Robotics 1

Direct kinematics

Prof. Alessandro De Luca
Kinematics of robot manipulators

- study of...
  geometric and timing aspects of robot motion, without reference to the causes producing it

- robot seen as...
  an (open) kinematic chain of rigid bodies interconnected by (revolute or prismatic) joints
Motivations

- functional aspects
  - definition of robot workspace
  - calibration

- operational aspects

  task execution (actuation by motors)
  task definition and performance

two different "spaces" related by kinematic (and dynamic) maps

- trajectory planning
- programming
- motion control
Kinematics
formulation and parameterizations

- choice of parameterization $q$
  - unambiguous and minimal characterization of robot configuration
  - $n = \# \text{ degrees of freedom (dof)} = \# \text{ robot joints ( rotational or translational)}$

- choice of parameterization $r$
  - compact description of position and/or orientation (pose) variables of interest to the required task
  - usually, $m \leq n$ and $m = 6$ (but none of these is strictly necessary)
Open kinematic chains

- $m = 2$
  - pointing in space
  - positioning in the plane
- $m = 3$
  - orientation in space
  - positioning and orientation in the plane

- $m = 5$
  - positioning and pointing in space
    - (like for spot welding)
- $m = 6$
  - positioning and orientation in space
  - positioning of two points in space
    - (e.g., end-effector and elbow)

- e.g., the relative angle between a link and the following one
- e.g., it describes the pose of frame $RF_E$

$r = (r_1, \ldots, r_m)$
Classification by kinematic type
first 3 dofs only

Cartesian or gantry (PPP)
cylindric (RPP)
polar or spherical (RRP)
articulated or anthropomorphic (RRR)

SCARA (RRP)

R = 1-dof rotational (revolute) joint
P = 1-dof translational (prismatic) joint
Direct kinematic map

- the structure of the **direct kinematics** function depends on the chosen \( r \)
  \[
  r = f_r(q)
  \]

- methods for computing \( f_r(q) \)
  - geometric/by inspection
  - systematic: assigning frames attached to the robot links and using homogeneous transformation matrices
Direct kinematics of 2R planar robot
just using inspection...

\[ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad n = 2 \]
\[ r = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} \quad m = 3 \]

\[ p_x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \]
\[ p_y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \]
\[ \phi = q_1 + q_2 \]

for more general cases, we need a ‘method’!
Numbering links and joints

icon representation of joint types for the manipulator skeleton

revolute

prismatic
Spatial relation between joint axes

axis of joint $i$

axis of joint $i + 1$

$90^\circ$

common normal (axis of link $i$)

$\alpha_i = \text{displacement } AB$ between joint axes (always well defined)

$\alpha_i = \text{twist angle}$ between joint axes — projected on a plane $\pi$ orthogonal to the link axis

always constant!

with sign (pos/neg)!
Spatial relation between link axes

\[ \text{displacement } CD \text{ (a variable if joint } i \text{ is prismatic)} \]

\[ \theta_i = \text{angle between link axes} \text{ (a variable if joint } i \text{ is revolute)} \]

— projected on a plane \( \sigma \) orthogonal to the joint axis

\[ \text{with sign (pos/neg)}! \]
Denavit-Hartenberg (DH) frames

- joint axis $i - 1$
- joint axis $i$
- joint axis $i + 1$

Link $i - 1$ is moved by joint $i - 1$

Frame $RF_i$ is attached to link $i$

Axis of joint $i$ as common normal to joint axes $i$ and $i + 1$

Axis of link $i$ around which link $i$ rotates or along which link $i$ slides
Definition of DH parameters

- unit vector $z_i$ along axis of joint $i + 1$
- unit vector $x_i$ along the common normal to joint $i$ and $i + 1$ axes ($i \rightarrow i + 1$)
- $a_i = \text{distance } DO_i$, + if oriented as $x_i$, always constant (= ‘length’ of link $i$)
- $d_i = \text{distance } O_{i-1}D$, + if oriented as $z_{i-1}$, variable if joint $i$ is PRISMATIC
- $\alpha_i = \text{twist angle from } z_{i-1} \text{ to } z_i \text{ around } x_i$, + if CCW, always constant
- $\theta_i = \text{angle from } x_{i-1} \text{ to } x_i \text{ around } z_{i-1}$, + if CCW, variable if joint $i$ is REVOLUTE
DH layout made simple
a popular 3-minute illustration...

