gamma) = c\beta = 0 \]
for $\beta = \pm \pi / 2$
(singularity of the RPY representation)

similar treatment for the other 11 minimal representations...
Robot Jacobian matrices

- **analytical** Jacobian (obtained by time differentiation)

  \[
  \begin{aligned}
  \mathbf{r} &= \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \\
  \dot{\mathbf{r}} &= \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q} \dot{q} = J_r(q) \dot{q} 
  \end{aligned}
  \]

- **geometric** Jacobian (no derivatives)

  \[
  \begin{aligned}
  \begin{pmatrix} \mathbf{v} \\ \mathbf{\omega} \end{pmatrix} &= \begin{pmatrix} \dot{p} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q) \dot{q} 
  \end{aligned}
  \]

- In both cases, the Jacobian matrix depends on the (current) configuration of the robot
Analytical Jacobian of planar 2R arm

Given $r$, this is a $3 \times 2$ matrix.

Direct kinematics:

\[
\begin{align*}
    p_x &= l_1 c_1 + l_2 c_{12} \\
    p_y &= l_1 s_1 + l_2 s_{12} \\
    \phi &= q_1 + q_2
\end{align*}
\]

\[
\begin{align*}
    \dot{p}_x &= -l_1 s_1 \dot{q}_1 - l_2 s_{12} (\dot{q}_1 + \dot{q}_2) \\
    \dot{p}_y &= l_1 c_1 \dot{q}_1 + l_2 c_{12} (\dot{q}_1 + \dot{q}_2) \\
    \dot{\phi} &= \omega_z = \dot{q}_1 + \dot{q}_2
\end{align*}
\]

Here, all rotations occur around the same fixed axis $z$ (normal to the plane of motion).
Analytical Jacobian of polar robot

direct kinematics (here, $r = p$)

\[
\begin{align*}
p_x &= q_3 c_2 c_1 \\
p_y &= q_3 c_2 s_1 \\
p_z &= d_1 + q_3 s_2
\end{align*}
\]

taking the time derivative

\[
v = \dot{p} = \begin{bmatrix}
-q_3 c_2 s_1 & -q_3 s_2 c_1 & c_2 c_1 \\
q_3 c_2 c_1 & -q_3 s_2 s_1 & c_2 s_1 \\
0 & q_3 c_2 & s_2
\end{bmatrix}
\]

\[
\dot{q} = J_r(q) \dot{q}
\]

Robotics 1
Geometric Jacobian

always a 6 x n matrix

end-effector instantaneous velocity

\[
\begin{pmatrix}
v_E \\
\omega_E
\end{pmatrix} =
\begin{pmatrix}
J_L(q) \\
J_A(q)
\end{pmatrix}
\dot{q} =
\begin{pmatrix}
J_{L1}(q) & \ldots & J_{Ln}(q) \\
J_{A1}(q) & \ldots & J_{An}(q)
\end{pmatrix}
\begin{pmatrix}
\dot{q}_1 \\
\vdots \\
\dot{q}_n
\end{pmatrix}
\]

superposition of effects

\[v_E = J_{L1}(q) \dot{q}_1 + \ldots + J_{Ln}(q) \dot{q}_n\]

\[\omega_E = J_{A1}(q) \dot{q}_1 + \ldots + J_{An}(q) \dot{q}_n\]

contribution to the linear e-e velocity due to \(\dot{q}_1\)

contribution to the angular e-e velocity due to \(\dot{q}_1\)

linear and angular velocity belong to (linear) vector spaces in \(\mathbb{R}^3\)
Contribution of a prismatic joint

Note: joints beyond the i-th one are considered to be “frozen”, so that the distal part of the robot is a single rigid body.

\[ J_{Li}(q) \dot{q}_i = z_{i-1} \dot{d}_i \]

<table>
<thead>
<tr>
<th>prismatic i-th joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_{Li}(q) \dot{q}_i )</td>
</tr>
<tr>
<td>( J_{Ai}(q) \dot{q}_i )</td>
</tr>
</tbody>
</table>
Contribution of a revolute joint

\[
q_i = \theta_i
\]

\[
J_{Li}(q) \dot{q}_i = z_{i-1} \dot{\theta}_i
\]

\[
J_{Ai}(q) \dot{q}_i = z_{i-1} \dot{\theta}_i
\]

