Robotics 1

Direct kinematics

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Kinematics of robot manipulators

- study of ...
  geometric and timing aspects of robot motion, without reference to the causes producing it

- robot seen as ...
  an (open) kinematic chain of rigid bodies interconnected by (revolute or prismatic) joints
Motivations

- functional aspects
  - definition of robot workspace
  - calibration
- operational aspects
  - task execution (actuation by motors)
  - task definition and performance

Two different "spaces" related by kinematic (and dynamic) maps

- trajectory planning
- programming
- motion control
Kinematics formulation and parameterizations

- choice of parameterization \( q \)
  - unambiguous and minimal characterization of robot configuration
  - \( n = \# \text{ degrees of freedom (dof)} = \# \text{ robot joints (rotational or translational)} \)

- choice of parameterization \( r \)
  - compact description of position and/or orientation (pose) variables of interest to the required task
  - usually, \( m \leq n \) and \( m \leq 6 \) (but none of these is strictly necessary)

\[ q = (q_1, \ldots, q_n) \quad \text{JOINT space} \]

\[ r = f(q) \quad \text{DIRECT}\]

\[ q = f^{-1}(r) \quad \text{INVERSE}\]

\[ r = (r_1, \ldots, r_m) \quad \text{TASK (Cartesian) space} \]
Open kinematic chains

- $m = 2$
  - pointing in space
  - positioning in the plane
- $m = 3$
  - orientation in space
  - positioning and orientation in the plane
- $m = 5$
  - positioning and pointing in space
- $m = 6$
  - positioning and orientation in space (like for spot welding)
  - positioning of two points in space (e.g., end-effector and elbow)

$\mathbf{r} = (r_1, \ldots, r_m)$

- e.g., it describes the pose of frame $\mathbf{RF}_E$

- e.g., the relative angle between a link and the following one
Classification by kinematic type
(first 3 dofs)

- Cartesian or gantry (PPP)
- Cylindric (RPP)
- Polar or spherical (RRP)
- SCARA (RRP)

R = 1-dof rotational (revolute) joint
P = 1-dof translational (prismatic) joint
Direct kinematic map

- the structure of the direct kinematics function depends from the chosen $r$

$$r = f_r(q)$$

- methods for computing $f_r(q)$
  - geometric/by inspection
  - systematic: assigning frames attached to the robot links and using homogeneous transformation matrices
Example: direct kinematics of 2R arm

\[ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \]
\[ n = 2 \]
\[ r = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} \]
\[ m = 3 \]

\[ p_x = l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \]
\[ p_y = l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \]
\[ \phi = q_1 + q_2 \]

for more general cases, we need a “method”!
Numbering links and joints

Joint 1
Link 0 (base)
Joint 2
Link 1
Joint i-1
Link i-1
Joint i
Link i
Joint i+1
Link i
Joint n
Link n (end effector)

Revolute
Prismatic
Spatial relation between joint axes

axis of joint i

90°

axis of joint i+1

90°

Common normal (axis of link i)

A

B

a_i = displacement AB between joint axes (always well defined)

\( \alpha_i \) = twist angle between joint axes — projected on a plane \( \pi \) orthogonal to the link axis

with sign (pos/neg)!
Spatial relation between link axes

\[ d_i = \text{displacement CD (a variable if joint i is prismatic)} \]
\[ \theta_i = \text{angle between link axes (a variable if joint i is revolute)} \]

- projected on a plane \( \sigma \) orthogonal to the joint axis

with sign (pos/neg)!
Denavit-Hartenberg (DH) frames

- Joint axis $i-1$
- Joint axis $i$
- Joint axis $i+1$

- Link $i-1$
- Link $i$
- Link $i+1$

- Frame RF$_i$ is attached to link $i$
- Frame RF$_{i-1}$ is moved by joint $i-1$

- Axis around which the link rotates or along which the link slides

- Common normal to joint axes $i$ and $i+1$
Denavit-Hartenberg parameters

- unit vector \( z_i \) along axis of joint \( i+1 \)
- unit vector \( x_i \) along the common normal to joint \( i \) and \( i+1 \) axes (\( i \rightarrow i+1 \))
- \( a_i \) = distance \( DO_i \) — positive if oriented as \( x_i \) (constant = “length” of link \( i \))
- \( d_i \) = distance \( O_{i-1}D \) — positive if oriented as \( z_{i-1} \) (variable if joint \( i \) is PRISMATIC)
- \( \alpha_i \) = twist angle between \( z_{i-1} \) and \( z_i \) around \( x_i \) (constant)
- \( \theta_i \) = angle between \( x_{i-1} \) and \( x_i \) around \( z_{i-1} \) (variable if joint \( i \) is REVOLUTE)
Denavit-Hartenberg layout made simple
(a popular 3-minute illustration...)