Denavit–Hartenberg Reference Frame Layout
Produced by Ethan Tira–Thompson

video

https://www.youtube.com/watch?v=rA9tm0gTln8

- note: the author of this video uses \( r \) in place of \( a \), and does not add subscripts!
Homogeneous transformation between successive DH frames (from frame $i - 1$ to frame $i$)

- roto-translation (screw motion) around and along $Z_{i-1}$

\[
i^{-1}A_i'(q_i) = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & 0 \\
\sin \theta_i & \cos \theta_i & 0 & 0 \\
0 & 0 & 1 & d_i \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

the product of these two matrices commutes!

rotational joint $\Rightarrow q_i = \theta_i$

prismatic joint $\Rightarrow q_i = d_i$

- roto-translation (screw motion) around and along $x_i$

\[
i'A_i = \begin{bmatrix}
1 & 0 & 0 & a_i \\
0 & \cos \alpha_i & -\sin \alpha_i & 0 \\
0 & \sin \alpha_i & \cos \alpha_i & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

always a constant matrix
Denavit-Hartenberg matrix


\[
i^{-1}A_i(q_i) = i^{-1}A_{i'}(q_i) i'\ A_i = \begin{bmatrix}
\cos \theta_i & -\cos \alpha_i \sin \theta_i & \sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \alpha_i \cos \theta_i & -\sin \alpha_i \cos \theta_i & a_i \sin \theta_i \\
0 & \sin \alpha_i & \cos \alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

compact notation: \( c = \cos \), \( s = \sin \)

super-compact notation (if feasible): \( c_i = \cos q_i \), \( s_i = \sin q_i \)
Direct kinematics of robot manipulators

description ‘internal’ to the robot using

- \( q = (q_1, \ldots, q_n) \)
- product of DH matrices

\[ 0A_1(q_1)A_2(q_2)\cdots A_{n-1}(q_{n-1}) \]

description ‘external’ to the robot using

\[ wT_E = \begin{bmatrix} wR_E & wP_{WE} \\ 0^T & 1 \end{bmatrix} \]

- \( r = f_r(q) \)
- \( r = (r_1, \ldots, r_m) \)
Direct kinematics of 2R planar robot using DH frame assignment...

\[
0A_1(\theta_1) = \begin{bmatrix}
  c\theta_1 & -s\theta_1 & 0 & l_1c\theta_1 \\
  s\theta_1 & c\theta_1 & 0 & l_1s\theta_1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
1A_2(\theta_2) = \begin{bmatrix}
  c\theta_2 & -s\theta_2 & 0 & l_2c\theta_2 \\
  s\theta_2 & c\theta_2 & 0 & l_2s\theta_2 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
0A_2(q) = \begin{bmatrix}
  c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\
  s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
0p_{\text{hom}} = \begin{bmatrix}
p_x \\
p_y \\
0 \\
1
\end{bmatrix} = 0A_2(q) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix}
l_1c_1 + l_2c_{12} \\
l_1s_1 + l_2s_{12}
\end{bmatrix}
\]

\[
\phi = q_1 + q_2 \quad \text{(extracted from } 0R_2(q)\text{)}
\]

\[
z_0, z_1, z_2 \text{ outgoing from plane}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
i & \alpha_i & a_i & d_i & \theta_i \\
\hline
1 & 0 & l_1 & 0 & q_1 \\
2 & 0 & l_2 & 0 & q_2 \\
\hline
\end{array}
\]

\[
\phi = q_1 + q_2
\]
Direct kinematics of 2R planar robot
TCP location on the robot end effector

\[
0A_2(q) = \begin{bmatrix}
    c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\
    s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