<table>
<thead>
<tr>
<th></th>
<th>revolute i-th joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{Li}(q) \dot{q}_i$</td>
<td>$(z_{i-1} \times p_{i-1,E}) \dot{\theta}_i$</td>
</tr>
<tr>
<td>$J_{Ai}(q) \dot{q}_i$</td>
<td>$z_{i-1} \dot{\theta}_i$</td>
</tr>
</tbody>
</table>
Expression of geometric Jacobian

\[
\begin{bmatrix}
\dot{p}_{0,E} \\ \omega_E
\end{bmatrix} =
\begin{bmatrix}
\nu_E \\ \omega_E
\end{bmatrix} =
\begin{bmatrix}
J_L(q) \\ J_A(q)
\end{bmatrix}
\dot{q} =
\begin{bmatrix}
J_{L1}(q) & \ldots & J_{L_n}(q) \\ J_{A1}(q) & \ldots & J_{A_n}(q)
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\ \vdots \\ \dot{q}_n
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>prismatic i-th joint</th>
<th>revolute i-th joint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{Li}(q)$</td>
<td>$Z_{i-1}$</td>
</tr>
<tr>
<td>$J_{Ai}(q)$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

This can be also computed as

\[ \frac{\partial p_{0,E}}{\partial q_i} = Z_{i-1} \times p_{i-1, E} \]

all vectors should be expressed in the same reference frame (here, the base frame $RF_0$)
Example: planar 2R arm

\[ J = \begin{bmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{bmatrix} \]

\[ 0A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ 0A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

DENAVIT-HARTENBERG table

<table>
<thead>
<tr>
<th>joint</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( a_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( l_1 )</td>
<td>( q_1 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( l_2 )</td>
<td>( q_2 )</td>
</tr>
</tbody>
</table>

\[ p_{1,E} = p_{0,E} - p_{0,1} \]
Geometric Jacobian of planar 2R arm

\[ J = \begin{bmatrix}
  z_0 \times p_{0,E} & z_1 \times p_{1,E} \\
  z_0 & z_1 \\
  -l_1s_1 - l_2s_{12} & -l_2s_{12} \\
  l_1c_1 + l_2c_{12} & l_2c_{12} \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  1 & 1
  \end{bmatrix} \]

**Note:** the Jacobian is here a 6 × 2 matrix, thus its maximum rank is 2.

at most 2 components of the linear/angular end-effector velocity can be independently assigned.

Compare rows 1, 2, and 6 with the analytical Jacobian in slide #12!
Transformations of the Jacobian matrix

\[
\begin{bmatrix}
0 v_n \\
0 \omega
\end{bmatrix} = ^0J_n(q) \dot{q}
\]

b) we may choose \( E \Rightarrow O_j(q) \)

\[
v_E = v_n + \omega \times r_{nE}
\]

\[
= v_n + S(\theta_{En}) \omega
\]

\[
\begin{bmatrix}
Bv_E \\
B\omega
\end{bmatrix} = \begin{bmatrix} B_r_0 & 0 \\ 0 & B_r_0 \end{bmatrix} \begin{bmatrix} I & S(0r_{En}) \\ 0 & I \end{bmatrix} \begin{bmatrix} 0v_n \\ 0\omega \end{bmatrix}
\]

\[
= \begin{bmatrix} B_r_0(q) & 0 \\ 0 & B_r_0(q) \end{bmatrix} \begin{bmatrix} I & S(0r_{En}(q)) \\ 0 & I \end{bmatrix} \begin{bmatrix} ^0J_n(q) \dot{q} \\ ^BJ_E(q) \dot{q} \end{bmatrix}
\]

a) we may choose \( RF_B \Rightarrow RF_i(q) \)

never singular!
Example: Dexter robot

- 8R robot manipulator with transmissions by pulleys and steel cables (joints 3 to 8)
  - lightweight: only 15 kg in motion
  - motors located in second link
  - incremental encoders (homing)
  - redundancy degree for e-e pose task: n-m=2
  - compliant in the interaction with environment
Mid-frame Jacobian of Dexter robot

- geometric Jacobian $^0J_8(q)$ is very complex
- "mid-frame" Jacobian $^4J_4(q)$ is relatively simple!