https://www.youtube.com/watch?v=rA9tm0gTln8

- **note**: the authors of this video use $r$ in place of $a$, and do not add subscripts!
Ambiguities in defining DH frames

- **frame<sub>0</sub>:** origin and x<sub>0</sub> axis are arbitrary
- **frame<sub>n</sub>:** z<sub>n</sub> axis is not specified (but x<sub>n</sub> must be orthogonal to and intersect z<sub>n-1</sub>)
- **positive** direction of z<sub>i-1</sub> (up/down on joint i) is arbitrary
  - choose one, and try to avoid “flipping over” to the next one
- **positive** direction of x<sub>i</sub> (on axis of link i) is arbitrary
  - we often take x<sub>i</sub> = z<sub>i-1</sub> × z<sub>i</sub> when successive joint axes are incident
  - when natural, we follow the direction “from base to tip”
- if z<sub>i-1</sub> and z<sub>i</sub> are **parallel:** common normal not uniquely defined
  - O<sub>i</sub> is chosen arbitrarily along z<sub>i</sub>, but try to “zero out” parameters
- if z<sub>i-1</sub> and z<sub>i</sub> are **coincident:** normal x<sub>i</sub> axis may be chosen at will
  - again, we try to use “simple” constant angles (0, π/2)
  - this case may occur only if the two joints are of different kind (P & R)
Homogeneous transformation
between successive DH frames (from frame $i-1$ to frame $i$)

- roto-translation around and along $z_{i-1}$

$$i^{-1}A_i' (q_i) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotational joint $\Rightarrow q_i = \theta_i$
prismatic joint $\Rightarrow q_i = d_i$

- roto-translation around and along $x_i$

$$i'A_i = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

always a constant matrix
Denavit-Hartenberg matrix


\[
i^{-1}A_i(q_i) = i^{-1}A_i'(q_i) i' A_i = \begin{bmatrix}
    c\theta_i & -c\alpha_i s\theta_i & s\alpha_i s\theta_i & a_i c\theta_i \\
    s\theta_i & c\alpha_i c\theta_i & -s\alpha_i c\theta_i & a_i s\theta_i \\
    0 & s\alpha_i & c\alpha_i & d_i \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

compact notation: \( c = \cos, \ s = \sin \)

super-compact notation: \( c_i = \cos q_i, \ s_i = \sin q_i \)
Direct kinematics of manipulators

• product $^0A_1(q_1)^1A_2(q_2)\ldots^{n-1}A_n(q_n)$
• $q = (q_1,\ldots,q_n)$

description “internal” to the robot using

- $r = f_r(q)$

description “external” to the robot using

- $^B_T^E = ^B_T^00A_1(q_1)^1A_2(q_2)\ldots^{n-1}A_n(q_n)^nT_E$
- $r = (r_1,\ldots,r_m)$

alternative descriptions of the direct kinematics of the robot
Example: SCARA robot

Sankyo SCARA 8438

Sankyo SCARA SR 8447

video
Step 1: joint axes

all parallel (or coincident)

twists $\alpha_i = 0$ or $\pi$

J1 shoulder

J2 elbow

J3 prismatic $\equiv$ J4 revolute
Step 2: link axes

the vertical “heights” of the link axes are arbitrary (for the time being)
Step 3: frames

axes $y_i$ for $i > 0$
are not shown
(nor needed; they form
right-handed frames)
Step 4: DH table of parameters

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$a_1$</td>
<td>$d_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$a_2$</td>
<td>0</td>
<td>$q_2$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$q_3$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\pi$</td>
<td>0</td>
<td>$d_4$</td>
<td>$q_4$</td>
</tr>
</tbody>
</table>

Note that:
- $d_1$ and $d_4$ could be set $= 0$
- Here, it is $d_4 < 0$
Step 5: transformation matrices

\[
0A_1(q_1) = \begin{bmatrix}
c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\
s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\
0 & 0 & 1 & d_1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
1A_2(q_2) = \begin{bmatrix}
c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\
s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
2A_3(q_3) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
q = (q_1, q_2, q_3, q_4) = (\theta_1, \theta_2, d_3, \theta_4)
\]

\[
3A_4(q_4) = \begin{bmatrix}
c\theta_4 & s\theta_4 & 0 & 0 \\
s\theta_4 & -c\theta_4 & 0 & 0 \\
0 & 0 & -1 & d_4 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Step 6a: direct kinematics
as homogeneous matrix $^B_T E$ (products of $^i A_{i+1}$)

\[ ^0 A_3(q_1,q_2,q_3) = \]
\[
\begin{bmatrix}
    c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\
    s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\
    0 & 0 & 1 & d_1 + q_3 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ ^3 A_4(q_4) = \]
\[
\begin{bmatrix}
    c_4 & s_4 & 0 & 0 \\
    s_4 & -c_4 & 0 & 0 \\
    0 & 0 & -1 & d_4 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ R(q_1,q_2,q_4) = [n \; s \; a] \]

\[ ^B T_E = ^0 A_4(q_1,q_2,q_3,q_4) = \]
\[
\begin{bmatrix}
    c_{124} & s_{124} & 0 & a_1 c_1 + a_2 c_{12} \\
    s_{124} & -c_{124} & 0 & a_1 s_1 + a_2 s_{12} \\
    0 & 0 & -1 & d_1 + q_3 + d_4 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