Tool Center Point \( TCP \) and associated end-effector frame \( RF_E \)

\[
2T_E = \begin{bmatrix}
    0 & 1 & 0 & \begin{bmatrix} 2TCP_x \end{bmatrix} \\
    0 & 0 & -1 & \begin{bmatrix} 2TCP_y \end{bmatrix} \\
    -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0TCP_x(q) \\
    0TCP_y(q) \\
    0 \\
    1
\end{bmatrix} = 0A_2(q) \begin{bmatrix}
    2TCP_x \\
    2TCP_y \\
    0 \\
    1
\end{bmatrix} = 0T_E(q) \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1
\end{bmatrix} = 0A_2(q) \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1
\end{bmatrix} = 0T_E(q) 2T_E
\]
Ambiguities in defining DH frames

- **frame 0**: origin and $x_0$ axis are arbitrary
- **frame $n$**: $z_n$ axis is not specified
  - however, $x_n$ must intersect and be chosen orthogonal to $z_{n-1}$
- **positive** direction of $z_{i-1}$ (up/down on axis of joint $i$) is arbitrary
  - choose one, and try to ‘avoid flipping over’ to the next one
- **positive** direction of $x_i$ (back/forth on axis of link $i$) is arbitrary
  - if successive joint axes are incident, we often take $x_i = z_{i-1} \times z_i$
  - when natural, follow the direction ‘from base to tip’
- if $z_{i-1}$ and $z_i$ are **parallel** (common normal not uniquely defined)
  - $O_i$ is chosen arbitrarily along $z_i$, still trying to ‘zero out’ parameters
- if $z_{i-1}$ and $z_i$ are **coincident**, normal $x_i$ axis can be chosen at will
  - this case occurs **only** if the two joints are of different kind (P/R or R/P)
  - again, try using ‘simple values’ (e.g., 0 or $\pm \pi/2$) for constant angles
DH assignment for a SCARA robot

Sankyo SCARA 8438

Sankyo SCARA SR 8447
Step 1: joint axes

all parallel (or coincident)

-twist angles $\alpha_i = 0$ or $\pi$

J1 shoulder

J2 elbow

J3 prismatic $\equiv$ J4 revolute
Step 2: link axes

the vertical ‘heights’ of the link axes are arbitrary (for the time being)
Step 3: frames

axes $y_i$ for $i > 0$
are not shown
(nor needed; they form right-handed frames)
Step 4: DH table of parameters

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$a_1$</td>
<td>$d_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$a_2$</td>
<td>0</td>
<td>$q_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$q_3$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\pi$</td>
<td>0</td>
<td>$d_4$</td>
<td>$q_4$</td>
</tr>
</tbody>
</table>

Note that:

- $d_1$ and $d_4$ could be set $= 0$
- $d_4 < 0$ (opposite to $z_3$)
- $q_3 < 0$ in this configuration
- Similarly, here $q_1 > 0$, $q_2 < 0$, $q_4 < 0$
Step 5: DH transformation matrices

\[ ^0A_1(q_1) = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ ^1A_2(q_2) = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ ^2A_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ ^3A_4(q_4) = \begin{bmatrix} c\theta_4 & s\theta_4 & 0 & 0 \\ s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & -1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ q = (q_1, q_2, q_3, q_4) \]
\[ = (\theta_1, \theta_2, d_3, \theta_4) \]
Step 6a: direct kinematics

homogeneous matrix $^wT_E$ as product of the $^iA_i(q_i)$’s

$^0A_2(q_1, q_2) = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$^0A_3(q_1, q_2, q_3) = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & d_1 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$^wT_E = ^0A_4(q_1, q_2, q_3, q_4) = \begin{bmatrix} c_{124} & s_{124} & 0 & a_1c_1 + a_2c_{12} \\ s_{124} & -c_{124} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 + q_3 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Step 6b: direct kinematics
as task vector $r \in \mathbb{R}^m$