6 rows, 8 columns
Summary of differential relations

\[ \dot{p} \rightarrow v \quad \dot{p} = v \]

\[ \dot{R} \rightarrow \omega \quad \dot{R} = S(\omega) R \quad \text{for each column } r_i \text{ of } R \text{ (unit vector of a frame), it is } \dot{r}_i = \omega \times r_i \]

\[ \dot{\phi} \rightarrow \omega \quad \omega = \omega_{\phi_1} + \omega_{\phi_2} + \omega_{\phi_3} = a_1 \dot{\phi}_1 + a_2(\phi_1) \dot{\phi}_2 + a_3(\phi_1, \phi_2) \dot{\phi}_3 = T(\phi)\dot{\phi} \]

(moving) axes of definition for the sequence of rotations \( \phi_i \)

\[ r = \begin{pmatrix} p \\ \phi \end{pmatrix} \quad J(q) = \begin{pmatrix} I & 0 \\ 0 & T(\phi) \end{pmatrix} J_r(q) \quad J_r(q) = \begin{pmatrix} I & 0 \\ 0 & T^{-1}(\phi) \end{pmatrix} J(q) \]

\( T(\phi) \) has always a singularity \( \iff \) singularity of the specific minimal representation of orientation

\( \text{Robotics 1} \)
Acceleration relations (and beyond...)  
Higher-order differential kinematics

- differential relations between motion in the joint space and motion in the task space can be established at the second order, third order, ...
- the analytical Jacobian always “weights” the highest-order derivative

- velocity \[ \dot{r} = J_r(q) \dot{q} \]  
- acceleration \[ \ddot{r} = J_r(q) \ddot{q} + J_r(q) \dot{q} \]  
- jerk \[ \dddot{r} = J_r(q) \dddot{q} + 2 J_r(q) \ddot{q} + J_r(q) \dot{q} \]  
- snap \[ \ddddot{r} = J_r(q) \ddddot{q} + \ldots \]

- the same holds true also for the geometric Jacobian \( J(q) \)
Primer on linear algebra

given a matrix $J$: $m \times n$ (m rows, n columns)

- **rank** $\rho(J) = \max \#$ of rows or columns that are linearly independent
  - $\rho(J) \leq \min(m,n)$ (if equality holds, $J$ has “full rank”)
  - if $m = n$ and $J$ has full rank, $J$ is “non singular” and the inverse $J^{-1}$ exists
  - $\rho(J) = \text{dimension of the largest non singular square submatrix of } J$

- **range** $\mathcal{R}(J) = \text{vector subspace generated by all possible linear combinations of the columns of } J$
  \[
  \mathcal{R}(J) = \{ v \in \mathbb{R}^m : \exists \xi \in \mathbb{R}^n, v = J\xi \}
  \]
  - $\dim(\mathcal{R}(J)) = \rho(J)$

- **kernel** $\mathcal{N}(J) = \text{vector subspace of all vectors } \xi \in \mathbb{R}^n \text{ such that } J\cdot\xi = 0$
  - $\dim(\mathcal{N}(J)) = n - \rho(J)$

\[
\mathcal{R}(J) + \mathcal{N}(J^T) = \mathbb{R}^m \quad \text{and} \quad \mathcal{R}(J^T) + \mathcal{N}(J) = \mathbb{R}^n
\]
  - sum of vector subspaces $V_1 + V_2 = \text{vector space where any element } v \text{ can be written as } v = v_1 + v_2, \text{ with } v_1 \in V_1, v_2 \in V_2$

- all the above quantities/subspaces can be computed using, e.g., Matlab

Robotics 1

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Robot Jacobian decomposition in linear subspaces and duality

\[ \mathbb{R}^n = \mathbb{R}^n(J^T) \oplus \mathbb{R}(J) \]

\[ \mathbb{R}^m = \mathbb{R}(J) \oplus \mathbb{R}(J^T) \]

(in a given configuration \( q \))
Mobility analysis

- \( \rho(J) = \rho(J(q)) \), \( \mathcal{R}(J) = \mathcal{R}(J(q)) \), \( \mathcal{N}(J^T) = \mathcal{N}(J^T(q)) \) are locally defined, i.e., they depend on the current configuration \( q \).

- \( \mathcal{R}(J(q)) \) = subspace of all “generalized” velocities (with linear and/or angular components) that can be instantaneously realized by the robot end-effector when varying the joint velocities at the configuration \( q \).

- If \( J(q) \) has max rank (typically = \( m \)) in the configuration \( q \), the robot end-effector can be moved in any direction of the task space \( \mathbb{R}^m \).

- If \( \rho(J(q)) < m \), there exist directions in \( \mathbb{R}^m \) along which the robot end-effector cannot move (instantaneously!)
  - these directions lie in \( \mathcal{N}(J^T(q)) \), namely the complement of \( \mathcal{R}(J(q)) \) to the task space \( \mathbb{R}^m \), which is of dimension \( m - \rho(J(q)) \).