$p = p(q_1,q_2,q_3)$

$^B T_0 = ^4 T_E = I$
Step 6b: direct kinematics
as task vector \( r \in \mathbb{R}^m \)

\[
0A_4(q_1,q_2,q_3,q_4) = \begin{bmatrix}
  c_{124} & s_{124} & 0 \\
  s_{124} & -c_{124} & 0 \\
  0 & 0 & -1 \\
  0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  a_1c_1 + a_2c_{12} \\
  a_1s_1 + a_2s_{12} \\
  d_1 + q_3 + d_4 \\
  1 \\
\end{bmatrix}
\]

extract \( \alpha_z \) from \( R(q_1,q_2,q_4) \)

\[
r = \begin{bmatrix}
  p_x \\
  p_y \\
  p_z \\
  \alpha_z \\
\end{bmatrix}
= f_r(q) = \begin{bmatrix}
  a_1c_1 + a_2c_{12} \\
  a_1s_1 + a_2s_{12} \\
  d_1 + q_3 + d_4 \\
  q_1 + q_2 + q_4 \\
\end{bmatrix}
\in \mathbb{R}^4
\]

take \( p(q_1,q_2,q_3) \) as such
Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

shoulder offset

“one possible” DH assignment of frames is shown
determine the associated
- DH parameters table
- homogeneous transformation matrices
- direct kinematics
write a program for computing the direct kinematics
- numerically (Matlab)
- symbolically (Mathematica, Maple, Symbolic Manipulation Toolbox of Matlab, ...)

Robotics 1
DH table for Stanford manipulator

- 6-dof: 2R-1P-3R (spherical wrist)

<table>
<thead>
<tr>
<th>i</th>
<th>$\alpha_i$</th>
<th>$a_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>$d_1&gt;0$</td>
<td>$q_1=0$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/2$</td>
<td>0</td>
<td>$d_2&gt;0$</td>
<td>$q_2=0$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$q_3&gt;0$</td>
<td>$-\pi/2$</td>
</tr>
<tr>
<td>4</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_4=0$</td>
</tr>
<tr>
<td>5</td>
<td>$\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$q_5=-\pi/2$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$d_6&gt;0$</td>
<td>$q_6=0$</td>
</tr>
</tbody>
</table>

Joint variables are in red, with their current value in the shown configuration.
KUKA LWR 4+

- 7R (no offsets, spherical shoulder and spherical wrist)

- determine frames and table of DH parameters
- homogeneous transformation matrices
- direct kinematics
- \( d_1 \) and \( d_7 \) can be set = 0 or not (as needed)

available at DIAG Robotics Lab

Robotics 1
KUKA KR5 Sixx R650

- 6R (offsets at shoulder and elbow, spherical wrist)

- Determine frames and table of DH parameters
- Homogeneous transformation matrices
- Direct kinematics

available at DIAG Robotics Lab
Appendix: Modified DH convention

- a modified version used in J. Craig’s book “Introduction to Robotics”, 1986
  - has $z_i$ axis on joint $i$
  - $a_{i-1}$ & $\alpha_{i-1} =$ distance & twist angle from $z_{i-1}$ to $z_i$, measured along & about $x_{i-1}$
  - $d_i$ & $\theta_i =$ distance & angle from $x_{i-1}$ to $x_i$, measured along & about $z_i$
  - source of much confusion... if you are not aware of it (or don’t mention it!)
  - convenient with link flexibility: a rigid frame at the base, another at the tip...

### Modified DH

\[
\begin{align*}
&\text{classical (or distal)} \\
&i^{-1} A_i = 
\begin{pmatrix}
c \theta_i & -c \alpha_i s \theta_i & s \alpha_i s \theta_i & a_i c \theta_i \\
s \theta_i & c \alpha_i c \theta_i & -s \alpha_i c \theta_i & a_i s \theta_i \\
0 & s \alpha_i & c \alpha_i & d_i \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&\text{modified (or proximal)} \\
&i^{-1} A_i^{\text{mod}} = 
\begin{pmatrix}
c \theta_i & -s \theta_i & 0 & a_{i-1} \\
c \alpha_i-1 s \theta_i & c \alpha_i-1 c \theta_i & -s \alpha_i-1 & -d_i s \alpha_i-1 \\
s \alpha_i-1 s \theta_i & s \alpha_i-1 c \theta_i & c \alpha_i-1 & d_i c \alpha_i-1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}
\]

Modified DH tends to place frames at the base of each link.

Planar 2R example.