$$0_A^4(q_1, q_2, q_3, q_4) = \begin{bmatrix}
  c_{124} & s_{124} & 0 \\
  s_{124} & -c_{124} & 0 \\
  0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
  a_1 c_1 + a_2 c_{12} \\
  a_1 s_1 + a_2 s_{12} \\
  d_1 + q_3 + d_4 \\
  q_1 + q_2 + q_4
\end{bmatrix}$$

extract $\alpha_z \in \mathbb{R}$
from
$R(q_1, q_2, q_4)$

$$r = \begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
  \alpha_z
\end{bmatrix} = f_r(q) = \begin{bmatrix}
  a_1 c_1 + a_2 c_{12} \\
  a_1 s_1 + a_2 s_{12} \\
  d_1 + q_3 + d_4 \\
  q_1 + q_2 + q_4
\end{bmatrix} \in \mathbb{R}^4$$

MATLAB code available on web site: dirkin_SCARA.m
Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

- robot with shoulder offset
- ‘one possible’ DH assignment of frames is shown
- determine the associated table of DH parameters
- homogeneous transformation matrices
- direct kinematics
- write a program for computing the direct kinematics
  - numerically (Matlab), given a \( q \)
  - symbolically (Mathematica, Maple, Symbolic Manipulation Toolbox of Matlab, ...)
DH table for Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>$d_1 &gt; 0$</td>
<td>$q_1 = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/2$</td>
<td>0</td>
<td>$d_2 &gt; 0$</td>
<td>$q_2 = 0$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$q_3 &gt; 0$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>4</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_4 = 0$</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_5 = -\pi/2$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$d_6 &gt; 0$</td>
<td>$q_6 = 0$</td>
</tr>
</tbody>
</table>

Joint variables are in red, while their values in the shown robot configuration are in blue.
KUKA LWR 4+

- 7R (no offsets, spherical shoulder and spherical wrist)

frames and table of DH parameters
homogeneous transformation matrices
direct kinematics
$d_1$ and $d_7$ can be set $= 0$ or not (as needed)

available at DIAG Robotics Lab

side view (from the left)  frontal view

Robotics 1
KUKA KR5 Sixx R650

- 6R (offsets at shoulder and elbow, spherical wrist)

- determine frames and table of DH parameters
- homogeneous transformation matrices
- direct kinematics (symbolic & numeric)

available at DIAG Robotics Lab
Appendix:
Modified DH convention

- a modified version introduced in J. Craig’s book “Introduction to Robotics” (1986) and aligned for the indexing by Khalil and Kleinfinger (ICRA, 1986)
- has \( z_i \) axis on joint \( i \)
- \( a_i \) & \( \alpha_i \) = distance & twist angle from \( z_{i-1} \) to \( z_i \), measured along & about \( x_{i-1} \)
- \( d_i \) & \( \theta_i \) = distance & angle from \( x_{i-1} \) to \( x_i \), measured along & about \( z_i \)
- source of much confusion... if you are not aware of it (or don’t mention it!)
- convenient with link flexibility: a rigid frame at the base, another at the tip...

\[
\begin{align*}
i^{-1}A_i &= \begin{bmatrix}
c\theta_i & -c\alpha_is\theta_i & s\alpha_is\theta_i & a_ic\theta_i \\
s\theta_i & c\alpha_ic\theta_i & -s\alpha_ic\theta_i & a_is\theta_i \\
0 & s\alpha_i & c\alpha_i & d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
i^{-1}A_i^{\text{mod}} &= \begin{bmatrix}
c\theta_i & -s\theta_i & 0 & a_i \\
c\alpha_is\theta_i & c\alpha_ic\theta_i & -s\alpha_i & -d_is\alpha_i \\
s\alpha_is\theta_i & s\alpha_ic\theta_i & c\alpha_i & d_i c\alpha_i \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]