- When \( \mathcal{N}(J(q)) \neq \{0\} \), there exist non-zero joint velocities that produce zero end-effector velocity (“self motions”)
  - this always happens for \( m < n \), i.e., when the robot is redundant for the task.
Kinematic singularities

- configurations where the Jacobian loses rank
  \[ \Leftrightarrow \text{loss of instantaneous mobility of the robot end-effector} \]

- for \( m = n \), they correspond to Cartesian poses at which the number of solutions of the inverse kinematics problem differs from the “generic” case

- “in” a singular configuration, we cannot find a joint velocity that realizes a desired end-effector velocity in an arbitrary direction of the task space

- “close” to a singularity, large joint velocities may be needed to realize some (even small) velocity of the end-effector

- finding and analyzing in advance all singularities of a robot helps in avoiding them during trajectory planning and motion control
  - when \( m = n \): find the configurations \( q \) such that \( \det J(q) = 0 \)
  - when \( m < n \): find the configurations \( q \) such that all \( m \times m \) minors of \( J \) are singular (or, equivalently, such that \( \det [J(q) J^T(q)] = 0 \))

- finding all singular configurations of a robot with a large number of joints, or the actual “distance” from a singularity, is a hard computational task
Singularities of planar 2R arm

- **Singularities**: arm is stretched \((q_2 = 0)\) or folded \((q_2 = \pi)\)
- **Singular configurations** correspond here to Cartesian points on the **boundary** of the workspace
- In many cases, these singularities **separate** regions in the joint space with **distinct** inverse kinematic solutions (e.g., “elbow up” or “down”)

**Direct Kinematics**
\[
\begin{align*}
    p_x &= l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\
    p_y &= l_1 \sin q_1 + l_2 \sin(q_1 + q_2)
\end{align*}
\]

**Analytical Jacobian**
\[
\dot{p} = \begin{bmatrix}
    -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) \\
    l_1 \cos q_1 + l_2 \cos(q_1 + q_2)
\end{bmatrix}
\begin{bmatrix}
    \dot{q}_1 \\
    \dot{q}_2
\end{bmatrix}
\]
\[
\det J(q) = l_1 l_2 \sin q_2
\]

Robotics 1
Singularities of polar (RRP) arm

**direct kinematics**

\[ p_x = q_3 \ c_2 \ c_1 \]
\[ p_y = q_3 \ c_2 \ s_1 \]
\[ p_z = d_1 + q_3 \ s_2 \]

**analytical Jacobian**

\[
\dot{p} = \begin{pmatrix}
-q_3 s_1 c_2 & -q_3 c_1 s_2 & c_1 c_2 \\
q_3 c_1 c_2 & -q_3 s_1 s_2 & s_1 c_2 \\
0 & q_3 c_2 & s_2
\end{pmatrix}
\]

**\( \dot{q} = J(q) \dot{q} \)**

- **singularities**
  - E-E is along the z axis (\( q_2 = \pm \pi/2 \)): *simple* singularity \( \Rightarrow \) rank \( J = 2 \)
  - third link is fully retracted (\( q_3 = 0 \)): *double* singularity \( \Rightarrow \) rank \( J \) drops to 1
  - all singular configurations correspond here to Cartesian points *internal* to the workspace (supposing *no limits* for the prismatic joint)
Singularities of robots with spherical wrist

- $n = 6$, last three joints are revolute and their axes intersect at a point
- without loss of generality, we set $O_6 = W =$ center of spherical wrist (i.e., choose $d_6 = 0$ in the DH table)

$$J(q) = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

- since $\det J(q_1,\ldots,q_5) = \det J_{11} \cdot \det J_{22}$, there is a decoupling property
  - $\det J_{11}(q_1,\ldots,q_3) = 0$ provides the arm singularities
  - $\det J_{22}(q_4, q_5) = 0$ provides the wrist singularities
- being $J_{22} = [z_3 \ z_4 \ z_5]$ (in the geometric Jacobian), wrist singularities correspond to when $z_3$, $z_4$ and $z_5$ become linearly dependent vectors
  \[\Rightarrow\] when either $q_5 = 0$ or $q_5 = \pm \pi/2$
- inversion of $J$ is simpler (block triangular structure)
- the determinant of $J$ will never depend on $q_1$: why?